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Existence of an internal quasimode for a sine-Gordon soliton

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We apply our recently derived Hamiltonian theory of constrained nonlinear Klein-Gordon systems to the problem of a single sine-Gordon (SG) kink and show that there exists a quasi-internal degree of freedom which we describe by a collective variable. We show that the collective variable used by Rice to describe the regular oscillations of a ϕ^4 and SG kink internal mode is actually, in the exact theory, coupled to the phonon field. In the ϕ^4 case, the internal mode is an exact eigenstate of the linearized ϕ^4 equation (when linearized about the single-kink solution) whose eigenfrequency lies in the gap below the phonon band edge. In the SG case there is no exact bound eigenstate (other than the zero-frequency Goldstone mode) and so the frequency calculated by Rice corresponding to the quasi-internal mode for the Sg system is in the phonon continuum. Therefore, any bound oscillation at the Rice frequency in the SG system decays via spontaneous emission of phonons. However, rather surprisingly, we find by numerical solution of the SG equation of motion that the internal mode is extremely long-lived with a lifetime of well over 300 oscillations at a frequency $\omega_s=(1.004\pm0.001)\Gamma_0$, where Γ_0 is the frequency at the phonon band edge. We calculate the phonon dressing of the bare kink ansatz using the collective variable theory in lowest order and show the renormalized "dressed" frequency Ω_d of the internal mode agrees with the frequency observed from simulation ω_s to within 5%. We calculate the linewidth of the radiation from simulation and obtain $1/\tau_s=(0.003\pm0.001)\Gamma_0$. Using collective variable theory and the simple model of radiation reaction we obtain for the lifetime the value $1/\tau=0.002\Gamma_0$. The physical observability and relationship to other investigations of collective variable treatments of internal modes are analyzed and discussed.

I. INTRODUCTION

In this paper we consider the continuum sine-Gordon (SG) equation and show that the single kink solution has a long-lived internal quasimode whose frequency from simulation is $\omega_s=(1.004\pm0.001)\Gamma_0$ (where Γ_0 is the frequency of the lower phonon band edge) and whose inverse lifetime from simulation is $1/\tau_s=(0.003\pm0.001)\Gamma_0$. The quasimode is that of an internal oscillation of the kink, that is, a temporal oscillation of the slope of the kink (at its center) about its static SG value. The original suggestion for an internal quasimode for the SG system was made by Rice[1,2] who introduced collective variables $X(t)$ for the center of mass of the kink, and $l(t)$ for the kink's length and derived equations of motion for the collective variables in the approximation that all the phonon degrees of freedom were set equal to zero. To be more explicit, in his derivation which is valid for the SG, ϕ^4, and double sine-Gordon systems, Rice considers the continuum steady-state single-kink solution which for the SG case is

$$\phi=4\tan^{-1}\exp\left[\frac{\pi}{l_0(1-v^2)^{1/2}}(x-vt)\right], \quad (1.1)$$

where $2\pi/l_0$ represents the slope of the kink evaluated at its center, l_0 a measure of the length of the kink, v the kink's constant velocity, and ϕ the single kink solution to the SG equation of motion

$$\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}+\left[\frac{\pi}{l_0}\right]^2\sin\phi=0. \quad (1.2a)$$

Equation (1.2a) is derivable from the Lagrangian density

$$\mathcal{L}=\frac{1}{2}\left[\frac{\partial\phi}{\partial t}\right]^2-\frac{1}{2}\left[\frac{\partial\phi}{\partial x}\right]^2-\left[\frac{\pi}{l_0}\right]^2(1-\cos\phi). \quad (1.2b)$$

Rice replaces vt by $X(t)$ and $l_0(1-v^2)^{1/2}$ by $l(t)$ and writes the solution as

$$\phi\to\sigma\equiv\sigma\{[\pi/l(t)][x-X(t)]\},$$

substitutes the new form σ for the solution into the Lagrangian density for ϕ, and integrates over the continuum variable x to obtain an effective Lagrangian $L_{\text{eff}}[X(t),l(t)]$. Next he derives the Hamiltonian and the equations of motion for $X(t)$, $l(t)$, and their respective conjugate momenta $P_X(t)$ and $P_l(t)$. The equations of motion for $X(t)$ and $l(t)$ are coupled to each other and since $X(t)$ is cyclic, P_X is a constant of the motion. P_X depends on $l(t)$ as well as the velocity of the center of mass of the kink $\dot{X}$. Therefore, since $l(t)$ is oscillatory, $\dot{X}$ must also be oscillatory in order for P_X to remain a constant of the motion.

We consider the center-of-mass frame of the kink in order to concentrate on the $l(t)$ motion and so we set $X(t)=\dot{X}(t)=0$ and, thus, $P_X=0$. The resultant equation of motion for $l(t)$ is nonlinear and Rice solved it obtaining the remarkable result that $l(t)$ oscillates harmonically about a center which depends on the constant energy and with a frequency which is independent of energy. His results are

$$\Omega_{SG}=\left[\frac{12}{\pi^2}\right]^{1/2}\frac{\pi}{l_0}\quad \text{(sine-Gordon)}\ , \tag{1.3a}$$

$$\Omega_{\phi^4}=[3/(\pi^2-6)]^{1/2}\frac{\pi}{l_0}\quad (\phi^4)\ . \tag{1.3b}$$

There is a fundamental question raised by Rice's work. We note that the equations of motion for $l(t)$ for the SG and ϕ^4 systems have exactly the same structure, the only difference is in the value of constant coefficients in each equation. The coefficients are different because of the different substrate potentials of the two systems. However, linearizing about the single-kink solution for each system shows that the eigenfunction spectrum of the SG and ϕ^4 systems are not the same. Namely, the ϕ^4 system possesses two localized modes and a continuum of linear phonon eigenstates. The two localized modes are the zero-frequency Goldstone mode and an internal oscillatory "shape mode" of the kink whose nonzero frequency lies below the lowest phonon frequency, that is, below the frequency of the phonon band edge. The SG system, on the other hand, possesses only a single localized mode which is the zero-frequency Goldstone mode, and a continuum of linear phonon states. The SG system does not support a localized oscillatory eigenmode with nonzero eigenvalue whereas the ϕ^4 system does. Therefore, in the ϕ^4 case, we expect the collective variable $l(t)$ to describe the oscillatory bound state and Rice's value[1] of Ω_{ϕ^4} is quite close to the exact frequency of that state. (We will show in Sec. II that it is possible to set up a collective variable theory for the ϕ^4 system which gives the exact ϕ^4 small oscillation frequency.) In the SG case, however, where there is no localized oscillatory collective mode, the question arises as to what the frequency Ω_{SG} corresponding to the $l(t)$ motion means.

We gain insight into the physics by first noting that the frequency Ω_{SG} is larger than the lowest-frequency phonon and therefore Ω_{SG} lies above phonon band edge. The "state" corresponding to Ω_{SG} thus resonates with phonon modes and radiates phonons. The direct resonance of the "state" or quasimode with the phonons is what prevents the mode from being an exact eigenstate of the linearized SG system.

The question that remains is whether or not the lifetime of the internal SG quasimode is significantly long enough to play any role in SG dynamics. Rice[1] was not able to answer this question since he did not possess a complete collective variable procedure that takes into account the phonon excitations. His calculation neglected the phonons completely. Using a method similar to Rice, Fernandez *et al.*[3] have investigated the relativistic dynamics of a SG kink that is immersed in a medium with a position-dependent index of refraction $n(x)$ and chose $n(x)$ to be such that the effect on the kink was that it was trapped in a parobolic potential well. They showed analytically that a stable constant solution existed for the function $l(t)$, as well as other time-dependent solutions. They also performed simulations where the kink oscillated inside the parabolic well. The functions $X(t)$ and $l(t)$ were monitored and oscillatory behavior was observed in both collective variables accompanied by phonons emitted by the kink. The phonons are easily discernible in the simulations [see Fig. (2b) of Ref. 3] where the length of the kink was given an initial value which deviates about 18% from its constant equilibrium value. Since Fernandez and co-workers,[3] like Rice,[1] completely neglected the phonon excitations in their analytic calculations, they were not able to account for dressing and radiation effects.

In a recent paper[4] we constructed an exact collective variable formalism which takes into account all dressing and radiation effects. In the formalism, the original or "old" field $\phi(x,t)$ is broken up into a set of "new" variables consisting of a single collective variable for each nonlinear collective mode [such as $X(t)$ and $l(t)$ discussed above] and a new field variable χ which represents the effects of phonons that are not necessarily perturbative.[4] We based the collective variable theory on Dirac's theory of constrained Hamiltonian systems because, for each collective variable introduced into the "new" set of variables, one must also introduce two constraints—one for the collective variable and the second for its conjugate momentum. Therefore, the number of degrees of freedom in transforming from the old to the new variables is conserved.

We frequently refer to our collective variable formalism as a projection operator method since the equations of motion for the collective variables and field χ are obtained by projecting the original or "old" equation of motion onto appropriate directions in Hilbert space, as we show in Sec. II. We have shown[4] that the equations of motion obtained using the projection method are identical with those obtained using the Dirac bracket procedure. The benefit of the projection approach is that the equations of motion are derived with much less work than with the Dirac procedure.

In Sec. II we derive the full collective variable theory for a kink with an internal or quasi-internal mode. The equations will be in terms of a general substrate potential V and therefore valid for systems such as the SG, ϕ^4, and double sine-Gordon systems. We will then specialize in the SG case and show that our equations reduce to those of Rice[1] and Fernandez *et al.*[3] when we set the phonon field χ to zero. We discuss the simulations of the SG quasimode in Sec. III. Section IV contains a derivation of a model for the radiation linewidth and in Sec. V we discuss our conclusions. We put the details of the derivation of the phonon dressed frequency, the calculation of the radiation linewidth, and the analytic evaluation of integrals in three Appendices.

II. EQUATIONS OF MOTION

The continuum SG field ϕ obeys the continuum SG equation given by Eq. (1.2a) with the steady-state solution Eq. (1.1). However, when the quasimode is excited by some process, Eq. (1.1) no longer correctly describes the evolution of the system. That is, the slope of the kink is no longer constant nor is the position of the kink given by the quantity vt. We therefore introduce time-dependent collective variable functions in order to describe the more general behavior that occurs when the quasimode is excited. Let $X(t)$ denote the center of mass of the kink and let $2\Gamma(t)$ denote the slope of the kink evaluated at its center. The new field variable will be denoted by χ. We set the speed of sound in our system to unity, i.e., $c=1$, so x and t, the continuous space and time variables, have the same units.

The first step in obtaining the Hamiltonian equations of motion for the collective variables is to define

$$\phi(x,t)=\sigma[\xi(t)]+\chi[\xi(t),t] , \tag{2.1a}$$

$$\xi(t)\equiv\Gamma(t)[x-X(t)] , \tag{2.1b}$$

where $\sigma[\xi(t)]$ is the steady-state single-kink solution corresponding to the substrate potential V in the equation of motion

$$\frac{\partial^2\phi}{\partial t^2}-\frac{\partial^2\phi}{\partial x^2}=-\frac{\partial V}{\partial\phi} \tag{2.1c}$$

and χ is the phonon field. Below, we will choose V to be the SG potential, but for now V is any potential that supports stationary single-kink solutions. Note that the argument of σ and χ is a function of the collective variables.

Since we have increased the number of degrees of freedom by four, namely $X(t)$, $\Gamma(t)$, and their conjugate momenta, we need to specify four constraints and they are

$$C_{1X}=\int\frac{\partial\sigma}{\partial X}\{\Gamma(t)[x-X(t)]\}\chi\{\Gamma(t)[x-X(t)],t\}dx$$
$$=-\langle\sigma'(\xi)|\chi(\xi,t)\rangle=0 , \tag{2.2a}$$

$$C_{1\Gamma}=\int\frac{\partial\sigma}{\partial\Gamma}\{\Gamma(t)[x-X(t)]\}\chi\{\Gamma(t)[x-X(t)],t\}dx$$
$$=\frac{1}{\Gamma^2}\langle\xi\sigma'(\xi)|\chi(\xi,t)\rangle=0 , \tag{2.2b}$$

$$C_{2X}=\int\frac{\partial\sigma}{\partial X}\{\Gamma(t)[x-X(t)]\}\pi\{\Gamma(t)[x-X(t)],t\}dx$$
$$=-\langle\sigma'(\xi)|\pi(\xi,t)\rangle=0 , \tag{2.2c}$$

$$C_{2\Gamma}=\int\frac{\partial\sigma}{\partial\Gamma}\{\Gamma(t)[x-X(t)]\}\pi\{\Gamma(t)[x-X(t)],t\}dx$$
$$=\frac{1}{\Gamma^2}\langle\xi\sigma'(\xi)|\pi(\xi,t)\rangle=0 , \tag{2.2d}$$

where $\pi\{\Gamma(t)[x-X(t)],t\}$ is the momentum conjugate to χ, the prime denotes the derivative with respect to ξ, and

$$\langle f|g\rangle\equiv\int f^*(\xi)g(\xi)d\xi .$$

Setting the constraints to zero determines the values of $X(t)$ and $\Gamma(t)$ that minimize the fluctuations about the kink form $\sigma[\xi(t)]$. It is in this manner that the collective variables $X(t)$ and $\Gamma(t)$ are given their physical meaning.

In Ref. 4 we proved the equivalence of the Hamiltonian equations of motion for $\dot\phi(x,t)$ and its conjugate momentum $\dot\Pi(x,t)$ to the collective variable equations of motion for $\dot X$, $\dot P_X$, $\dot\Gamma$, $\dot P_\Gamma$, $\partial\chi/\partial t$, and $\partial\pi/\partial t$ where the overdot indicates differentiation with respect to time. Furthermore, in Ref. 4 we proved that the equations of motion for $\ddot X$, $\ddot\Gamma$, and $\partial^2\chi/\partial t^2$, which are obtained after eliminating the momenta, are more directly obtained by substituting Eq. (2.1a) for $\phi(x,t)$ into the "original" equation of motion Eq. (2.1c) and projecting the resultant equation along $\partial\sigma/\partial X$ to derive the equation of motion for $\ddot X$, along $\partial\sigma/\partial\Gamma$ to derive the equation of motion for $\ddot\Gamma$, and projecting along the space orthogonal to the directions $\partial\sigma/\partial X$ and $\partial\sigma/\partial\Gamma$ to derive the equation of motion for $\partial^2\chi/\partial t^2$. To this end we substitute Eq. (2.1a) into Eq. (2.1c) and obtain

$$\frac{\partial^2\chi}{\partial t^2}-\chi''\left[\Gamma^2(1-\dot X^2)+2\xi\dot X\dot\Gamma-\left[\frac{\dot\Gamma}{\Gamma}\right]^2\xi^2\right]+2\frac{\partial\chi'}{\partial t}\left[\left[\frac{\dot\Gamma}{\Gamma}\right]\xi-\dot X\Gamma\right]+\chi'\left[\left[\frac{\ddot\Gamma}{\Gamma}\right]\xi-2\dot X\dot\Gamma-\ddot X\Gamma\right]+\frac{\partial V(\sigma+\chi)}{\partial\sigma}$$
$$=\sigma''\left[\Gamma^2(1-\dot X^2)+2\xi\dot X\dot\Gamma-\left[\frac{\dot\Gamma}{\Gamma}\right]^2\xi^2\right]-\sigma'\left[\left[\frac{\ddot\Gamma}{\Gamma}\right]\xi-2\dot X\dot\Gamma-\ddot X\Gamma\right] . \tag{2.3}$$

When we multiply Eq. (2.3) by $(\partial\sigma/\partial X)dx=-\sigma'd\xi$, integrate over ξ, and solve for $\ddot X$ we obtain

$$\ddot X=-\frac{\dot X\dot\Gamma}{\Gamma(1-b_X)}-\frac{1}{M_X(1-b_X)}\left[\langle\sigma'|\chi''\rangle\Gamma^2(1-\dot X^2)+2\langle\sigma'|\xi\chi''\rangle\dot X\dot\Gamma-\left[\frac{\dot\Gamma}{\Gamma}\right]^2\langle\sigma'|\xi^2\chi''\rangle\right.$$
$$\left.-2\frac{\dot\Gamma}{\Gamma}\left\langle\sigma'|\xi\frac{\partial\chi'}{\partial t}\right\rangle+2\dot X\Gamma\left\langle\sigma'\left|\frac{\partial\chi'}{\partial t}\right.\right\rangle-\frac{\ddot\Gamma}{\Gamma}\langle\sigma'|\xi\chi'\rangle+2\langle\sigma'|\chi'\rangle\dot X\dot\Gamma-\left\langle\sigma'\left|\frac{\partial V}{\partial\sigma}\right.\right\rangle\right] , \tag{2.4a}$$

where

$$M_X\equiv\Gamma\langle\sigma'|\sigma'\rangle \tag{2.4b}$$

is the bare mass of the kink associated with the X motion and

$$b_X \equiv \Gamma\langle \sigma''|\chi\rangle / M_X \ . \tag{2.4c}$$

Note that the term proportional to $\partial^2\chi/\partial t^2$ does not appear in Eq. (2.4a) because

$$\left\langle \sigma' \left| \frac{\partial^2 \chi}{\partial t^2} \right.\right\rangle = \frac{\partial^2}{\partial t^2}\langle \sigma'|\chi\rangle = 0$$

by virtue of the constraint $C_{1X}=0$. In deriving Eq. (2.4a) we have also made use of the general result $\Gamma\langle \sigma'|\xi\sigma''\rangle = -M_X/2$ which is obtained from an integration by parts.

Likewise, when we multiply Eq. (2.3) by $(\partial\sigma/\partial\Gamma)dx = (\xi/\Gamma^2)\sigma' d\xi$, integrate over ξ, and solve for $\ddot{\Gamma}$, we obtain

$$\ddot{\Gamma} = \frac{3\dot{\Gamma}^2}{2\Gamma(1-b_\Gamma)} - \frac{M_X(1-\dot{X}^2)}{2\Gamma M_\Gamma(1-b_\Gamma)} + \frac{1}{M_\Gamma(1-b_\Gamma)}\left[\langle \xi\sigma'|\chi''\rangle(1-\dot{X}^2) + \frac{2\dot{X}\dot{\Gamma}}{\Gamma^2}\langle \xi\sigma'|\chi''\xi\rangle - \frac{1}{\Gamma^2}\left[\frac{\dot{\Gamma}}{\Gamma}\right]^2 \langle \xi\sigma'|\xi^2\chi''\rangle \right.$$
$$\left. - \frac{2}{\Gamma^2}\frac{\dot{\Gamma}}{\Gamma}\left\langle \xi\sigma' \left| \xi\frac{\partial \chi'}{\partial t}\right.\right\rangle + \frac{2\dot{X}}{\Gamma}\left\langle \xi\sigma' \left| \frac{\partial \chi'}{\partial t}\right.\right\rangle + \left[\frac{2\dot{X}\dot{\Gamma}}{\Gamma^2} + \frac{\ddot{X}}{\Gamma}\right]\langle \xi\sigma'|\chi'\rangle - \frac{1}{\Gamma^2}\left\langle \xi\sigma' \left| \frac{\partial V}{\partial \sigma}\right.\right\rangle \right] , \tag{2.5a}$$

where

$$M_\Gamma \equiv \frac{1}{\Gamma^3}\langle \xi\sigma'|\xi\sigma'\rangle \tag{2.5b}$$

is the bare mass of the kink associated with the Γ motion and

$$b_\Gamma \equiv \frac{\langle \xi^2\sigma''|\chi\rangle}{\Gamma^3 M_\Gamma} \ . \tag{2.5c}$$

In order to arrive at the form for Eq. (2.5a) we used

$$\Gamma\langle \sigma'|\xi\sigma''\rangle = -M_X/2$$

and

$$\langle \xi\sigma'|\xi^2\sigma''\rangle/\Gamma^3 = -3M_\Gamma/2 \ .$$

We also used $\langle \xi\sigma'|\xi\chi'\rangle = -\langle \xi^2\sigma''|\chi\rangle$ which is merely an integration by part with an application of the constraint $C_{1\Gamma}=0$.

Next we need to project Eq. (2.3) onto a direction that is orthogonal to the $\partial\sigma/\partial X$ and $\partial\sigma/\partial\Gamma$ directions in order to obtain the equation of motion for χ. However, it is not necessary to do so as long as we consider the system of three equations [Eqs. (2.3), (2.4a), and (2.5a)] simultaneously. The projection needed to obtain the equation of motion for χ is then carried out implicitly (see Ref. 4 for a detailed explanation). Therefore, Eqs. (2.3), (2.4a), and (2.5a) are the exact system of equations that govern, respectively, the dynamics of the phonon field χ, the center of mass $X(t)$, and shape $\Gamma(t)$ of a single-kink system with potential V for which the field ϕ may be broken up as in Eq. (2.1a). We point out that we cannot really associate Eq. (2.4a), say, as *the* equation of motion for X since it is coupled to the other two equations (2.3) and (2.5a). Nevertheless, we refer to the equations according to the variable on their left-hand sides for convenience. Equations (2.3), (2.4a), and (2.5a) are identical to the equations of motion obtained using the Dirac bracket formalism, as shown in Ref. 4, and therefore satisfy the constraints of Eqs. (2.2a)–(2.2d) rigorously.

We obtain Rice's equations of motion for $\ddot{X}$ and $\ddot{\Gamma}$ when we set $\chi=0$ and $V=\Gamma_0^2[1-\cos(\sigma)]$ in Eqs. (2.4a) and (2.5a), where $\Gamma_0 \equiv \pi/l_0$ represents the frequency of the phonon band edge, i.e., the lowest-frequency phonon. Equations (2.4a) and (2.5a) for the case $\chi=0$ were obtained by Fernandez *et al.*[3] where they used

$$V=[1+n(x)][1-\cos\sigma] \ .$$

To facilitate the study of the quasi-internal mode we now set $X=\dot{X}=0$ for the remainder of the paper. Therefore, the only coupling we are concerned with is that between $\Gamma(t)$ and $\chi(x,t)$ of which there are two consequences: (1) dressing of the mode $\partial\sigma/\partial\Gamma$ and (2) radiation of phonons by the kink. For the remainder of Sec. II we discuss the dressing problem and the consequent renormalization of the small oscillation frequency of the quasi-internal mode. We will address the radiation problem in Sec. IV.

We begin our discussion of the dressing problem by first looking at the ϕ^4 system where there exists, in contrast to the SG system, an exact bound state whose eigenfunction of the linearized ϕ^4 equation has the form

$$\psi_b(\xi) \sim \tanh(\xi)\text{sech}(\xi)$$

but whose shape mode derived from the ansatz function (the steady-state ϕ^4 kink)

$$\sigma \sim \tanh(\Gamma x) = \tanh(\xi)$$

has the form $\partial\sigma/\partial\Gamma \sim \xi\,\text{sech}^2(\xi)$. We see that ψ_b and $\partial\sigma/\partial\Gamma$ are approximately equal for small ξ but, in general, they are not the same. Since ψ_b is the exact functional form of the small oscillation solution it does not have to be dressed. On the other hand, if we use, instead, $\partial\sigma/\partial\Gamma$ in our calculations, then, since $\partial\sigma/\partial\Gamma$ is only an approximation to the exact shape mode, it must necessarily be dressed by χ in order to correct for its inadequacy. If we do not allow χ to dress $\partial\sigma/\partial\Gamma$ then our calculations will be inaccurate. However, since we know the exact small oscillation eigenfunction for the bound state of the ϕ^4 case, we can bypass having to work with $\partial\sigma/\partial\Gamma$ and its dressing by constructing a more accurate ansatz to begin with, namely

$$\phi(x,t)=\hat{\sigma}[\Gamma(t)x]+\chi[\Gamma(t)x,t] , \tag{2.6a}$$

$$\hat{\sigma}[\Gamma(t)x]\equiv\sigma_{\phi^4}[\Gamma_0 x]+\int_{\Gamma_0}^{\Gamma}\psi_b(\Gamma' x)d\Gamma' , \tag{2.6b}$$

where $\sigma_{\phi^4}(\Gamma_0 x)=\tanh(\Gamma_0 x)$. Now we have the result from Eq. (2.6b) that the eigenfunction and shape mode are equal $\partial\hat{\sigma}/\partial\Gamma=\psi_b$. Therefore, when we use the ansatz of Eq. (2.6a) and set $\chi=0$, we still retain the exact bound eigenstate and its corresponding eigenfrequency, so the linear dressing vanishes. We, therefore, need not worry about dressing (at least for small oscillations) as long as our ansatz includes the exact ψ_b as in Eq. (2.6). In the SG system, however, there is no exact oscillatory bound state and so we cannot construct a $\hat{\sigma}$ for the SG system that is analogous to Eq. (2.6a). Therefore, we must use $\partial\sigma/\partial\Gamma$ and its dressing since we expect that if there is some kind of quasi bound state in the SG case then the approximate shape mode $\partial\sigma/\partial\Gamma$ will be dressed by χ.

In order to calculate the dressing for the small oscillations of the quasi-internal mode of the SG system, we set

$$V=\Gamma_0^2[1-\cos(\sigma+\chi)]$$

in Eqs. (2.3) and (2.5a). We then linearize the resulting equations to first order in χ and, in doing so for Eq. (2.3), we obtain

$$\frac{\partial^2\chi}{\partial t^2}-\chi''\left[\Gamma^2-\left[\frac{\dot{\Gamma}}{\Gamma}\right]^2\xi^2\right]+2\frac{\dot{\Gamma}}{\Gamma}\frac{\partial\chi'}{\partial t}\xi+\frac{\ddot{\Gamma}}{\Gamma}\xi\chi'+\Gamma_0^2\sin\sigma+\Gamma_0^2\chi\cos\sigma=\sigma''\left[\Gamma^2-\left[\frac{\dot{\Gamma}}{\Gamma}\right]^2\xi^2\right]-\xi\sigma'\frac{\ddot{\Gamma}}{\Gamma} \tag{2.7a}$$

and for Eq. (2.5a) we obtain

$$\ddot{\Gamma}=\left[\frac{3\dot{\Gamma}^2}{2\Gamma}-\frac{M_X}{2\Gamma M_\Gamma}\right](1+b_\Gamma)+\frac{\langle\xi\sigma'|\chi''\rangle}{M_\Gamma}-\frac{1}{M_\Gamma\Gamma^2}\left[\frac{\dot{\Gamma}}{\Gamma}\right]^2\langle\xi\sigma'|\xi^2\chi''\rangle$$
$$-\frac{2}{M_\Gamma\Gamma^2}\frac{\dot{\Gamma}}{\Gamma}\left\langle\xi\sigma'|\xi\frac{\partial\chi'}{\partial t}\right\rangle-\frac{\Gamma_0^2}{M_\Gamma\Gamma^2}\langle\xi\sigma'|\sin\sigma\rangle(1+b_\Gamma)-\frac{\Gamma_0^2}{M_\Gamma\Gamma^2}\langle\xi\sigma'|\chi\cos\sigma\rangle . \tag{2.7b}$$

Since we are considering small oscillations of the quasi-internal mode, we further linearize Eqs. (2.7a) and (2.7b) in $\delta\Gamma\equiv\Gamma(t)-\Gamma_0$ discarding terms of second order such as χ^2, $(\delta\Gamma)^2$, $\chi\delta\Gamma$, $(\delta\dot{\Gamma})^2$, and higher. Upon linearizing with respect to $\delta\Gamma$, Eqs. (2.7a) and (2.7b) become, respectively,

$$\frac{\partial^2\chi}{\partial t^2}-\Gamma_0^2\chi''+\Gamma_0^2\chi\cos\sigma_0=2\Gamma_0\delta\Gamma\sigma_0''-\xi_0\sigma_0'\frac{\delta\ddot{\Gamma}}{\Gamma_0} , \tag{2.7a$'$}$$

$$\delta\ddot{\Gamma}=-\Omega_{SG}^2\delta\Gamma+\frac{1}{M_{\Gamma_0}}\langle\xi_0\sigma_0'|\chi''\rangle$$
$$-\frac{1}{M_{\Gamma,0}}\langle\xi_0\sigma_0'|\chi\cos\sigma_0\rangle , \tag{2.7b$'$}$$

where

$$\sigma_0\equiv\sigma|_{\Gamma=\Gamma_0},\quad \xi_0\equiv\Gamma_0 x ,$$

Ω_{SG} is given by Eq. (1.3a) and we used $\sigma''=\sin\sigma$. In going from Eq. (2.7b) to Eq. (2.7b′) we used the fact that the terms proportional to b_Γ cancel exactly to first order and that

$$-\frac{2}{\Gamma_0 M_{\Gamma_0}}\langle\xi\sigma'|\sigma''\rangle=\frac{M_X(\Gamma=\Gamma_0)}{\Gamma_0^2 M_{\Gamma_0}}=\frac{12\Gamma_0^2}{\pi^2}=\Omega_{SG}^2 . \tag{2.8}$$

Equation (2.7b′) indicates that, when $\chi\to 0$, the quasi-internal mode oscillates with frequency Ω_{SG} that Rice found. We point out that Rice found Ω_{SG} to be the frequency at which the quasimode oscillates no matter what the amplitude. However, there is about a 10% discrepancy between the value of Ω_{SG} and the value obtained from simulation ω_s. In our collective variable theory, the linear dressing which appears explicitly on the right-hand side of Eq. (2.7b′) will modify the theoretical value of the frequency of oscillation of the quasimode and hence improve the agreement with simulation.

We now eliminate $\delta\ddot{\Gamma}$ by substituting Eq. (2.7b′) into Eq. (2.7a′), and using Eq. (2.8) once more to obtain

$$\left|\frac{\partial^2\chi}{\partial t^2}\right\rangle+(1-\mathcal{P}_{\Gamma_0})\mathcal{L}_0|\chi\rangle=2\Gamma_0\delta\Gamma(1-\mathcal{P}_{\Gamma_0})|\sigma_0''\rangle , \tag{2.9a}$$

where we have made use of bra and ket notation. The operator $\mathcal{L}_0$ is defined as

$$\mathcal{L}_0\equiv\Gamma_0^2\mathcal{L}_\xi\equiv-\Gamma_0^2\frac{\partial^2}{\partial\xi^2}+\Gamma_0^2\cos\sigma_0 . \tag{2.9b}$$

The spectrum of the operator $\mathcal{L}_\xi$ has one localized mode which is

$$\frac{1}{2\sqrt{2}}|\sigma_0'\rangle=\frac{1}{\sqrt{2}}\text{sech}\xi \tag{2.9c}$$

with eigenvalue zero and an infinity of extended "phonon" states

$$|\psi_{\bar{k}}\rangle=\frac{1}{\sqrt{2\pi}\omega_{\bar{k}}}e^{i\bar{k}\xi}(i\bar{k}-\tanh\xi) \tag{2.9d}$$

whose eigenvalues are $\omega_{\bar{k}}^2 \equiv \bar{k}^2+1$, where $\bar{k}\equiv k/\Gamma_0$ is the normalized phonon wave number. The projection operator $\mathcal{P}_{\Gamma_0}$ is defined as

$$\mathcal{P}_{\Gamma_0} \equiv |\xi_0\sigma_0'\rangle \frac{1}{\Gamma_0^3 M_{\Gamma_0}} \langle \xi_0\sigma_0'| \ . \tag{2.9e}$$

We note that Eq. (2.9a) is identical to operating on Eq. (2.7a′) with the operator $1-\mathcal{P}_{\Gamma_0}$ which projects orthogonal to $|\xi_0\sigma_0'\rangle$. The factor $1/\Gamma_0^3$ in Eq. (2.9e) is present because the bracket notation represents integration over $d\xi$ and not dx.

We calculate the static dressing of the kink by neglecting $|\partial^2\chi/\partial t^2\rangle$ in Eq. (2.9a). We then solve the static approximation to Eq. (2.9a) by first explicitly bringing the projection operator terms on the right-hand side

$$\mathcal{L}_0|\tilde{\chi}_d\rangle = 2\Gamma_0\delta\Gamma|\sigma_0''\rangle + |\xi_0\sigma_0'\rangle \left[\frac{\Omega_{SG}^2}{\Gamma_0}\delta\Gamma + \frac{1}{\Gamma_0^3 M_{\Gamma_0}} \langle \xi_0\sigma_0'|\mathcal{L}_0|\tilde{\chi}_d\rangle \right] , \tag{2.10a}$$

where we used the notation $\tilde{\chi}_d$ to explicitly indicate the dressing. We evaluate the contribution to the source of the matrix element $\langle \xi_0\sigma_0'|\mathcal{L}_0|\tilde{\chi}_d\rangle$ by first rewriting Eq. (2.10a) as

$$\mathcal{L}_0|\tilde{\chi}_d\rangle = \Gamma_0\delta\Gamma(2|\sigma_0''\rangle + \lambda|\xi_0\sigma_0'\rangle) , \tag{2.10a′}$$

where

$$\lambda \equiv \left[\frac{\Omega_{SG}}{\Gamma_0} \right]^2 + \frac{1}{\Gamma_0^4 M_{\Gamma_0}\delta\Gamma} \langle \xi_0\sigma_0'|\mathcal{L}_0|\tilde{\chi}_d\rangle \ . \tag{2.10b}$$

Note that both terms in Eq. (2.10b) are $O(1)$. Next, operating with $\mathcal{L}_0^{-1}=(1/\Gamma_0^2)\mathcal{L}_\xi^{-1}$ on Eq. (2.10a′), we obtain

$$|\tilde{\chi}_d\rangle = \frac{\delta\Gamma}{\Gamma_0}(2\mathcal{L}_\xi^{-1}|\sigma_0''\rangle + \lambda\mathcal{L}_\xi^{-1}|\xi_0\sigma_0'\rangle) \ . \tag{2.10c}$$

We evaluate λ by operating on Eq. (2.10c) with $\langle \xi_0\sigma_0'|$ and invoking the constraint $C_{1\Gamma}=0$ which, in the present limit, is $\langle \xi_0\sigma_0'|\tilde{\chi}_d\rangle=0$. (We used this same method to obtain the static dressing of a discrete SG kink.[5,6])

Operating on Eq. (2.10c) with $\langle \xi_0\sigma_0'|$ and invoking the constraint, we obtain

$$\langle \xi_0\sigma_0'|\tilde{\chi}_d\rangle = 0 = \frac{\delta\Gamma}{\Gamma_0}(2\langle \xi_0\sigma_0'|\mathcal{L}_\xi^{-1}|\sigma_0''\rangle + \lambda\langle \xi_0\sigma_0'|\mathcal{L}_\xi^{-1}|\xi_0\sigma_0'\rangle) \ . \tag{2.11}$$

When we solve Eq. (2.11) for λ we obtain

$$\lambda = -2\frac{\langle \xi_0\sigma_0'|\mathcal{L}_\xi^{-1}|\sigma_0''\rangle}{\langle \xi_0\sigma_0'|\mathcal{L}_\xi^{-1}|\xi_0\sigma_0'\rangle} \ . \tag{2.12}$$

In order to evaluate the right-hand side of Eq. (2.12) we represent the Green's function $\mathcal{L}_\xi^{-1}$ as

$$\mathcal{L}_\xi^{-1} = \int_{-\infty}^{\infty} d\bar{k}\, |\psi_{\bar{k}}(\xi)\rangle \frac{1}{\omega_{\bar{k}}^2} \langle \psi_{\bar{k}}(\xi')| \ . \tag{2.13}$$

The Goldstone mode $|\sigma_0'\rangle$ is orthogonal to $|\xi_0\sigma_0'\rangle$ in Eq. (2.12) and makes no contribution. Upon inserting Eq. (2.13) for $\mathcal{L}_\xi^{-1}$ into Eq. (2.12), we obtain

$$\lambda = \frac{I_2}{I_4} , \tag{2.14a}$$

where

$$I_n \equiv \int_{-\infty}^{\infty} d\bar{k} \frac{\operatorname{sech}^2(\bar{k}\pi/2)}{\omega_{\bar{k}}^n} \ . \tag{2.14b}$$

To obtain Eq. (2.14a) we made use of

$$\langle \xi_0\sigma_0'|\psi_{\bar{k}}\rangle = -\sqrt{2\pi}\frac{\operatorname{sech}(\bar{k}\pi/2)}{\omega_{\bar{k}}} \tag{2.15a}$$

and

$$\langle \sigma_0''|\psi_{\bar{k}}\rangle = \sqrt{\pi/2}\,\omega_{\bar{k}}\operatorname{sech}(\bar{k}\pi/2) \ . \tag{2.15b}$$

The integrals I_2 and I_4 are evaluated in Appendix A and the result for λ in Eq. (2.14a) is

$$\lambda = \frac{2\zeta(2)}{\zeta(2)+\zeta(3)} \approx 1.1556 , \tag{2.16a}$$

where

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}, \quad \operatorname{Re}p>0 \tag{2.16b}$$

is the Riemann ζ function[7] and Re is the real part.

We are now in a position to evaluate Eq. (2.10c) for $|\tilde{\chi}_d\rangle$. Upon substituting Eq. (2.13) into Eq. (2.10c) we obtain

$$\begin{aligned} |\tilde{\chi}_d\rangle &\equiv \delta\Gamma|\chi_d\rangle \\ &= \frac{\delta\Gamma}{\Gamma_0}\int_{-\infty}^{\infty} d\bar{k}\,|\psi_{\bar{k}}\rangle \frac{1}{\omega_{\bar{k}}^2}(2\langle\psi_{\bar{k}}|\sigma_0''\rangle + \lambda\langle\psi_{\bar{k}}|\xi_0\sigma_0'\rangle) \\ &= \sqrt{2\pi}\frac{\delta\Gamma}{\Gamma_0}\int_{-\infty}^{\infty} d\bar{k}\,|\psi_{\bar{k}}\rangle \frac{\operatorname{sech}(\bar{k}\pi/2)}{\omega_{\bar{k}}}\left[1-\frac{\lambda}{\omega_{\bar{k}}^2}\right] . \end{aligned} \tag{2.17}$$

Note that $|\tilde{\chi}_d\rangle$ is $O(\delta\Gamma)$ but $|\chi_d\rangle$ is $O(1)$.

In order to calculate the effect of the dressing $|\chi_d\rangle$ on dynamical quantities, such as the renormalization of the small oscillation frequency of the quasi bound state, we construct the new ansatz

$$\phi(x,t) = \hat{\sigma} + \chi = \sigma[\Gamma(t)x] + \delta\Gamma\chi_d + \chi \tag{2.18a}$$

from which we derive the new shape mode

$$\left|\frac{\partial\hat{\sigma}}{\partial\Gamma}\right\rangle = \left|\frac{\partial\sigma}{\partial\Gamma}\right\rangle + |\chi_d\rangle \ . \tag{2.18b}$$

In order to calculate the renormalized or "dressed" frequency, which we denote by Ω_d, we substitute the new ansatz Eq. (2.18a) with $\chi=0$ into the original equation of motion Eq. (1.2a) and linearize in $\delta\Gamma$. We obtain

$$\delta\ddot{\Gamma}\left|\frac{\xi_0}{\Gamma_0}\sigma_0' + \chi_d\right\rangle + \delta\Gamma|\mathcal{L}_0\chi_d - 2\Gamma_0\sigma_0''\rangle = 0 \ . \tag{2.19}$$

We now need to project Eq. (2.19) into the direction given by the new ansatz vector $\langle \xi_0\sigma_0'/\Gamma_0+\chi_d|$ which is the right-hand side of Eq. (2.18b) in the present limit of small $\delta\Gamma$. Note that, according to the general theory of Ref. 4, projecting in the new direction implies that we are now operating under the constraints in Eq. (2.2) in which σ is replaced by the $\hat{\sigma}$ of Eq. (2.18a). The projection on Eq. (2.19) yields

$$\delta\ddot{\Gamma}+\Omega_d^2\delta\Gamma=0\ , \tag{2.20a}$$

$$\Omega_d^2\equiv\frac{1}{\hat{M}_{\Gamma_0}\Gamma_0}\left\langle\frac{\xi_0}{\Gamma_0}\sigma_0'+\chi_d\left|\mathcal{L}_0\chi_d-2\Gamma_0\sigma''\right.\right\rangle\ , \tag{2.20b}$$

$$\hat{M}_{\Gamma_0}\equiv\frac{1}{\Gamma_0}\left\langle\frac{\partial\hat{\sigma}}{\partial\Gamma}\left|\frac{\partial\hat{\sigma}}{\partial\Gamma}\right.\right\rangle\Bigg|_{\Gamma=\Gamma_0}\ . \tag{2.20c}$$

$\hat{M}_{\Gamma_0}$ is the renormalized or dressed mass. In Appendix B we show that

$$\hat{M}_{\Gamma_0}=\frac{2\pi}{\Gamma_0^3}\lambda^2 I_6\approx\frac{6.7864}{\Gamma_0^3}\ , \tag{2.21}$$

$$\Omega_d^2=\frac{I_4}{I_6}\Gamma_0^2=\frac{4\Gamma_0^2}{3}\frac{\zeta(2)+\zeta(3)}{\zeta(2)+\zeta(3)+\frac{1}{2}\zeta(4)}\approx(1.0585)^2\Gamma_0^2\ . \tag{2.22}$$

We see that the dressed mass differs from the bare mass $M_{\Gamma_0}=2\pi^2/(3\Gamma_0^3)=6.5797/\Gamma_0^3$ by only about 3% and depends quadratically on λ. The dressed frequency Ω_d is about 4% smaller than Rice's bare frequency

$$\Omega_{\rm SG}=(12\Gamma_0^2/\pi^2)^{1/2}=1.1027\Gamma_0\ ,$$

is closer to the phonon band edge, and is independent of λ.

In the next section we carry out simulations and find that there is a long-lived quasimode with a frequency very near the phonon band edge which differs by 10% from Rice's value $\Omega_{\rm SG}$ and by 5% from our dressed frequency Ω_d. We also discuss in Sec. III ways to improve agreement between the value of Ω_d and the value ω_s from simulation.

III. RESULTS OF SIMULATIONS

We discretize the SG equation of motion in the following manner:

$$\frac{u_i^{j+1}-2u_i^j+u_i^{j-1}}{(\Delta t)^2}-\frac{u_{i+1}^j-2u_i^j+u_{i-1}^j}{(\Delta x)^2}=-\Gamma_0^2\sin u_i^j\ , \tag{3.1}$$

where u_i^j is the phase of the ith discretized segment of the field at the jth time step. The magnitude of the time step is $\Delta t=0.01$, the length of each discretized segment is $\Delta x=0.02$, and the length of the system is 1000 units so that the system is divided into 50 000 segments. We treat the ends of the system as free. Solving Eq. (3.1) for u_i^{j+1} we obtain

$$u_i^{j+1}=(u_{i+1}^j-2u_i^j+u_{i-1}^j)\left[\frac{\Delta t}{\Delta x}\right]^2-(\Delta t)^2\Gamma_0^2\sin u_i^j+2u_i^j-u_i^{j-1}\ . \tag{3.2}$$

We use Eq. (3.2) in our numerical solution of the SG equation in order to follow the evolution of the quasimode from various initial conditions.

We impose initial conditions in our simulations in two different ways. The first way is by specifying the field u_i^j at $t=0$ to be $|\sigma_0\rangle$ and at $t=\Delta t$ to be $|\sigma_0\rangle+\epsilon|\xi\sigma_0'\rangle$, where ϵ is a small number thus giving the equilibrium kink a nonzero initial velocity. The second way is to start the kink from rest with an initial kink length l_0 such that $\pi/l_0\neq\Gamma_0$ and then let the field evolve according to Eq. (3.2). Either way, the resulting oscillation of the quasi-internal mode corresponds to a roughly constant amplitude (after initial transients have vanished) which we denote by $\delta\gamma_0$. We will indicate explicitly below which method we invoke in order to gather data for the following figures. In our simulations we monitor the functions $\Gamma(t)$ and $\chi=\phi(25,t)-\sigma_0$, that is χ evaluated 25 units away from the kink.

In Fig. 1(a) we plot the simulation results for $\Gamma(t)$ where the initial configuration σ_0 is given a nonzero initial velocity as described above so that $\delta\gamma_0\approx0.01$. We have set Γ_0 [the parameter in the SG potential in Eq. (3.2)] equal to unity. Figure 1(a) is thus representative of the small oscillation regime. After the initial buildup of phonons due to the imposition of the initial conditions, the variable $\Gamma(t)$ settles down to a quasimonochromatic oscillation of frequency $\omega_s=(1.004\pm0.001)\Gamma_0$ with a linewidth of $1/\tau_s=(0.003\pm0.001)\Gamma_0$. The magnitude of the Fourier transform of $\Gamma(t)$ is plotted in Fig. 2. In Sec. IV we model the decay exponentially but the actual decay is not a simple exponential, rather it has some power-law character. In Fig. 1(b) we plot the amplitude of χ as a function of time at a distance of about eight kink lengths from the kink at the origin. We can see $\chi(t)$ build up from zero as the kink starts oscillating and then after four or five oscillations the amplitude of $\chi(t)$ starts to decay at the same frequency and lifetime as $\Gamma(t)$.

The frequency of oscillation in the simulation, $\omega_s=1.004\Gamma_0$, is about 10% less than the bare Rice frequency $\Omega_{\rm SG}$ in Eq. (1.3a) and about 5% less than our dressed frequency Ω_d. Thus, our static dressing χ_d accounts for half of the difference between Rice's frequency and the value obtained from simulation. Better agreement might be obtained by picking a different function $\partial\sigma/\partial\Gamma$ with which to represent the internal oscillation of the kink, that is, a function whose linear dressing would account for more of the difference between Rice's value of the frequency and the simulation value. However, it is difficult to construct a suitable functional form for $\partial\sigma/\partial\Gamma$, if it exists, in order to obtain a value for Ω_d that is less than 5% from the simulation frequency value. It may be possible to find such a function by invoking a variational procedure with the constraint being that the value of the simulation frequency is obtained, but we have not investigated this possibility. On the other hand,

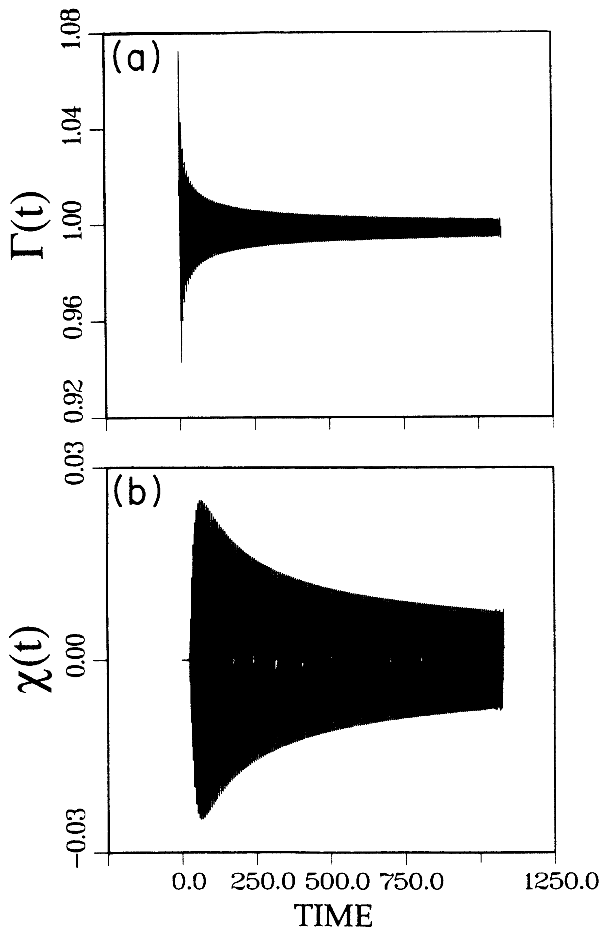

FIG. 1. $\delta\Gamma_0 \approx 0.01$, $\Gamma_0 = 1$. (a) shows $\Gamma(t)$ and (b) shows $\chi(t) = \phi - \sigma$ measured 25 units away from a kink. The initial condition for $t=0$ is a kink $|\sigma_0\rangle$ with velocity determined by specifying the kink shape at $t = \Delta t$ as $|\sigma_0\rangle + 0.001 |\partial\sigma_0/\partial\Gamma_0\rangle$. Motion evolves according to Eq. (3.2). The length of the system is 1000 units.

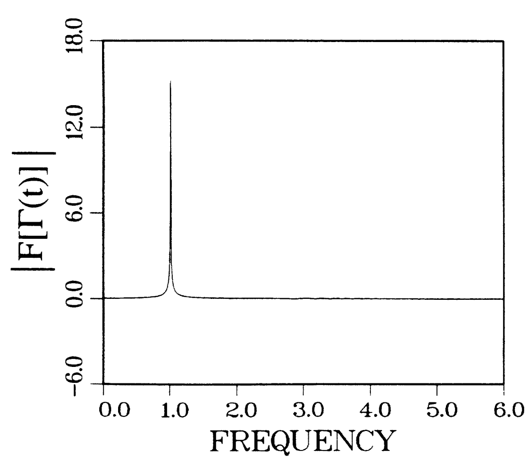

FIG. 2. Magnitude of the Fourier transform of $\Gamma(t)$ in Fig. 1(a). The transform started at $t \approx 200$.

one may not be able to obtain better than about 5% agreement with the simulation frequency by just using the static dressing.[5] The dynamical dressing must then be taken into account. In fact, in Ref. 6, we have theoretically calculated the Peierls-Nabarro frequency of a discrete SG kink using two methods: (1) using an ansatz analogous to Eq. (2.6b), where the ψ_b we used was the eigenfunction of the linearized discrete SG equation corresponding to the lowest eigenvalue, and (2) dressing the continuum kink form using a Lagrange multiplier method just as we did in the previous section. The result was that method (1) gave essentially exact agreement with simulation for the Peierls-Nabarro frequency whereas method (2) agreed to within 5% of the simulation value for values of Γ_0 which corresponded to maximum dressing of the continuum kink form. Thus, for the discrete SG, a 5% discrepancy is obtained with method (2) even when there is an exact eigenstate corresponding to the mode in question. This indicates that the 5% discrepancy using the static dressing is most likely due to an insufficient incorporation of dynamical effects which are, in fact, taken into account using method (1). Since we cannot set up an ansatz using method (1) for the analysis of the present paper, we are forced to use method (2).

The measured linewidth in Fig. 2 does not agree well with the linewidth calculated from our dressed frequency Ω_d. However, when we use the simulation value of the frequency ω_s in the theoretical calculation of the linewidth in Sec. IV, we obtain essentially exact agreement with the simulation value. As we see in Sec. IV and Appendix C, the essential reason for the long lifetime is the fact that the linewidth is proportional to k_s [where $k_s = \Gamma_0(\omega_s^2/\Gamma_0^2 - 1)^{1/2}$] so that the closer ω_s is to the phonon band edge the more narrow the linewidth.

In Figs. 3(a), 3(b), and (4) we plot, respectively, $\Gamma(t)$, $\chi(t)$, and the Fourier transform of $\Gamma(t)$ where the initial condition corresponds to giving an equilibrium kink $|\sigma_0\rangle$ a nonzero initial velocity such that $\delta\gamma_0 \approx 0.1$. Within the limits of accuracy of our simulations, the frequency and linewidth for the case $\delta\gamma_0 \approx 0.1$ are the same as in the case $\delta\gamma_0 \approx 0.01$. In Fig. 4, for $\delta\gamma_0 \approx 0.1$, we can see a small peak due to the second harmonic at $\omega \approx 2.008\Gamma_0$. Note the magnitude of the large peak in Fig. 4 (corresponding to $\delta\gamma_0 \approx 0.1$) is about 10 times the magnitude of the corresponding peak in Fig. 2 (corresponding to $\delta\gamma_0 \approx 0.01$).

In Figs. 5(a) and 5(b) the results of simulation for $\delta\gamma_0 \approx 1$ are plotted where the initial kink form with zero initial velocity corresponds to a kink length such that $\pi/l_0 = 0.005$. The initial condition is then allowed to evolve according to Eq. (3.2) where $\Gamma_0 = 1$. Consequently, the kink is almost flat initially. Yet after five or six periods the kink oscillates at the frequency $\omega_s = 1.004\Gamma_0$ with a linewidth of $1/\tau_s = 0.003\Gamma_0$. The only qualitative difference of the nonlinear oscillation in Figs. 5(a) and 5(b) and the linear oscillations of Figs. 1 and 3 is that, in the first few periods in Fig. 5, the kink oscillates very nonlinearly and radiates over 75% of its oscillation energy in phonons and then settles down into the same quasi-

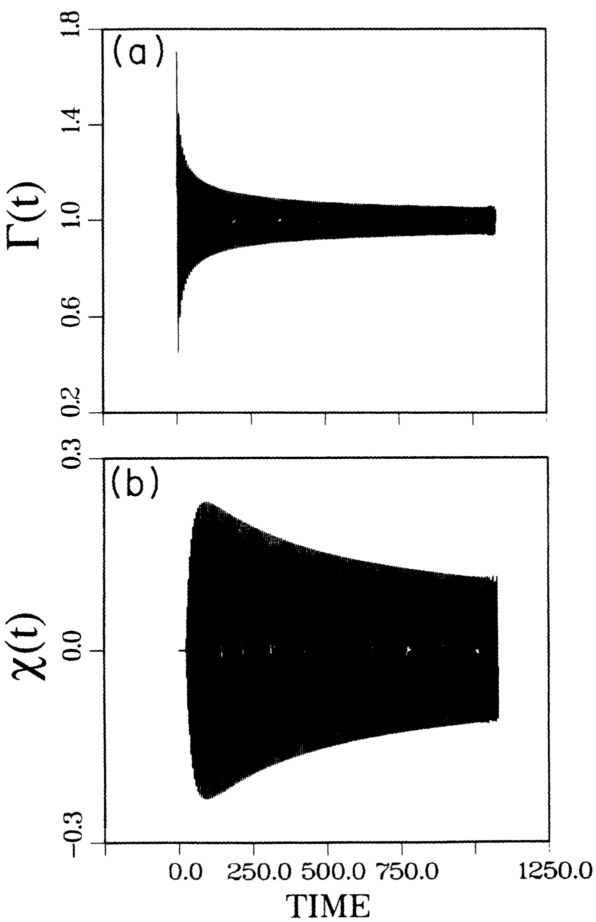

FIG. 3. $\delta\Gamma_0 \approx 0.1$, $\Gamma_0 = 1$. (a) shows $\Gamma(t)$ and (b) shows $\chi(t) = \phi - \sigma$ measured 25 units away from a kink. The initial condition for $t = 0$ is a kink $|\sigma_0\rangle$ with velocity determined by specifying the kink shape at $t = \Delta t$ as $|\sigma_0\rangle + 0.01|\partial\sigma_0/\partial\Gamma_0\rangle$. Motion evolves according to Eq. (3.2). The length of the system is 1000 units.

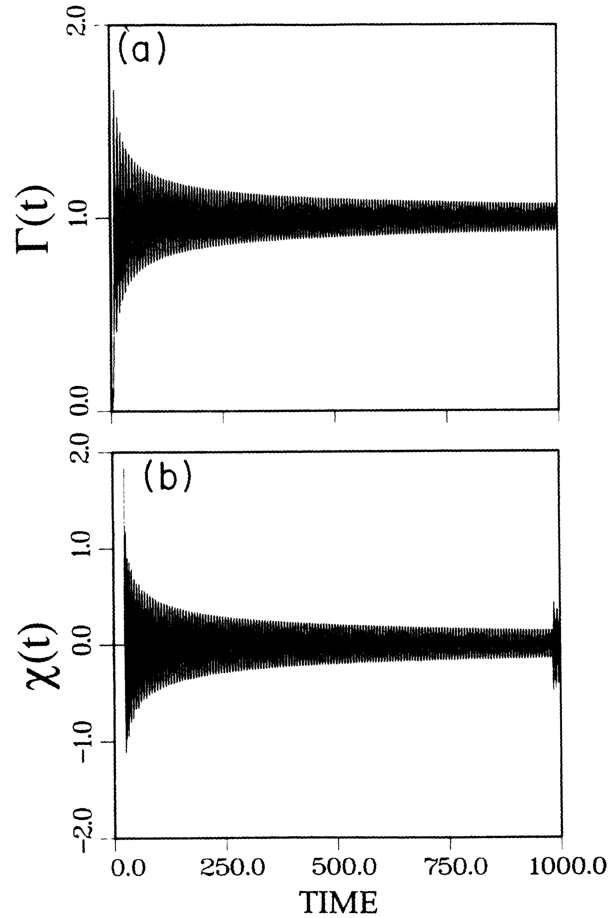

FIG. 5. $\delta\Gamma_0 \approx 0.1$, $\Gamma_0 = 1$. (a) shows $\Gamma(t)$ and (b) shows $\chi(t) = \phi - \sigma$ measured 25 units away from a kink. Initial condition for $t = 0$ is a kink with $\pi/l_0 = 0.005$ with zero velocity. Motion evolves according to Eq. (3.2). Note the reflection in (b) because of a shorter system (200 units long) than in Figs. 1 and 3.

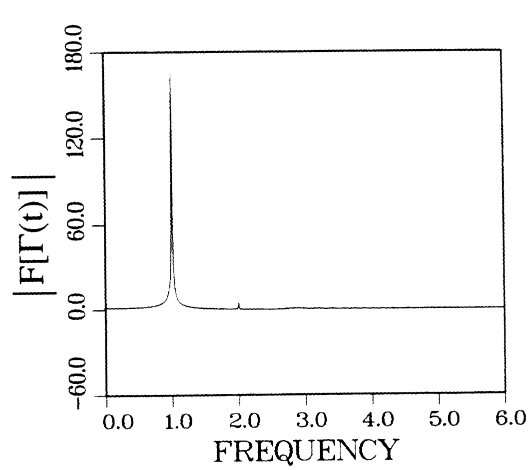

FIG. 4. Magnitude of the Fourier transform of $\Gamma(t)$ in Fig. 3(a). Transform started at $t \approx 200$.

mode as in the linear oscillation case. The reason for the large initial radiation burst is that the initial $\Gamma(t)$ motion has many higher-order harmonics which radiate orders of magnitude more rapidly because the higher harmonics have much shorter lifetimes, e.g., the lifetime of the second harmonic is more than ten times shorter than the lifetime of the fundamental. Consequently, for nonlinear excitations we have rapid radiation of phonons of wavelengths the size of the soliton and shorter and then the SG kink oscillates at the quasimode frequency with narrow linewidth, as in Figs. 1 and 3.

IV. RADIATION LINEWIDTH

In this section we apply the standard perturbation calculation used in electromagnetic theory to calculate the radiative lifetime of a radiating oscillator.[8] The calculation assumes one has an undamped radiating oscillator at frequency ω_0 and one calculates the power radiated by integrating Poynting's flux over a sphere at a distance large compared with the wavelength. The expression for the resultant power in Gaussian units is

$$P=\tfrac{1}{3}ck^4|p|^2=\frac{\omega_0^4e^2|x_0|^2}{3c^3}\ , \tag{4.1}$$

where $p=ex_0$ is the dipole moment and x_0 is the initial displacement of the oscillator from its equilibrium position. We calculate, in the absence of damping, the time average (denoted by an overbar) of the quantity $\dot{x}d\ddot{x}/dt$ where we assume

$$x(t)=x_0\exp(-i\omega_0 t)$$

and obtain

$$\overline{\dot{x}\frac{d\ddot{x}}{dt}}=-\tfrac{1}{2}|x_0|^2\omega_0^4\ , \tag{4.2a}$$

which yields

$$P=-\frac{2e^2}{3c^3}\overline{\dot{x}\frac{d\ddot{x}}{dt}}\ . \tag{4.2b}$$

We next assume that the damping rate is not zero but small compared with its oscillation frequency ω_0 and that it is equal to the time rate of change of the energy of the oscillator

$$\frac{d}{dt}(\tfrac{1}{2}m\dot{x}^2+\tfrac{1}{2}m\omega_0^2x^2)=-P\ , \tag{4.3a}$$

which becomes, after simplifying,

$$\ddot{x}+\frac{1}{\tau}\dot{x}+\omega_0^2x=0\ , \tag{4.3b}$$

where the damping rate is

$$1/\tau\equiv 2e^2\omega_0^2(3mc^3)^{-1}\ .$$

The above argument uses only conservation of energy and the fact that the damping is perturbative. The relation between $1/\tau$ and the power is[8]

$$\frac{1}{\tau}=\frac{2}{\omega_0^2m}\frac{P}{|x_0|^2}\ . \tag{4.4a}$$

Now, although in the case of the SG system the decay is not exponential, we will nevertheless model the decay as

$$\delta\ddot{\Gamma}+\frac{1}{\tau}\delta\dot{\Gamma}+\omega_s^2\delta\Gamma=0\ ,$$

where $\omega_s=\Gamma_0\bar{\omega}_s$ is the frequency of the quasimode obtained from simulation. (A bar over a variable denotes that the variable is normalized to Γ_0.) We, therefore, make the association with Eq. (4.4a) for the inverse lifetime

$$\frac{1}{\tau}=\frac{2}{\omega_s^2\hat{M}_{\Gamma_d}}\frac{\langle P\rangle_{\text{avg}}}{|\delta\gamma_0|^2}=\frac{2}{\Gamma_0^2\bar{\omega}_s^2\hat{M}_{\Gamma_0}}\frac{\langle P\rangle_{\text{avg}}}{|\delta\gamma_0|^2}\ , \tag{4.4b}$$

where $\delta\gamma_0$ is the initial displacement of the internal quasimode, $\hat{M}_{\Gamma_0}$ is the dressed mass given by Eq. (2.21), and $\langle P\rangle_{\text{avg}}$ is the spatial average of the radiated power. Equation (4.4b) is correct in the units of the present paper.

In order to utilize Eq. (4.4b), we calculate P for our system by considering the conservation of energy for a one-dimensional system

$$\frac{\partial h}{\partial t}+\frac{\partial j}{\partial x}=0,\quad j=\dot{\phi}\frac{\partial\phi}{\partial x}\ , \tag{4.5}$$

where h is the SG energy per unit length and j is the energy flux. We integrate Eq. (4.5) from $-\hat{x}$ to $\hat{x}$ (where $\hat{x}$ is a distance large compared with the size l_0 of the kink and the wavelength of the radiation) and obtain

$$P=\frac{dE}{dt}=-\dot{\phi}\frac{\partial\phi}{\partial x}\Bigg|_{x=-\hat{x}}^{x=\hat{x}}=-2\dot{\phi}(\hat{x})\frac{\partial\phi(\hat{x})}{\partial x}\ , \tag{4.6}$$

where $E=\int_{-\hat{x}}^{\hat{x}}h\,dx$ and where we used the fact that the Poynting's flux at $-\hat{x}$ is the negative of the flux at $\hat{x}$. Since $\phi=\sigma+\chi$, and since the flux is evaluated at a distance far from the kink, we have

$$P=-2\dot{\chi}\frac{\partial\chi}{\partial x}=-2\Gamma_0\dot{\chi}\chi'\ , \tag{4.7}$$

where the prime indicates the derivative with respect to ξ.

We now briefly describe how to calculate the radiation χ and, consequently, the Poynting's flux given by Eq. (4.7) but give the details of the calculation in Appendix C. In order to calculate χ we begin by substituting the ansatz of Eq. (2.18c), which includes the dressing into the original SG equation of motion, for ϕ

$$\frac{\partial^2\phi}{\partial\phi^2}-\frac{\partial^2\phi}{\partial x^2}+\Gamma_0^2\sin\phi=0 \tag{4.8a}$$

to obtain an equation identical in form to Eq. (2.3) but with $\sigma\rightarrow\hat{\sigma}$ and

$$V=\Gamma_0^2[1-\cos(\hat{\sigma}+\chi)]\ .$$

Then, linearizing the resulting equation in $\delta\Gamma$ and χ, we obtain

$$\left|\frac{\partial^2\chi}{\partial t^2}\right\rangle+\mathcal{L}_0|\chi\rangle=\delta\Gamma(2\Gamma_0|\sigma_0''\rangle-\mathcal{L}_0|\chi_d\rangle)-\delta\ddot{\Gamma}\left|\frac{\xi_0\sigma_0'}{\Gamma_0}+\chi_d\right\rangle\ . \tag{4.8b}$$

Of course, Eq. (4.8b) does not contain any information about the constraints yet since all we have done is substitute the ansatz into the original equation of motion and linearize in $\delta\Gamma$ and χ. We effectively impose the constraints by projecting out of Eq. (4.8b) the mode corresponding to $\delta\ddot{\Gamma}(t)$ so that what we are left with is an equation of motion for $\partial^2\chi/\partial t^2$. We accomplish the projection by operating on Eq. (4.8b) with $1-\hat{\mathcal{P}}_{\Gamma_0}$, where

$$\hat{\mathcal{P}}_{\Gamma_0}\equiv\left|\frac{\xi_0\sigma_0'}{\Gamma_0}+\chi_d\right\rangle\frac{1}{\Gamma_0\hat{M}_{\Gamma_0}}\left\langle\frac{\xi_0\sigma_0'}{\Gamma_0}+\chi_d\right| \tag{4.9}$$

and where $\hat{M}_{\Gamma_0}$ is given by Eq. (2.21). Recall that the constraints effectively invoked when operating on Eq. (4.8b) with $1-\hat{\mathcal{P}}_{\Gamma_0}$ are those in Eqs. (2.2a)–(2.2d) in which we must make the replacement $\sigma\to\hat{\sigma}$. After projecting, we solve the resulting equation of motion for χ and then calculate the spatial average of the Poynting's flux defined in Eq. (4.7) (see Appendix C) to obtain

$$\langle P\rangle_{\rm avg}=\frac{k_s\Gamma_0}{2}\lambda^2\bar{\omega}_s^2\left[\frac{\delta\gamma_0}{\Gamma_0}\right]^2\mathcal{J}\,, \tag{4.10}$$

where

$$\mathcal{J}=\int_0^\infty d\bar{k}\,\frac{\bar{k}\,\mathrm{sech}^2(\bar{k}\pi/2)}{\omega_{\bar{k}}^7}\approx 0.0987$$

and $k_s=\Gamma_0\bar{k}_s=\Gamma_0(\bar{\omega}_s^2-1)^{1/2}$. Then substituting Eq. (4.10) into Eq. (4.4b) we obtain, for the lifetime,

$$\frac{1}{\tau}=\frac{k_s\Gamma_0\lambda^2\mathcal{J}}{\Gamma_0^4\hat{M}_{\Gamma_0}}=\frac{k_s}{2\pi}\frac{\mathcal{J}}{I_6}=\frac{1}{\lambda_s}\frac{\mathcal{J}}{I_6}\approx 0.1220\frac{1}{\lambda_s}\,, \tag{4.11}$$

where $\lambda_s=2\pi/k_s$ is the radiated wavelength from simulation and the value of I_6 is given by Eq. (A11). We have calculated $\mathcal{J}$ numerically. The wavelength corresponding to the simulation frequency $\omega_s=1.004\Gamma_0$ is $\lambda_s\approx 70.18\Gamma_0^{-1}$ and thus we obtain

$$\frac{1}{\tau}=0.0017\Gamma_0\approx 0.002\Gamma_0$$

which agrees well with our simulation value of $1/\tau_s=0.003\pm0.001)\Gamma_0$. We see that the damping is extremely small because it depends inversely on the wavelength which is very long (over 22 kink lengths according to simulation) because the frequency of radiation is very close to the band edge.

In order to understand more qualitatively the $1/\lambda_s$ dependence of the linewidth, we now perform an approximate calculation of $1/\tau$ by calculating the power radiated by a single phonon of frequency $\omega_s=\bar{\omega}_s\Gamma_0$ with amplitude $\delta\gamma_0$. Using Eq. (2.9d) we express the phonon as

$$\begin{aligned}\chi_{\rm ph}&=\frac{\delta\gamma_0}{\sqrt{2\pi}\bar{\omega}_s\Gamma_0}\mathrm{Re}(i\bar{k}_s-\tanh\xi)e^{i(\bar{k}_s\xi-\bar{\omega}_s\Gamma_0 t)}\\&=-\frac{\delta\gamma_0}{\sqrt{2\pi}\bar{\omega}_s\Gamma_0}[\cos(\bar{k}_s\xi-\bar{\omega}_s\Gamma_0 t)\\&\qquad+\bar{k}_s\sin(\bar{k}_s\xi-\bar{\omega}_s\Gamma_0 t)]|_{\tanh\xi=1}\,.\end{aligned} \tag{4.12}$$

The factor of Γ_0 in the denominator of Eq. (4.12) is present in order to maintain the correct units. Since we are interested in the radiated power far from the kink, we have set the tanh to unity in Eq. (4.12). Next we calculate the space (or time) average of the flux and obtain

$$\begin{aligned}-2\Gamma_0\left\langle\frac{\partial\chi_{\rm ph}}{\partial t}\chi'_{\rm ph}\right\rangle&\equiv\langle P\rangle_{\rm avg}\\&=\frac{2\Gamma_0(\delta\gamma_0)^2}{2\pi\bar{\omega}_s^2\Gamma_0^2}\frac{\bar{\omega}_s}{2}(\bar{k}_s+\bar{k}_s^3)\Gamma_0\\&=\frac{(\delta\gamma_0)^2}{2\pi}\bar{k}_s\bar{\omega}_s\,.\end{aligned} \tag{4.13}$$

Then substituting Eq. (4.13) into the expression for the inverse lifetime defined in Eq. (4.4b), we obtain

$$\begin{aligned}\frac{1}{\tau}&=\frac{2}{(\Gamma_0\bar{\omega}_s)^2\hat{M}_{\Gamma_0}}\frac{\langle P\rangle_{\rm avg}}{(\delta\gamma_0)^2}\\&=\frac{2\bar{k}_s\Gamma_0}{(2\pi)^2\bar{\omega}_s\lambda^2 I_6}=\frac{1}{\lambda_s}\left[\frac{1}{\pi\bar{\omega}_s\lambda^2 I_6}\right]\end{aligned} \tag{4.14}$$

which agrees to within a factor of 2 with simulations. [In the rigorous calculation of Appendix C, the inverse lifetime is proportional to an integral over all k's, which is the reason for the difference in magnitude between Eqs. (4.11) and (4.14).]

From the above heuristic argument, we see again that the dominant cause of the smallness of $1/\tau$ is the very long wavelength of the radiation emitted by the internal oscillation. For example, if the bare Rice frequency is used [Eq. (1.3a)] instead of the observed ω_s, we find that $1/\tau$ is approximately an order of magnitude larger.

V. DISCUSSION AND CONCLUSION

We have used a projection operator collective variable formalism in order to study the internal quasimode for a SG soliton which we found to have a frequency $\omega_s=(1.004\pm0.001)\Gamma_0$ with an inverse lifetime of $1/\tau_s=(0.003\pm0.001)\Gamma_0$. Our theoretical values for the frequency and inverse lifetime are $\Omega_d=1.0585\Gamma_0$ and $1/\tau=0.0017\Gamma_0\approx0.002\Gamma_0$. The system in our simulations was 1000 units long corresponding to a lowest phonon frequency of

$$\omega_{\rm min}=[(2\pi/1000)^2+1]^{1/2}\approx 1.000\,02\,.$$

Our simulation value for the frequency ω_s, with uncertainty included, is well above $\omega_{\rm min}$ and so our value of ω_s is not due to finite-size effects, nor does ω_s correspond to an excited phonon state since the shape of the quasimode is truly that of a localized internal vibration of the soliton and not that of an extended long-wavelength phonon.

Thus, collective variable descriptions of kink phenomena (such as kink internal modes, kink-kink collisions, and kink impurity scattering) are very useful because they allow a particlelike description of phenomena, which are often complicated, to describe in terms of the original nonlinear field. A particularly useful example of the application of collective variables is, as we have seen in the present paper, the case of internal degrees of freedom such as the double sine-Gordon (DSG) and ϕ^4 systems. In the DSG and ϕ^4 systems, however, one finds that the collective variable for the exact radiationless linear internal mode can also describe highly nonlinear anharmonic oscillations of the internal mode where the anharmonicity couples the internal oscillations to the phonon field. Often the coupling to the phonon field is relatively weak and the collective variable description of a "particle" with an internal anharmonic mode is a very good zeroth-order description. Examples are Rice's[2] work on polyacetylene, Campbell and co-workers[9,10] on ϕ^4, and DSG kink-kink collisions and Ref. 11 on DSG dynamics. Segur[12] showed that the ϕ^4 has a wobbling kink solution for a relatively long time. In a collective variable treatment Segur's result would be described as a kink with an excited internal anharmonic degree of freedom that radiates phonons only weakly. Segur has also constructed an exact three-soliton solution of the SG equation (which may be unstable[12]) that also has the properties of a wobbling kink. However, unstable or not, the quasimode of the SG system of the present paper is a single soliton solution and bears no relationship to Segur's SG wobbling kink solution. Nevertheless, our SG quasimode is closely related to his ϕ^4 wobbling kink solution with the shape of the modes being very close to each other. The fundamental difference is that an infinitesimally small amplitude ϕ^4 oscillation does not radiate at all but the SG soliton does. If, on the other hand, the SG "internal mode" solution behaved such that $\bar{\omega}_s$ were equal to 1, the band edge, instead of 1.004, then it would be a true mode, i.e., it would have an infinite lifetime. Peyrard and Campbell[13] studied the interactions of a kink and an antikink in a parametrically modified SG model with the potential

$$V(\phi)=(1-r^2)(1-\cos\phi)/(1+r^2+2r\cos\phi)\ ,$$

with the pure SG case corresponding to the parameter value $r=0$. For r negative they found a series of bound states where the band edge of their continuum is located at $\omega_c=(1-r)/(1+r)$. Their lowest bound state as $r\to 0$ through negative values is very close in frequency and shape to the SG quasimode which suggests the possibility that the internal quasimode of the present paper might be related to an analytic continuation of the Peyrard-Campbell parametrically modified SG model.

We, and the authors of Refs. 3, found the SG quasimode by changing the slope of the static kink initially and following its weakly damped oscillations. The question arises as to which physical phenomena will actually excite the SG quasimode. The first case that should be studied is the interaction of the SG soliton with an impurity whose potential varies on the same length scale l_0 as the size of the kink. Preliminary simulations show the kink slope deforming. The important signature of the excitation of the SG quasimode is radiation at the frequency ω_s corresponding to a wavelength λ_s which is about 22 times larger in length than the length l_0 over which the potential varies. A second important case where the SG quasimode should be excited is the discrete SG where the kink is trapped in a Peierls-Nabarro well and there is a resonance between a harmonic of the nonlinear Peierls-Nabarro frequency $\omega_{\rm PN}$ and ω_s, e.g., $3\omega_{\rm PN}\approx\omega_s$.

An important question about the internal quasimode of the SG that remains to be answered is why are there no effects of the mode in SG soliton-soliton and soliton-antisoliton collisions where we know analytically that, although the slopes of the solitons change as they pass through each other, the resultant shapes after collision are the same as the shapes before collision and nothing else has changed. In particular, there is no radiation because such a collision is an exact solution of the SG equation. In some sense the change of slope has to occur in such a manner that no radiation is emitted. The problem is very difficult to solve analytically because the soliton-soliton collisions are highly nonlinear and they are relativistic because the potential well depth is comparable to the rest energy of the solitons.

A completely different exact collective variable approach to the problem of the SG internal mode is to not introduce the second variable $\Gamma(t)$ and just keep the single collective variable X. However, then

$$\sigma\{(\pi/l_0)[x-X(t)]/(1-\dot{X}^2)^{1/2}\}$$

depends on X and $\dot{X}$. Since the Lagrangian depends on $\dot{\phi}^2$ and $\dot{\phi}$, in turn, depends on $\ddot{X}$, it follows that the Lagrangian equation of motion for X is a fourth derivative equation d^4X/dt^4 which can be thought of as two coupled second-order differential equations. It is possible that, in problems where the relativistic behavior is important, such as in kink-kink collisions, the d^4X/dt^4 collective variable description might have advantages over the present coupled $\ddot{X}$ and $\ddot{\Gamma}$ equations when carrying out calculations that are based on approximations to the exact equations of motion.

APPENDIX A

The purpose of this appendix is to evaluate the integrals I_0, I_2, I_4, and I_6 defined in Eq. (2.14b) which we rewrite as

$$I_{2m}\equiv\int_{-\infty}^{\infty}d\bar{k}\frac{\text{sech}^2(\bar{k}\pi/2)}{\omega_k^{2m}}\ . \qquad (A1)$$

I_0 is elementary and we obtain $I_0=4/\pi$. For the other three integrals $m=1$, 2, and 3 we choose a contour in the complex plane that lies along the real axis and is closed by a semicircle of infinite radius in the upper half plane. The integrand along the semicircle vanishes for I_2, I_4, and I_6 and so we write

$$I_{2m}=\oint dz\frac{\text{sech}^2(z\pi/2)}{\omega_z^{2m}}=2\pi i\sum_{l=0}^{\infty}R_l^{(2m)}\ , \qquad (A2)$$

where $R_l^{(2m)}$ is the residue of the lth singularity enclosed by the contour for the integral I_{2m}. The singularities are located at $z_l=(2l+1)i$ for $l=0,\ldots,\infty$. The contour for $m=1,2,3$ thus encloses an infinite number of singularities, the sum of whose residues we express in closed form.

The singularity in I_{2m} at $z_0=i$ is of order $m+2$ [m from

$$\omega_z^{2m}=(z+i)^m(z-i)^m$$

and 2 from the square of the sech]. The singularities in I_{2m} at z_l for $l>0$ are second order (from the sech). We define the variable u by $z\equiv z_l+u$, substitute into the integrand in Eq. (A2), and expand the integrand of I_{2m}

about $u=0$ to evaluate the residues at z_l for $l=0,\ldots,\infty$.

We find

$$R_0^{(2)}=\frac{1}{(2\pi i)\pi}\left[\frac{\pi^2}{3}+1\right],$$
$$R_{l>0}^{(2)}=-\frac{1}{(2\pi i)\pi}\frac{2l+1}{l^2(l+1)^2}, \tag{A3}$$

$$R_0^{(4)}=\frac{1}{(2\pi i)\pi}\left[\frac{\pi^2}{6}+1\right],$$
$$R_{l>0}^{(4)}=\frac{1}{(2\pi i)\pi}\frac{2l+1}{l^3(l+1)^3}, \tag{A4}$$

$$R_0^{(6)}=\frac{1}{(2\pi i)\pi}\left[\frac{\pi^4}{240}+\frac{\pi^2}{8}+\frac{15}{16}\right],$$
$$R_{l>0}^{(6)}=\frac{-3}{(2\pi i)16\pi}\frac{2l+1}{l^4(l+1)^4}. \tag{A5}$$

We need to evaluate the sum of the residues for $m=1,2,3$. We find it convenient to express the sums in terms of the Riemann ζ function[7] defined in Eq. (2.16b). Values of the Riemann ζ function that we use are $\zeta(2)=\pi^2/6$, $\zeta(3)\approx 1.2021$ (which is, at present, not known analytically), and $\zeta(4)=\pi^4/90$.

I_2 is evaluated by noting that

$$\sum_{l=1}^{\infty}\frac{2l+1}{l^2(l+1)^2}=\sum_{l=1}^{\infty}\frac{1}{l^2}-\sum_{l=1}^{\infty}\frac{1}{(l+1)^2}$$
$$=\zeta(2)-[\zeta(2)-1]=1 \tag{A6}$$

and therefore, using Eq. (A3), we obtain

$$I_2=2\pi i\sum_{l=0}^{\infty}R_l^{(2)}=\frac{\pi}{3}\approx 1.0472\ . \tag{A7}$$

In order to evaluate I_4 we must compute $\sum_{l=1}^{\infty}R_l^{(4)}$. To this end, we consider the expression $\zeta(3)-1$ and, using Eqs. (2.16b) and (A6), write $\zeta(3)-1$ as

$$\zeta(3)-1=\sum_{l=1}^{\infty}\frac{1}{l^3}-\sum_{l=1}^{\infty}\frac{2l+1}{l^2(l+1)^2}=\sum_{l=1}^{\infty}\frac{-l^3+2l+1}{l^3(l+1)^3}$$
$$=-\sum_{l=1}^{\infty}\frac{1}{(l+1)^3}+\sum_{l=1}^{\infty}\frac{2l+1}{l^3(l+1)^3}$$
$$=-[\zeta(3)-1]+\sum_{l=1}^{\infty}\frac{2l+1}{l^3(l+1)^3}$$

from which we obtain

$$\sum_{l=1}^{\infty}\frac{2l+1}{l^3(l+1)^3}=2[\zeta(3)-1]\ . \tag{A8}$$

Therefore, using Eq. (A4), we obtain

$$I_4=2\pi i\sum_{l=0}^{\infty}R_l^{(4)}=\frac{1}{\pi}[\zeta(2)+\zeta(3)]\approx 0.9062\ . \tag{A9}$$

To calculate I_6 we use Eq. (A8) and consider the expression

$$\zeta(4)-4[\zeta(3)-1]=\sum_{l=1}^{\infty}\frac{1}{l^4}-2\sum_{l=1}^{\infty}\frac{2l+1}{l^3(l+1)^3}$$
$$=\sum_{l=1}^{\infty}\frac{l^4+2l+1}{l^4(l+1)^4}$$
$$=\sum_{l=1}^{\infty}\frac{1}{(l+1)^4}+\sum_{l=1}^{\infty}\frac{2l+1}{l^4(l+1)^4}$$
$$=[\zeta(4)-1]+\sum_{l=1}^{\infty}\frac{2l+1}{l^4(l+1)^4}$$

from which we obtain

$$\sum_{l=1}^{\infty}\frac{2l+1}{l^4(l+1)^4}=5-4\zeta(3)\ . \tag{A10}$$

Therefore, using Eq. (A5), we obtain

$$I_6=2\pi i\sum_{l=0}^{\infty}R_l^{(6)}=\frac{3}{4\pi}[\zeta(2)+\zeta(3)+\tfrac{1}{2}\zeta(4)]$$
$$\approx 0.8089\ . \tag{A11}$$

Using Eqs. (A7), (A9), and (A11) we are then able to evaluate the quantities λ [Eq. (2.14a)], the dressed mass $\hat{M}_{\Gamma_0}$ [Eq. (2.21)], and the dressed frequency Ω_d^2 [Eq. (2.22)].

APPENDIX B: DERIVATION OF DRESSED FREQUENCY Ω_d AND DRESSED MASS $\hat{M}_{\Gamma_0}$

We derive in this appendix the expressions for Ω_d [Eq. (2.22)] and $\hat{M}_{\Gamma_0}$ [Eq. (2.21)], which are expressed in terms of the integral I_n defined in Eq. (2.14b) and evaluated in Appendix A for $n=0$, 2, 4, and 6. In order to calculate Ω_d^2 defined by Eq. (2.20b), we first need to calculate four integrals, namely $\langle\partial\sigma_0/\partial\Gamma_0|\mathcal{L}_0|\chi_d\rangle$, $\langle\chi_d|\mathcal{L}_0|\chi_d\rangle$, $\langle\partial\sigma_0/\partial\Gamma_0|\sigma_0''\rangle$, and $\langle\chi_d|\sigma_0''\rangle$ which appear in the numerator of Eq. (2.20b).

From the definition of χ_d in Eq. (2.17) and $\mathcal{L}_\xi$ in Eq. (2.9b), we obtain

$$\mathcal{L}_0|\chi_d\rangle = \Gamma_0^2\mathcal{L}_\xi|\chi_d\rangle$$
$$= \sqrt{2\pi}\Gamma_0\int_{-\infty}^{\infty} d\bar{k}\,|\psi_{\bar{k}}\rangle\omega_{\bar{k}}\operatorname{sech}\left[\frac{\bar{k}\pi}{2}\right]\left[1-\frac{\lambda}{\omega_{\bar{k}}^2}\right], \tag{B1}$$

where we used $\mathcal{L}_\xi|\psi_{\bar{k}}\rangle = \omega_{\bar{k}}^2|\psi_{\bar{k}}\rangle$. Then, since

$\langle \partial\sigma_0/\partial\Gamma_0|\psi_{\bar{k}}\rangle = \langle \xi_0\sigma_0'/\Gamma_0|\psi_{\bar{k}}\rangle$

is given by Eq. (2.15a), we have the result

$$\left\langle \frac{\partial\sigma_0}{\partial\Gamma_0}\left|\mathcal{L}_0\right|\chi_d\right\rangle = -2\pi\int_{-\infty}^{\infty} d\bar{k}\operatorname{sech}^2\left[\frac{\bar{k}\pi}{2}\right]\left[1-\frac{\lambda}{\omega_{\bar{k}}^2}\right]$$
$$= -2\pi(I_0-\lambda I_2)\ . \tag{B2}$$

Next, taking the inner product of Eq. (2.17) for $\langle\chi_d|$ with Eq. (B1), we obtain

$$\langle\chi_d|\mathcal{L}_0|\chi_d\rangle = 2\pi\int_{-\infty}^{\infty} d\bar{k}\operatorname{sech}^2\left[\frac{\bar{k}\pi}{2}\right]\left[1-\frac{\lambda}{\omega_{\bar{k}}^2}\right]^2$$
$$= 2\pi(I_0-2\lambda I_2+\lambda^2 I_4)\ . \tag{B3}$$

We also obtain

$$\left\langle \frac{\partial\sigma_0}{\partial\Gamma_0}\middle|\sigma_0''\right\rangle = -\frac{4}{\Gamma_0}\ . \tag{B4}$$

Finally, we evaluate the integral $\langle\chi_d|\sigma_0''\rangle$ by using Eqs. (2.15b) and (2.17) to obtain

$$\langle\chi_d|\sigma_0''\rangle = \frac{\pi}{\Gamma_0}\int_{-\infty}^{\infty} d\bar{k}\operatorname{sech}^2\left[\frac{\bar{k}\pi}{2}\right]\left[1-\frac{\lambda}{\omega_{\bar{k}}^2}\right]$$
$$= \frac{\pi}{\Gamma_0}(I_0-\lambda I_2)\ . \tag{B5}$$

Using the above integrals we now calculate the quantity which appears in the numerator of Eq. (2.20b), which is the expression for Ω_d^2,

$$\left\langle \frac{\partial\sigma_0}{\partial\Gamma_0}+\chi_d\left|\mathcal{L}_0\chi_d-2\Gamma_0\sigma_0''\right.\right\rangle = 8-2\pi I_0+2\pi\lambda^2 I_4$$
$$= 2\pi\lambda^2 I_4\ , \tag{B6}$$

where we used

$$2\pi\int_{-\infty}^{\infty} d\bar{k}\operatorname{sech}^2(\bar{k}\pi/2) = 2\pi I_0 = 8$$

and the value of I_4 is given by Eq. (A9).

To calculate

$$\hat{M}_{\Gamma_0} = \frac{1}{\Gamma_0}\left\langle \frac{\partial\hat{\sigma}}{\partial\Gamma}\middle|\frac{\partial\hat{\sigma}}{\partial\Gamma}\right\rangle\Bigg|_{\Gamma=\Gamma_0}$$

defined in Eq. (2.20c) [which appears in the denominator of Eq. (2.20b)] we need to calculate

$\langle \partial\sigma_0/\partial\Gamma_0+\chi_d|\partial\sigma_0/\partial\Gamma_0+\chi_d\rangle$

which requires the integrals

$$\left\langle \frac{\partial\sigma_0}{\partial\Gamma_0}\middle|\frac{\partial\sigma_0}{\partial\Gamma_0}\right\rangle = \Gamma_0 M_{\Gamma_0}\ , \tag{B7}$$

$$\left\langle \frac{\partial\sigma_0}{\partial\Gamma_0}\middle|\chi_d\right\rangle = \frac{1}{\Gamma_0}\langle\xi_0\sigma_0'|\chi_d\rangle$$
$$= -\frac{2\pi}{\Gamma_0^2}\int_{-\infty}^{\infty} d\bar{k}\frac{\operatorname{sech}^2(\bar{k}\pi/2)}{\omega_{\bar{k}}^2}\left[1-\frac{\lambda}{\omega_{\bar{k}}^2}\right]$$
$$= -\frac{2\pi}{\Gamma_0^2}(I_2-\lambda I_4)\ , \tag{B8}$$

and

$$\langle\chi_d|\chi_d\rangle = \frac{2\pi}{\Gamma_0^2}\int_{-\infty}^{\infty} d\bar{k}\frac{\operatorname{sech}^2(\bar{k}\pi/2)}{\omega_{\bar{k}}^2}\left[1-\frac{\lambda}{\omega_{\bar{k}}^2}\right]^2$$
$$= \frac{2\pi}{\Gamma_0^2}(I_2-2\lambda I_4+\lambda^2 I_6)\ , \tag{B9}$$

where $|\chi_d\rangle$ is defined in Eq. (2.17) and we used Eq. (2.15a) in order to obtain Eq. (B8). We then obtain, for the dressed mass $\hat{M}_{\Gamma_0}$ defined in Eq. (2.20c), the result

$$\hat{M}_{\Gamma_0} = M_{\Gamma_0} - 2\left[\frac{2\pi}{\Gamma_0^3}\right](I_2-\lambda I_4)$$
$$+\left[\frac{2\pi}{\Gamma_0^3}\right](I_2-2\lambda I_4+\lambda^2 I_6) = \frac{2\pi}{\Gamma_0^3}\lambda^2 I_6 \tag{B10}$$

which is Eq. (2.21). We used $M_{\Gamma_0} = (2\pi/\Gamma_0^3)I_2$ to obtain the last equality in Eq. (B10) and I_6 is given by Eq. (A11).

Finally, upon substituting Eqs. (B6) and (B10) into Eq. (2.20b) for Ω_d^2, we obtain

$$\Omega_d^2 = \frac{(2\pi/\Gamma_0)\lambda^2 I_4}{(2\pi/\Gamma_0^3)\lambda^2 I_6} = \frac{I_4}{I_6}\Gamma_0^2 \tag{B11}$$

which is Eq. (2.22).

APPENDIX C

In this appendix we solve Eq. (4.8b) for χ and calculate the spatially averaged radiated power

$$\langle P\rangle = -2\Gamma_0\left\langle \frac{\partial\chi}{\partial t}\chi'\right\rangle\ .$$

For convenience we reproduce Eq. (4.8b):

$$\left|\frac{\partial^2\chi}{\partial t^2}\right\rangle + \mathcal{L}_0|\chi\rangle = \delta\Gamma(2\Gamma_0|\sigma_0''\rangle - \mathcal{L}_0|\chi_d\rangle)$$
$$-\delta\ddot{\Gamma}\left|\frac{\xi_0\sigma_0'}{\Gamma_0}+\chi_d\right\rangle\ . \tag{4.8b}$$

Projecting out the $\delta\ddot{\Gamma}$ term by operating on Eq. (4.8b) with $1-\hat{\mathcal{P}}_{\Gamma_0}$ [where $\hat{\mathcal{P}}_{\Gamma_0}$ is defined in Eq. (4.9)], we obtain

$$\left|\frac{\partial^2\chi}{\partial t^2}\right\rangle + (1-\hat{\mathcal{P}}_{\Gamma_0})\mathcal{L}_0|\chi\rangle = \delta\Gamma(1-\hat{\mathcal{P}}_{\Gamma_0})(2\Gamma_0|\sigma_0''\rangle$$
$$-\mathcal{L}_0|\chi_d\rangle)$$
$$\equiv (1-\hat{\mathcal{P}}_{\Gamma_0})|\rho\rangle\ , \tag{C1}$$

where we used

$$(1-\hat{\mathcal{P}}_{\Gamma_0})\left|\frac{\xi_0\sigma_0}{\Gamma_0}+\chi_d\right\rangle=0$$

and where we have denoted the right-hand side of Eq. (C1) by $(1-\hat{\mathcal{P}}_{\Gamma_0})|\rho\rangle$, which defines $|\rho\rangle$. We use Eq. (4.9) to obtain

$$(1-\hat{\mathcal{P}}_{\Gamma_0})|\rho\rangle=\delta\Gamma(2\Gamma_0|\sigma_0''\rangle-\mathcal{L}_0|\chi_d\rangle)+\delta\Gamma\left|\frac{\xi_0\sigma_0'}{\Gamma_0}+\chi_d\right\rangle\frac{1}{\Gamma_0\hat{M}_{\Gamma_0}}\times\left\langle\frac{\xi_0\sigma_0'}{\Gamma_0}+\chi_d\left|\mathcal{L}_0\chi_d-2\Gamma_0\sigma_0''\right.\right\rangle. \quad \text{(C2)}$$

We see that the last term in Eq. (C2) is directly proportional to the expression for Ω_d^2 which was already derived in Eq. (2.20b). Therefore, substituting Eq. (2.20b) into Eq. (C2) we obtain

$$(1-\hat{\mathcal{P}}_{\Gamma_0})|\rho\rangle=\delta\Gamma(2\Gamma_0|\sigma_0''\rangle-\mathcal{L}_0|\chi_d\rangle)+\delta\Gamma\Omega_d^2\left|\frac{\xi_0\sigma_0'}{\Gamma_0}+\chi_d\right\rangle. \quad \text{(C3)}$$

Equation (C3) is the expression for the source that appears on the right-hand side of Eq. (C1).

In order to find the solution χ we need to integrate the source in Eq. (C3) against the Green's function for the operator that appears on the left-hand side of Eq. (C1). The construction of the Green's function is difficult, however, because of the presence of the projection operator—the eigenfunctions of the operator $\partial^2/\partial t^2+(1-\hat{\mathcal{P}}_{\Gamma_0})\mathcal{L}_0$ are unknown. Therefore, to simplify the calculation of the Green's function we will ignore the projection operator on the left-hand side of Eq. (C1). Thus, for the equation of motion for χ we obtain

$$\left|\frac{\partial^2\chi}{\partial t^2}\right\rangle+\mathcal{L}_0|\chi\rangle=\delta\Gamma(2\Gamma_0|\sigma_0''\rangle-\mathcal{L}_0|\chi_d\rangle)+\delta\Gamma\Omega_d^2\left|\frac{\xi_0\sigma_0'}{\Gamma_0}+\chi_d\right\rangle. \quad \text{(C4)}$$

Then, the Green's function for the operator on the left-hand side of Eq. (C4) is

$$G(\xi,\xi'|\tau)=\frac{1}{\Gamma_0}\int_{-\infty}^{\infty}d\bar{k}\,\psi_{\bar{k}}(\xi)\frac{\sin(\Gamma_0\omega_{\bar{k}}\tau)}{\omega_{\bar{k}}}\psi_{\bar{k}}^*(\xi')=\frac{2}{\Gamma_0}\mathrm{Re}\int_0^{\infty}d\bar{k}\,\psi_{\bar{k}}(\xi)\frac{\sin(\Gamma_0\omega_{\bar{k}}\tau)}{\omega_{\bar{k}}}\psi_{\bar{k}}^*(\xi'), \quad \text{(C5)}$$

where $\tau=t-t'$. The formal solution is then

$$\chi=\frac{2}{\Gamma_0}\mathrm{Re}\int_0^{\infty}d\bar{k}\frac{\psi_{\bar{k}}(\xi)}{\omega_{\bar{k}}}\int_0^{\infty}dt'\sin[\Gamma_0\omega_{\bar{k}}(t-t')]\times\langle\psi_{\bar{k}}(\xi')|1-\hat{\mathcal{P}}_{\Gamma_0}|\rho(\xi',t')\rangle. \quad \text{(C6)}$$

In order to evaluate the last term (the bracket) in Eq. (C6) we note, using Eq. (C3), that

$$\langle\psi_{\bar{k}}(\xi')|1-\hat{\mathcal{P}}_{\Gamma_0}|\rho(\xi',t')\rangle=\delta\Gamma\langle\psi_{\bar{k}}|2\Gamma_0\sigma_0''-\mathcal{L}_0\chi_d\rangle+\delta\Gamma\Omega_d^2\left\langle\psi_{\bar{k}}\left|\frac{\xi_0}{\Gamma_0}\sigma_0'+\chi_d\right.\right\rangle. \quad \text{(C7)}$$

Then, using Eqs. (2.15a)–(2.15b), (2.17), and (B1) to perform the integrals in Eq. (C7), we see that the right-hand side of Eq. (C7) becomes

$$\langle\psi_{\bar{k}}(\xi')|1-\hat{\mathcal{P}}_{\Gamma_0}|\rho(\xi',t')\rangle=\sqrt{2\pi}\Gamma_0\delta\Gamma\lambda\frac{\mathrm{sech}(\bar{k}\pi/2)}{\omega_{\bar{k}}^3}\left[\omega_{\bar{k}}^2-\frac{\Omega_d^2}{\Gamma_0^2}\right]. \quad \text{(C8a)}$$

Next we substitute Eq. (C8a) back into the expression for χ, Eq. (C6). At the same time we make the substitution, we express $\psi_{\bar{k}}$ explicitly using Eq. (2.9d) and also set

$$\delta\Gamma(t')\equiv\delta\gamma_0\sin(\Omega_d t') \quad \text{(C8b)}$$

which, analogous to the electromagnetic calculation outlined in Sec. IV, indicates that we are calculating χ for an undamped radiating oscillator where $\delta\gamma_0$ represents the amplitude of oscillation. Carrying out the substitutions we see that Eq. (C6) becomes

$$\chi=2\delta\gamma_0\lambda\,\mathrm{Re}\int_0^{\infty}d\bar{k}e^{i\bar{k}\xi}(i\bar{k}-\tanh\xi)\frac{\mathrm{sech}(\bar{k}\pi/2)}{\omega_{\bar{k}}^5}\left[\omega_{\bar{k}}^2-\frac{\Omega_d^2}{\Gamma_0^2}\right]\int_0^{\infty}dt'\sin[\Gamma_0\omega_{\bar{k}}(t-t')]\sin(\Omega_d t'), \quad \text{(C9)}$$

where we have extended the upper limit of the t' integral to infinity. The t' integral is

$$\int_0^{\infty}dt'\sin[\Gamma_0\omega_{\bar{k}}(t-t')]\sin(\Omega_d t')=-\frac{\pi}{2}\delta(\Omega_d-\Gamma_0\omega_{\bar{k}})\cos(\Gamma_0\omega_{\bar{k}}t)+\Omega_d\frac{P}{\Omega_d^2-\Gamma_0^2\omega_{\bar{k}}^2}\sin(\Gamma_0\omega_{\bar{k}}t), \quad \text{(C10)}$$

where the Dirac δ function appears in the first term on the right-hand side of Eq. (C10) and P denotes the Cauchy principle value in the second term. When we substitute Eq. (C10) back into Eq. (C9) we see that the δ-function term gives zero when the integral over $\bar{k}$ is performed because of the factor $[\omega_{\bar{k}}^2-(\Omega_d^2/\Gamma_0^2)]$ in the integrand. On the other hand, the same factor $[\omega_{\bar{k}}^2-(\Omega_d^2/\Gamma_0^2)]$ in the integrand just cancels the denominator of the principle part in Eq. (C10) (leaving only a factor of $-1/\Gamma_0^2$). Therefore, Eq. (C9) becomes

$$\chi=-2\lambda\left[\frac{\Omega_d}{\Gamma_0}\right]\left[\frac{\delta\gamma_0}{\Gamma_0}\right]\mathrm{Re}\int_0^\infty d\bar{k}e^{i\bar{k}\xi}(i\bar{k}-\tanh\xi)\frac{\mathrm{sech}(\bar{k}\pi/2)}{\omega_{\bar{k}}^5}\sin(\Gamma_0\omega_{\bar{k}}t)\ . \tag{C11}$$

Next we consider only waves traveling to the right, so Eq. (C11) becomes

$$\begin{aligned}\chi&=-2\lambda\left[\frac{\Omega_d}{\Gamma_0}\right]\left[\frac{\delta\gamma_0}{\Gamma_0}\right]\mathrm{Re}\int_0^\infty d\bar{k}\frac{\mathrm{sech}(\bar{k}\pi/2)}{\omega_{\bar{k}}^5}\left[-\frac{1}{2i}e^{i(\bar{k}\xi-\Gamma_0\omega_{\bar{k}}t)}\right](i\bar{k}-\tanh\xi)\\&=\lambda\left[\frac{\Omega_d}{\Gamma_0}\right]\left[\frac{\delta\gamma_0}{\Gamma_0}\right]\int_0^\infty d\bar{k}\frac{\mathrm{sech}(\bar{k}\pi/2)}{\omega_{\bar{k}}^5}[\bar{k}\cos(\bar{k}\xi-\Gamma_0\omega_{\bar{k}}t)-\sin(\bar{k}\xi-\Gamma_0\omega_{\bar{k}}t)]\ ,\end{aligned} \tag{C12}$$

where we have set $\tanh\xi$ equal to one since we will be evaluating the radiation and, hence, the Poynting's flux far away from the kink.

In order to calculate the Poynting's flux we calculate the quantity $\chi'(\partial\chi/\partial t)$. The result is a double integral over $\bar{k}$ and $\bar{k}'$ of four trigonometric functions. When we space average the twofold integration over k space only the following integral survives:

$$\left\langle\frac{\partial\chi}{\partial t}\chi'\right\rangle_{\mathrm{avg}}=-\tfrac{1}{2}\Gamma_0\lambda^2\left[\frac{\Omega_d}{\Gamma_0}\right]^2\left[\frac{\delta\gamma_0}{\Gamma_0}\right]^2\int d\bar{k}\,d\bar{k}'\frac{\mathrm{sech}(\bar{k}\pi/2)}{\omega_{\bar{k}}^4}\frac{\mathrm{sech}(\bar{k}'\pi/2)}{\omega_{\bar{k}'}^5}\bar{k}'(\bar{k}\bar{k}'+1)\langle\cos[(\bar{k}-\bar{k}')\xi]\rangle_{\mathrm{avg}}\ , \tag{C13}$$

where the factor of $-\frac{1}{2}$ out front appears as a consequence of multiplying the trigonometric functions together in order to obtain the argument $(\bar{k}-\bar{k}')\xi$ in the cos in Eq. (C13) and the definition of the spatial average is

$$\begin{aligned}\langle\cos[(\bar{k}-\bar{k}')\xi]\rangle_{\mathrm{avg}}&\equiv\frac{1}{2\bar{\lambda}_d}\int_{\xi-\bar{\lambda}_d}^{\xi+\bar{\lambda}_d}d\xi\cos[(\bar{k}-\bar{k}')\xi]\\&=\frac{1}{\bar{\lambda}_d}\cos[(\bar{k}-\bar{k}')\xi]\frac{\sin[(\bar{k}-\bar{k}')\bar{\lambda}_d]}{\bar{k}-\bar{k}'}\\&\to\frac{\pi}{\bar{\lambda}_d}\delta(\bar{k}-\bar{k}')\ ,\end{aligned} \tag{C14}$$

where $\bar{\lambda}_d=\Gamma_0\lambda_d$ and λ_d is the wavelength corresponding to the frequency Ω_d. That is, $\lambda_d=2\pi/k_d$, where

$$k_d=\Gamma_0(\Omega_d^2/\Gamma_0^2-1)^{1/2}\ .$$

The value of ξ where the space average is performed is a distance from the soliton at the origin that is large compared with the size of the soliton and large compared with $\bar{\lambda}_d$. The δ function is exact in the limit $\bar{\lambda}_d\to\infty$ and in our units the observed radiated wavelength is greater than 70 (greater than 22 kink lengths). Note that the space average has suppressed the time dependence in the trigonometric functions.

When we substitute Eq. (C14) into Eq. (C13) we obtain

$$\begin{aligned}-2\Gamma_0\left\langle\frac{\partial\chi}{\partial t}\chi'\right\rangle_{\mathrm{avg}}\equiv\langle P\rangle_{\mathrm{avg}}&=\frac{k_d}{2}\Gamma_0\lambda^2\left[\frac{\Omega_d}{\Gamma_0}\right]^2\left[\frac{\delta\gamma_0}{\Gamma_0}\right]^2\int_0^\infty d\bar{k}\,\bar{k}(\bar{k}^2+1)\frac{\mathrm{sech}^2(\bar{k}\pi/2)}{\omega_{\bar{k}}^9}\\&=\frac{k_d}{2}\Gamma_0\lambda^2\left[\frac{\Omega_d}{\Gamma_0}\right]^2\left[\frac{\delta\gamma_0}{\Gamma_0}\right]^2\mathcal{J}\ ,\end{aligned} \tag{C15a}$$

where

$$\mathcal{J}=\int_0^\infty d\bar{k}\,\bar{k}\frac{\mathrm{sech}^2(\bar{k}\pi/2)}{\omega_{\bar{k}}^7}\approx0.0987\ , \tag{C15b}$$

and where $\langle P\rangle_{\mathrm{avg}}$ is the space averaged power. In order to obtain better agreement with the simulation value of the linewidth we express the average power $\langle P\rangle_{\mathrm{avg}}$ defined in Eq. (C15a) in terms of the simulation frequency, i.e., $\Omega_d\to\omega_s=\Gamma_0\bar{\omega}_s$, and the simulation wave number, i.e., $k_d\to k_s$, and obtain

$$\langle P\rangle_{\mathrm{avg}}=\frac{k_s}{2}\Gamma_0\lambda^2\bar{\omega}_s^2\left[\frac{\delta\gamma_0}{\Gamma_0}\right]^2\mathcal{J}\ . \tag{C15a$'$}$$

Finally, substituting Eq. (C15a′) into Eq. (4.4b) for the inverse lifetime $1/\tau$ we obtain

$$\begin{aligned}\frac{1}{\tau}&=\frac{2}{(\Gamma_0\bar{\omega}_s)^2\hat{M}_{\Gamma_0}}\frac{\langle P\rangle_{\mathrm{avg}}}{(\delta\gamma_0)^2}\\&=\frac{k_s\Gamma_0\lambda^2\mathcal{J}}{\Gamma_0^4\hat{M}_{\Gamma_0}}=\frac{k_s}{2\pi}\frac{\mathcal{J}}{I_6}=\frac{1}{\lambda_s}\frac{\mathcal{J}}{I_6}\approx0.1220\frac{1}{\lambda_s}\end{aligned} \tag{C16}$$

which is Eq. (4.11) where we used Eq. (2.21) for $\hat{M}_{\Gamma_0}$ to obtain the third equality in Eq. (C16).

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[1]M. J. Rice, Phys. Rev. B **28**, 3587 (1983).
[2]M. J. Rice and E. J. Mele, Solid State Commun. **35**, 487 (1980).
[3]J. C. Fernandez, M. J. Goupil, O. Legrand, and G. Reinsch, Phys. Rev. B **34**, 6207 (1986).
[4]R. Boesch, P. Stancioff, and C. R. Willis, Phys. Rev. B **38**, 6713 (1988), and references therein.
[5]P. Stancioff, C. R. Willis, M. El-Batanouny, and S. Burdick, Phys. Rev. B **33**, 1912 (1987).
[6]R. Boesch and C. R. Willis, Phys. Rev. B **39**, 361 (1989), and references therein.
[7]*Handbook of Mathematical Functions,* Natl. Bur. of Stand. Appl. Math. Ser. No. 55, edited by M. Abromowitz and I. Stegun (U.S. GPO, Washington, D.C., 1965).
[8]For instance, R. Good, Jr. and T. Nelson, *Classical Theory of Electric and Magnetic Fields* (Academic, New York, 1971).
[9]D. Campbell, J. F. Schonfeld, and C. A. Wingate, Physica D **9**, 1 (1983).
[10]D. Campbell, M. Peyrard, and P. Sodano, Physica D **19**, 165 (1986).
[11]C. R. Willis, M. El-Batanouny, S. Burdick, R. Boesch, and P. Sodano, Phys. Rev. B **35**, 3496 (1987); S. Burdick, C. R. Willis, and M. El-Batanouny, *ibid.* **36**, 6920 (1987); R. Ravelo, M. El-Batanouny, and C. R. Willis, *ibid.* **38**, 4817 (1988).
[12]J. Segur, J. Math. Phys. **24**, 1439 (1983).
[13]M. Peyrard and D. K. Campbell, Physica D **9**, 33 (1983).

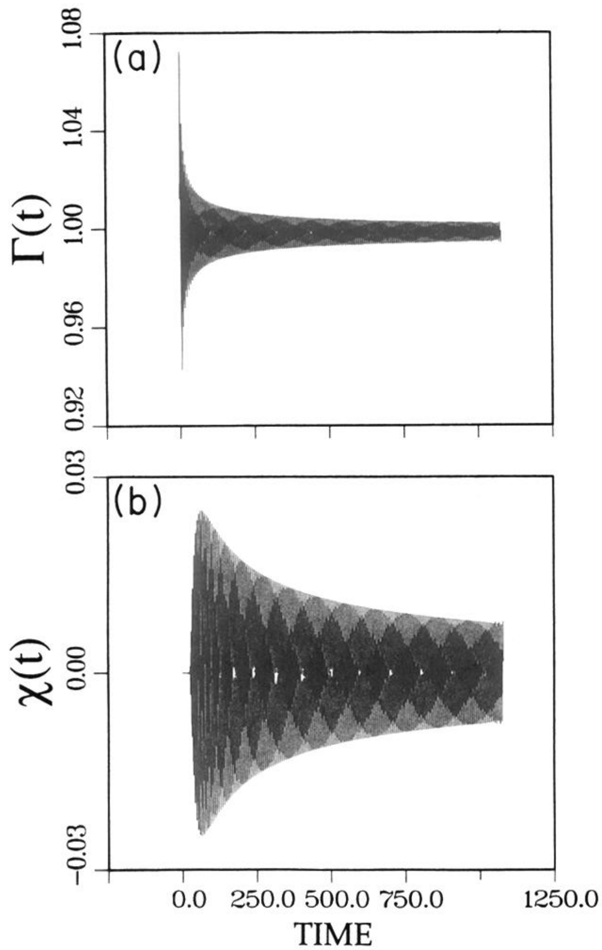

FIG. 1. $\delta\Gamma_0 \approx 0.01$, $\Gamma_0 = 1$. (a) shows $\Gamma(t)$ and (b) shows $\chi(t) = \phi - \sigma$ measured 25 units away from a kink. The initial condition for $t = 0$ is a kink $|\sigma_0\rangle$ with velocity determined by specifying the kink shape at $t = \Delta t$ as $|\sigma_0\rangle + 0.001|\partial\sigma_0/\partial\Gamma_0\rangle$. Motion evolves according to Eq. (3.2). The length of the system is 1000 units.

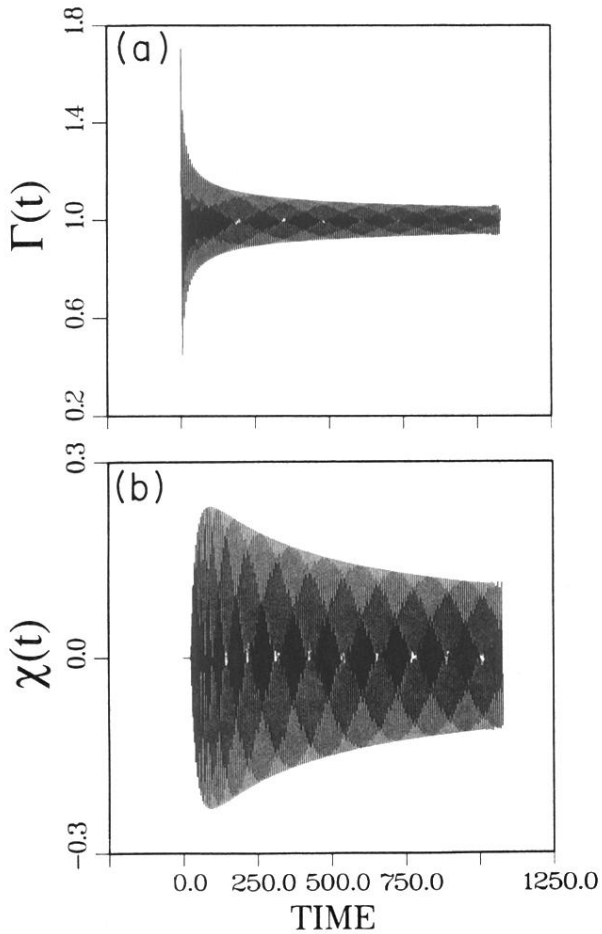

FIG. 3. $\delta\Gamma_0 \approx 0.1$, $\Gamma_0 = 1$. (a) shows $\Gamma(t)$ and (b) shows $\chi(t) = \phi - \sigma$ measured 25 units away from a kink. The initial condition for $t = 0$ is a kink $|\sigma_0\rangle$ with velocity determined by specifying the kink shape at $t = \Delta t$ as $|\sigma_0\rangle + 0.01|\partial\sigma_0/\partial\Gamma_0\rangle$. Motion evolves according to Eq. (3.2). The length of the system is 1000 units.

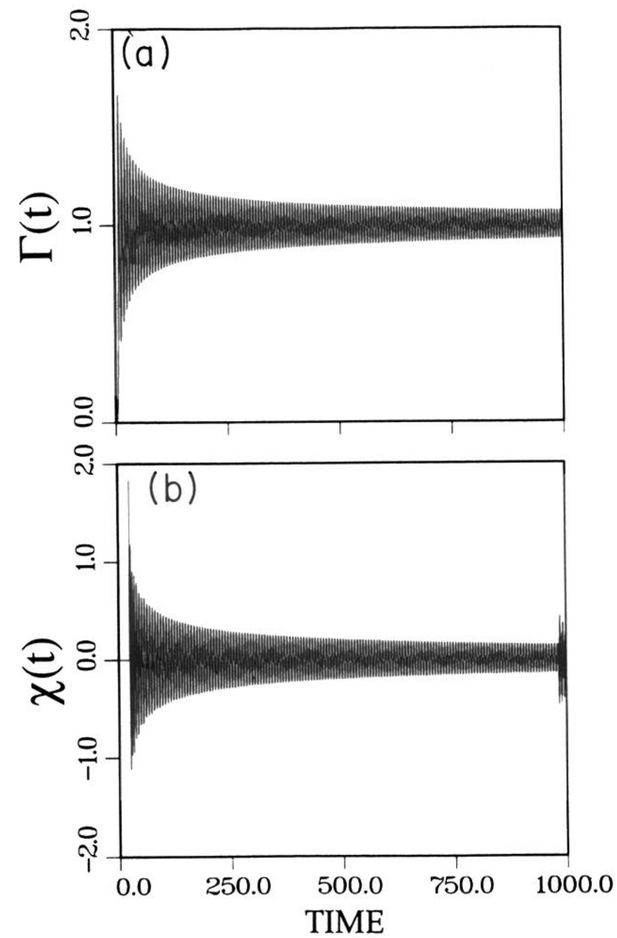

FIG. 5. $\delta\Gamma_0 \approx 0.1$, $\Gamma_0 = 1$. (a) shows $\Gamma(t)$ and (b) shows $\chi(t) = \phi - \sigma$ measured 25 units away from a kink. Initial condition for $t = 0$ is a kink with $\pi / l_0 = 0.005$ with zero velocity. Motion evolves according to Eq. (3.2). Note the reflection in (b) because of a shorter system (200 units long) than in Figs. 1 and 3.