Stretched-exponential behavior in Ising critical dynamics

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By use of Monte Carlo simulations in 1000^2 and 100^3 Ising systems in the para phase, magnetic relaxation is shown to have a Kohlrausch-type stretched-exponential behavior, $M(t) \sim \exp(-t/\tau)^{\alpha}$, $0<\alpha<1$, for $t<t_c$ and normal Debye relaxation with $\alpha=1$ for $t>t_c$; the crossover time t_c diverges near the transition point T_c. The average relaxation time τ shows normal critical slowing down: $\tau \sim (T-T_c)^{-\nu z}$, $\nu z \simeq 1.8$ and 1.1 in dimensions $d=2$ and 3, respectively. We find $\alpha \simeq 0.33$ for $d=2$ and 0.4 for $d=3$. These are compared with previous observations of stretched-exponential relaxation and critical slowing-down behavior of Ising critical dynamics.

The study of relaxation of some average macroscopic variables, before the (many-body) system reaches equilibrium (thermodynamic or statistical), has recently become a subject of great interest. The phenomenon has been studied, experimentally and using computer simulations, in a widely different class of systems. The systems studied include polymers, glasses, spin-glasses, etc.[1]

The critical dynamics near ordinary (continuous) phase transitions, which have been extensively studied, show a typical relaxation behavior[2]

$$\eta_t(T) = \eta_\infty(T) - A(T)\exp(-t/\tau) \tag{1}$$

for the response function $\eta_t(T)$. The relaxation time $\tau(T)$ shows a critical slowing down, as the critical point T_c is approached, with an exponent νz

$$\tau \sim \xi^z \sim (T-T_c)^{-\nu z}, \tag{2}$$

where z is the dynamic exponent and ν is the exponent with which the correlation length ξ diverges near T_c. For a random statistical system near its fluctuation driven singular (percolation) point[3] p_c,

$$\tau \sim \xi^z \sim (p-p_c)^{-\nu z}. \tag{3}$$

The value of the exponent νz depends on dimension (d), as well as on the symmetry.

An altogether different critical dynamics was suggested to characterize relaxation phenomena of glassy systems where, near the glass transition temperature T_g, the response function $\eta_t(T)$ is observed to behave as follows:

$$\eta_t(T) = \eta_\infty(T) - A(T)\exp(-t/\tau)^{\alpha}, \tag{4}$$

where $0<\alpha<1$. A careful analysis of the different observations[4] indicates that α tends to a constant value $\frac{1}{3}$ as T approaches the glass transition temperature T_g. The form of Eq. (4) is commonly known as the Kohlrausch stretched-exponential form. The relaxation time $\tau(T)$ in some cases (of glass) is observed to show the so-called Vogel-Fulcher[5] behavior which is considered to be the most important characteristics of the following glassy relaxation behavior:

$$\tau \sim \exp[1/(T-T_0)], \tag{5}$$

where $T_0 < T_g$ in general. Different novel mechanisms[6] have been suggested to explain this anomalous dynamical behavior [Kohlrausch stretched-exponential (4) and Vogel-Fulcher behavior (5)] in glasses using models involving a hierarchically organized set of free-energy barriers or using Lifshitz-like arguments for the density of states in the tail of random matrices. Accurate (simulation) data[7-9] for spin-glass relaxations indicate, however, a critical slowing-down behavior [cf. Eq. (2)], rather than Vogel-Fulcher-type behavior, for the temperature variation of τ.

An important observation found in recent years was that this stretched-exponential [$\alpha<1$ in (4)] behavior is also found in pure Ising dynamics,[10,11] and in simple disordered (percolating) systems.[12] Although the stretched-exponential behavior is observed in these systems, none of these systems obeys Vogel-Fulcher law for the average relaxation time τ; rather, normal critical slowing down ($\tau \sim \xi^z$) is observed.

The critically slow dynamics [of, say, the magnetism $M(t)$] of Ising systems near T_c has been extensively studied using Monte Carlo, etc., simulations.[2,13] Such relaxation data, for normal critical phenomena, have traditionally been fitted[2,13] to the simple exponential form, with $\alpha=1$ [in (4)], and one estimates the dynamic exponent z [in (2)]. Indeed, systematic deviations (indicating $\alpha<1$) were clearly observed[13] for early (small-time) relaxation data, when fitted to the simple exponential form. These deviations (which become prominent as T approaches T_c) were, in fact, attributed as errors (which is rather systematic than random) or nonlinear relaxation behavior in relaxation time τ.[13] As mentioned before, a stretched-exponential relaxation behavior ($\alpha \ll 1$) has recently been observed and recognized for Ising critical dynamics very near the transition point T_c.[10,11] These observations are, however, restricted to the determination of α and not extended to that of z. We repeated the study of dynamics for fairly large Ising systems and we show that the Monte Carlo simulation data for (pure) Ising systems all fit with the relaxation type (4) with $\alpha \simeq 0.33$ and 0.40 for $d=2$ and 3, respectively, for times (Monte Carlo steps) t below a crossover time $t_c(T)$, above which a crossover to $\alpha=1$ occurs. t_c is usually very small (of the order of 10 Monte Carlo steps for Ising systems for $T/T_c \simeq 1.1$ in 3D) and diverges near T_c [$t_c \propto \epsilon^{-x}$, $\epsilon \equiv (T-T_c)/T_c$; we could not estimate the exponent x from our data]. The relaxation time τ has the usual critical slowing-down divergence $\tau \sim \xi^z \sim \epsilon^{-\nu z}$ [as in (2)]; $\nu z = 1.8$ and 1.1 in $d=2$ and 3, respectively.

We have studied the dynamics of two- and three-dimensional Ising systems with system size 1000^2 and 100^3, respectively, and the study has essentially been confined to the para state ($T/T_c > 1.0$). Our range of observation has been $1.01 \le T/T_c \le 1.3$ in $d=2$ and $1.005 \le T/T_c \le 1.100$ in $d=3$. The critical temperatures were taken as equal to $2/\ln(1+\sqrt{2})$ ($d=2$) and equal to 4.511 149 3 ($d=3$).[13]

The simulation results are shown in Figs. 1-3 for a square lattice and in Figs. 4-6 for a simple cubic lattice. The results show that for both $d=2$ and 3 we get a deviation from the straight line in the first few steps (~ 40 for $d=2$ and ~ 10 for $d=3$ for $T/T_c = 1.10$) in the $\ln M(t)$ vs t plot (Figs. 1 and 4). This clearly shows that the relaxation is certainly not simple exponential (in at least the first few steps up to t_c) and cannot be associated with a single value of relaxation time τ. However, after these few (t_c) steps, the magnetization shows an exponential decay, and this t_c increases as $T/T_c \rightarrow 1.0$.

The best fit of $\ln M(t)$ variations for $t < t_c$ seem to be obtained with $t^{0.33}$ in two dimensions and $t^{0.40}$ in three dimensions (Figs. 2 and 5), indicating that the possible fitting is of the stretched-exponential type having exponent $\alpha \simeq 0.33$ for $d=2$ and 0.40 for $d=3$. We do not find the value of α to depend on temperature and converge to the value $\frac{1}{3}$ as $T \rightarrow T_c$ as observed in spin glasses.[4] It seems that for pure Ising systems (as in percolating systems[12]) the exponent α is only dimension dependent.

In the $\ln M(t)$ vs t curves (Figs. 1 and 4), we see a systematic increase in the curved portion (for which $\alpha < 1$) as T/T_c approaches unity. This indicates that the crossover time t_c, after which $\alpha = 1$, increases as $T \rightarrow T_c$ (although no exponent value could be determined for its divergence near T_c). The slope of the curve for $t > t_c$ gives $1/\tau$. We have plotted τ (Figs. 3 and 6) and get 1.8 and 1.1 for the exponent (νz) values for the τ divergence near T_c in $d=2$ and 3, respectively (compared to[2,13] $\nu z \simeq 2.0$ and 1.4 in $d=2$ and 3). Vogel-Fulcher-type behavior is neither expected nor observed.

Although the effects are thus clearly there, the origin of the stretched-exponential relaxation behavior (even for disordered or para phase at $T > T_c$) for normal critical dynamics of (pure) Ising systems is still not very clear. It may be mentioned that in the ferro phase ($T < T_c$) a stretched-exponential behavior ($\alpha = \frac{1}{2}$ for $d=2$ and $\alpha = 1$ for $d \ge 3$) has been argued[14] for asymptotic relaxation ($t \rightarrow \infty$) over droplet fluctuations; specifically, stretched-exponential behavior should occur for $t > t_c'$; where t_c' is some appropriate crossover time such that $\alpha = 1$ for $t < t_c'$ and $\alpha < 1$ for $t > t_c'$. Apart from numerical disagree-

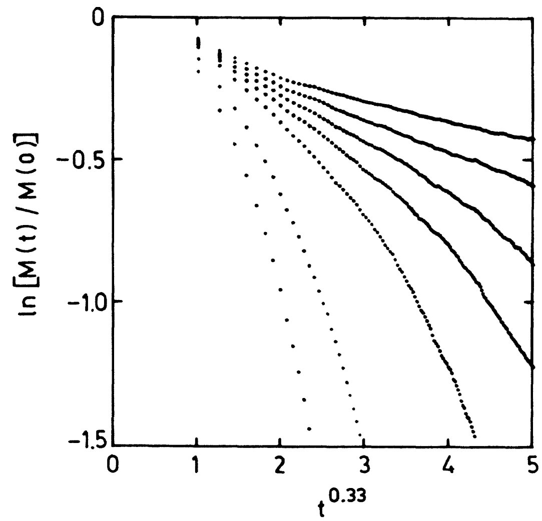

FIG. 2. A possible fitting of $M(t)/M(0)$ vs t^{α} to an exponent value $\alpha = 0.33$ in $d=2$.

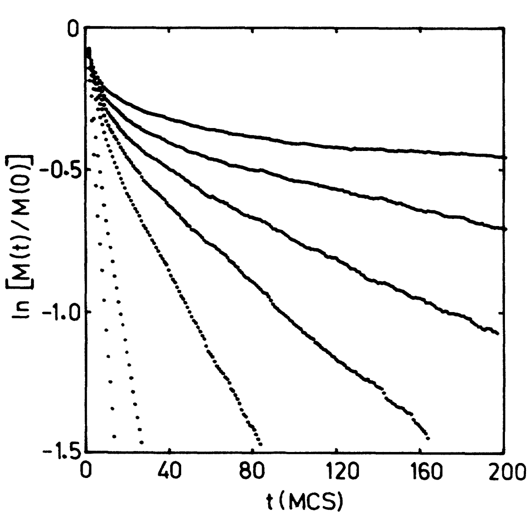

FIG. 1. Time development of magnetization $M(t)/M(0)$ for different temperatures, $T/T_c = 1.01$, 1.03, 1.05, 1.07, 1.10, 1.20, and 1.30, respectively (from top to bottom). Lattice size is 1000^2.

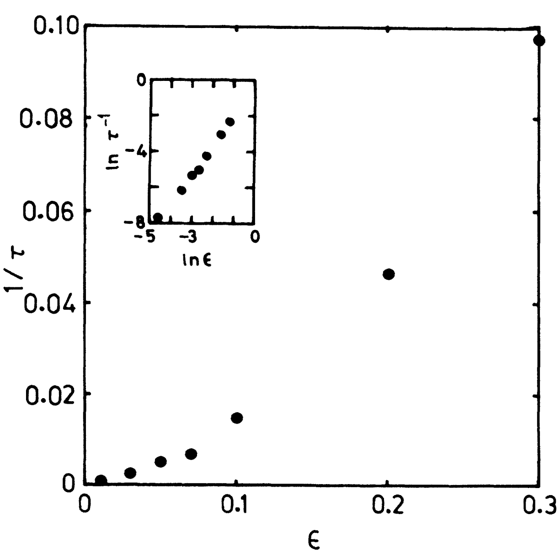

FIG. 3. Variation of relaxation time τ^{-1} against ϵ. Inset: The log-log plot giving the exponent $\nu z \simeq 1.8$ in 2D.

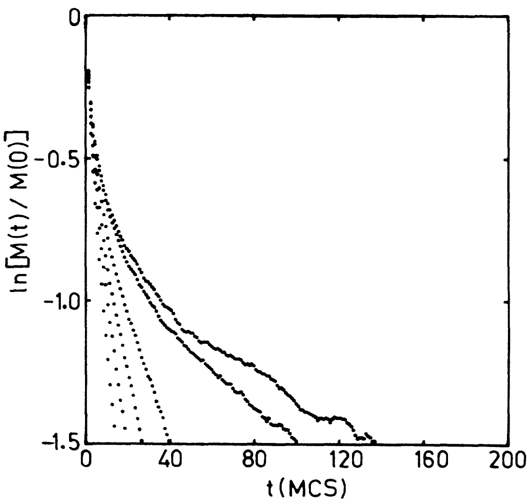

FIG. 4. Time development of magnetization $M(t)/M(0)$ for different temperatures; $T/T_c = 1.005$, 1.010, 1.030, 1.050, 1.070, and 1.100, respectively (from top to bottom). Lattice size is 100^3.

ments[10,11] (in values of α) the qualitative behavior seems to be exactly opposite to what we observe. It seems, however, that the individual spins diffuse in the thermally produced dynamic (spin-flipped) fractal produced by other spins. When the diffusion spread ($\sim t^{1/d_w}$) is less than the thermal correlation length ξ, the solution of the diffusion equation gives the anomalous behavior,[3] and magnetization has a time dependence of the form $\exp(-D)$, where spin-diffusion spread $D \sim t^{2/d_w}$. When diffusion spread is greater than ξ, the diffusion does not see the fractal and in such cases we get normal diffusion, with $\alpha = 1$. The crossover time t_c is determined by $t_c^{1/d_w} \sim \xi$ or $t_c \sim \epsilon^{-x}$, where $x = \nu d_w$. Thus, anomalous diffusion or classical localization may also be considered[12] here to be the cause of the fractional value of α. The value of d_w on such Ising (correlated) clusters, however, are not known.[15]

We thus see that the stretched-exponential behavior is

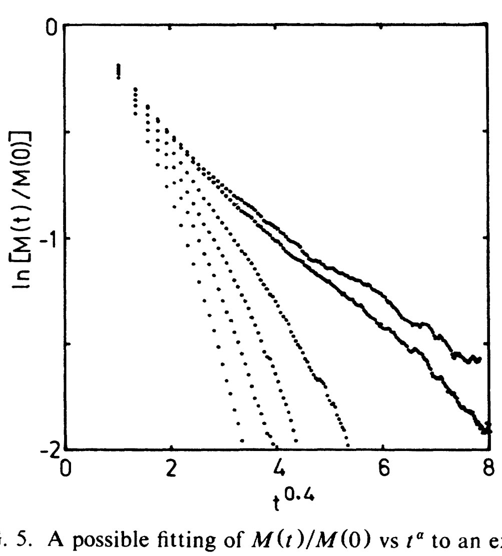

FIG. 5. A possible fitting of $M(t)/M(0)$ vs t^α to an exponent value $\alpha = 0.4$ in $d = 3$.

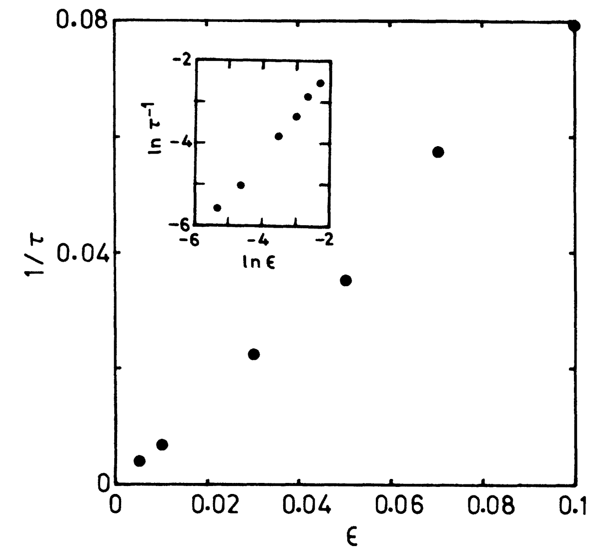

FIG. 6. Variation of relaxation time τ^{-1} against ϵ. Inset: The log-log plot giving the exponent $\nu z \simeq 1.1$ in 3D.

rather general and occurs even in the critical dynamics of simple Ising-like systems. In fact, the important point we note is the existence of a crossover time t_c, below which the relaxation is stretched exponential (with $\alpha \simeq 2/d_w < 1$) and simple exponential ($\alpha = 1$); above it $t_c \sim \epsilon^{-x}$, with $x \simeq \nu d_w$, so that the stretched-exponential region tends to dominate as one approaches the critical point. One also observes critical slowing-down behavior of average relaxation time τ ($\tau \sim \epsilon^{-\nu z}$). Normally this region ($t < t_c$), for which $\alpha < 1$ is observed, is very small (e.g., $t_c \sim 20$ for 3D Ising systems, even for $T/T_c = 1.05$; of course $t_c \rightarrow \infty$ as $T \rightarrow T_c$), while for glasses, this region may be normally and routinely very large (e.g., $t_c \sim 10^3$ for $T/T_g = 1.05$ for 3D Ising spin glass[8]). In none of these well-studied cases is Vogel-Fulcher behavior for $\tau(T)$ observed (see also Ref. 1 for comments on the lack of clear evidence of Vogel-Fulcher behavior even in standard glasses); rather, the ordinary critical slowing-down-type behavior is observed.

Two established different kinds of relaxation behaviors are thus observed in many-body systems:

(a) Kohlraush stretched-exponential relaxation with critical slowing down: $\eta_t(T) \sim \exp(-t/\tau)^\alpha$; $\alpha < 1$ and $\tau(T) \sim \xi^z \sim \epsilon^{-\nu z}$; $\alpha < 1$ and νz, depending on the dimension and symmetry of the order parameter. For Ising systems $\alpha \simeq 0.33$, 0.4, and 0.5 for $d = 2$, 3, and 4, respectively,[10,11] with $\nu z \simeq 2.0$, 1.4, and 1.0 (exact).[2] For percolating systems[12] $\alpha \simeq 0.6$ and $\nu z \simeq 4.0$ for $d = 2$. Above the lower critical dimensions, $\alpha \simeq \frac{1}{3}$ and $\nu z \simeq 7.9$ for Ising spin glass[8] and $\simeq 8.54$ for XY spin glass[9] in $d = 3$.

(b) Kohlrausch stretched-exponential relaxation with Vogel-Fulcher behavior for relaxation time: $\eta_t(T) \sim \exp(-t/\tau)^\alpha$, $\alpha < 1$ and $\tau(T) \sim \exp[1/(T - T_0)]$. This type of behavior now seems clearly ruled out for spin-glass dynamics,[8] although, for some dipolar glass this Vogel-Fulcher–like behavior for τ is traditionally being discussed.[16] In fact, in glass, where the free energy has many metastable (local) minima, the relaxation time

$\tau \sim$ (hopping diffusion constant)$^{-1}$ comes from thermally activated hopping over "typical" barrier heights h_0 $[\tau \sim \exp(-h_0/T)]$.[12] In cases where there is a thermodynamic rearrangement of the barrier heights due to cooporative structural rearrangements, the typical barrier height may diverge as $h_0 \sim \xi' \sim (T-T_0)^{-\nu'}$ near the structural-rearrangement transition point T_0. This would give a Vogel-Fulcher-like relaxation behavior[17] ($\tau \sim \exp[A/(T-T_0)^{\nu'}]$). Observation of such behavior (in standard glass, for example) would then indicate the existence and divergence of another correlation length near the barrier-height rearrangement transition point.

We would like to thank M. Barma for bringing Refs. 10, 11, and 14 to our attention.

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