Finite-size studies of particles obeying fractional statistics

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(Received 30 March 1990)

The ground-state properties of particles obeying fractional statistics, on lattices with spherical topology, are studied numerically. The possible existence of superfluidity in semion systems is examined by studies of pairing energies, flux quantization, and the effects of applied magnetic fields. Evidence of superfluidity is observed, in agreement with previous numerical studies on cylindrical topologies. Comparison of the results for these different topologies allows some conclusions to be made about finite-size and boundary effects. In particular, we argue that there is no constant shift in the flux-quantization values.

I. INTRODUCTION

The discovery of high-temperature oxide superconductors has stimulated interest in the study of superconductivity in quasi-two-dimensional systems, since the conduction electrons in these oxides are essentially constrained to move in two-dimensional sheets. A unique feature of two-dimensional systems is that they can support quasiparticles obeying fractional statistics (referred to as anyons)[1] while only Fermi and Bose statistics are allowed in three dimensions.[2] Fractional statistics can be parametrized by a continuous variable α such that the anyon wave function acquires a phase factor $e^{i\alpha\pi}$ from interchange of two identical anyons. Thus, fractional statistics interpolate between Bose ($\alpha=0$) and Fermi ($\alpha=1$) statistics.

Quantum statistics have profound consequences for condensed matter systems at low temperatures, giving rise to the concepts of a Fermi surface for fermions and a condensate for bosons. It is pedagogically very interesting to study the consequences of fractional statistics at low temperatures in physical systems where the dynamics of the particles are constrained to two dimensions. The electrons in an inversion layer at low temperatures provide an example of such a system, and in a large transverse magnetic field under certain conditions, are expected to have quasiparticle excitations which obey fractional statistics.[3] High-temperature oxide superconductors also appear to be highly two dimensional, and Laughlin[4] has proposed a theory of two-dimensional superconductivity which is driven completely by the fractional statistical nature of the charge cariers. In this theory, the charged carriers obey fractional statistics with $\alpha=\frac{1}{2}$. Such particles are referred to as semions.

There is in fact strong evidence that certain anyon systems may form a superfluid ground state at low temperatures. Anyons may be described as charged bosons or fermions with a magnetic flux tube attached to each particle.[5] When particles move around each other, each particle acquires a Bohm-Ahoronov phase due to the other particles, which is precisely the phase that would be acquired due to the fractional statistics. Therefore, bosons (fermions) carrying flux α $(1-\alpha)$ correspond to α statistics. Arovas *et al.*[6] calculated the second virial coefficient of a free-anyon gas and found that when anyons are treated as fermions with a gauge interaction, the second virial coefficient is reduced from that of free fermions. This suggests that there are some similarities between free anyons and fermions with attractive interactions. This similarity can also be seen by considering the anyon propagator. When an anyon moves around other anyons its propagator will acquire a path-dependent phase which leads to interference between different paths. However, the energy will be lower if anyons correlate their motion to form bosonic quasiparticles and hence avoid the destructive interference. In particular, a pair of semions form a boson and hence it is possible that semions may pair to form a charge $2e$ superfluid due to their statistical correlations as proposed by Laughlin[4] on the basis of a Hartree-Fock calculation for semions. Furthermore, Fetter, Laughlin, and Hanna[7] calculated the collective excitations of a semion gas in the random phase approximation and found a linear mode corresponding to a compressible sound mode suggesting that the system behaves like a superfluid. Canright, Girvin, and Brass[8] also found numerical evidence for semion pairing from exact diagonalization studies of semions on a square lattice with cylindrical symmetry (i.e., periodic boundary conditions were applied in one direction).

In this paper, we report numerical results which we have obtained for semions on lattices superimposed on a sphere. The spherical topology was chosen in order to eliminate edge effects associated with an open topology. Edge effects, or effects of boundary conditions in general, may be large for a finite system of anyons due to the long-range gauge interactions. Also, since the infinite systems are expected to be compressible the sensitivity to boundary conditions will be much larger, for example, than that found in numerical studies of the fractional quantum Hall effect.[9] Therefore, it is desirable to compare results with different boundary conditions imposed and, where possible, our results are compared to those of Canright, Girvin, and Brass.[8] A torus, or periodic boundary conditions in both directions, is the usual choice for eliminating edge effects in numerical studies. However, there are some subtleties in applying two-dimensional periodic boundary conditions to a system of particles obeying fractional statistics.[10] On the other hand, the sphere is particularly simple in this regard.[11]

The remainder of this paper is organized as follows: The numerical technique for finding the low-lying energy states of anyons on a lattice with a spherical topology is described in Sec. II. In Sec. III, we identify and check various signatures for superfluidity such as the pairing energy, flux quantization, and critical field by studying systems of hard-core bosons and of fermions with attractive interactions. The case of semions is discussed in Sec. IV. Section V contains a discussion of all of the numerical results.

II. NUMERICAL TECHNIQUE

Following Wilczek,[5] we treat anyons with α statistics as hard-core bosons with flux tubes of strength $\alpha\phi_0$, where $\phi_0 = hc/e$ is the flux quantum. Thus the Hamiltonian is

$$\mathcal{H} = \frac{1}{2m} \sum_i \left[\mathbf{p}_i + \frac{e}{c} \sum_{j<i} \mathbf{A}_{ij} \right]^2 + \sum_{j<i} V(\mathbf{r}_i - \mathbf{r}_j) , \tag{2.1}$$

where V is the interparticle interaction which is zero for free anyons and the gauge interaction is

$$\mathbf{A}_{ij} = \frac{\alpha\phi_0(\mathbf{r}_i - \mathbf{r}_j)\times\hat{\mathbf{z}}}{|\mathbf{r}_i - \mathbf{r}_j|^2} = \alpha\phi_0 \nabla\theta_{ij} . \tag{2.2}$$

Here θ_{ij} is the angle between particles labeled by i and j with respect to some fixed coordinate system. Since $\mathbf{A}_{ij}$ is a total derivative, the gauge interaction exerts no forces on anyons but provides the phase correlations between them as dictated by their statistics. Thus, we can always choose a gauge in which $\mathbf{A}_{ij}$ is zero everywhere except along the branch cut of θ_{ij}. In such a gauge, each anyon can be thought of as having a string attached to it. This "string gauge" is a convenient gauge for numerical calculations and is the gauge chosen for this work. It is clear from the Hamiltonian that even the problem of free anyons is intrinsically a many-body problem. This makes it difficult to obtain analytic solutions and, in fact, exact solutions exist only for the case of two anyons and for a few special cases of three anyons.[6,12]

We study the problem of anyons on a finite lattice since this results in a finite Hilbert space which can be studied numerically. At low densities the results should be similar to those of the continuum problem but near half filling there typically will be noticeable lattice effects.[13] For anyons confined to lattice sites the Hamiltonian becomes

$$\mathcal{H} = -t \sum_{<lm>} e^{i\phi_m} a_l^\dagger a_m + V \sum_{<lm>} n_l n_m , \tag{2.3}$$

where $a_l^\dagger$ creates a boson plus string on lattice site l if the site is initially empty, $n_l = a_l^\dagger a_l$, and we have specialized to the case of nearest-neighbor interactions of strength V. The statistical phase associated with hopping from site l to m is

$$\phi_{lm} = \alpha\phi_0 \int_l^m \mathbf{A}\cdot d\mathbf{l} , \tag{2.4}$$

where $\mathbf{A}$ is the statistical vector potential due to all the other particles. Without loss of generality we take $t=1$. As explained previously we study lattices with spherical topology. There are five regular polyhedra whose vertices lie on the surface of a sphere. The two largest are of interest: the icosahedron with 12 vertices which are taken to be the lattice sites and the dodecahedron with 20 vertices or sites. Thus, we can study up to 11 anyons on the icosahedron and up to 19 anyons on the dodecahedron. The largest Hilbert space is for ten anyons on 20 sites which has 184 756 states. Using the fivefold rotational symmetry we are able to diagonalize $\mathcal{H}$ even for this largest manifold of states. The diagonalization is performed using a modified Lanczos method on the Cray-MP/24 at the Ontario Centre for Large Scale Computation in Toronto.

The advantages to working on a regular polyhedron are that each site is equivalent and fractional statistics are particularly simple to formulate in this topology. However, on the sphere there is a constraint on the allowed values of the statistical parameter α

$$\alpha(N-1) = I , \tag{2.5}$$

where N is the total number of anyons and I is an integer. This constraint is easily seen to arise from considering a path where one particle encircles the other $N-1$ particles. On the surface of a sphere, this path can also be considered to enclose no particles and hence must give rise to a phase which is simply a multiple of 2π, as happens if Eq. (2.5) is satisfied. This constraint can also be thought of as arising from the monopole constraint for the total flux through a closed surface, keeping in mind that each particle sees only the flux attached to the other particles and does not see its own flux.

In the thermodynamic limit $N \to \infty$ this constraint is insignificant. However, it does restrict the statistics one may consider for a finite system. In particular, for the case of semions, only odd numbers of semions are allowed. In order to study the possible pairing of semions it is also necessary to study even numbers of semions. This can still be done in the spherical geometry in two different ways. One way is to introduce a small uniform magnetic field. In the presence of a magnetic field the constraint becomes

$$\alpha(N-1) + \phi/\phi_0 = I , \tag{2.6}$$

where ϕ is the total flux due to the external field. Thus, one may study an even number of semions in a field of strength $\phi_0/2$. One may think of this as the particle interacting with its own flux tube in a mean-field sense. For a large system this field is very small since ϕ is the total flux over the entire surface. Alternatively, one can study an odd number of semions but with the position of one semion fixed at the north pole, for example, so that it has no dynamics. This is equivalent to an even number of semions in the presence of a defect which restores the monopole constraint. We will use both of these procedures in order to interpret the results for even numbers

bers of semions.

For anyons confined to the surface of a sphere the vector potential in the string guage may be written as

$$\mathbf{A}_{ij}=\alpha\phi_0\delta(\phi_i-\phi_j)[\Theta(\theta_i-\theta_j)-1/2] + \sum_{m=1}^{M}\frac{\alpha\phi_0}{M}\delta(\phi_i-2\pi m/M) , \quad (2.7)$$

where Θ is the heaviside theta function. The first term corresponds to a string emanating from each particle and terminating at the south pole. Every time a particle crosses another particle's string the wave function, and hence the hopping matrix element, picks up a phase of $\pm\alpha\pi$. The second term corresponds to M strings each of weight $(N_p-1)\alpha\phi_0/2M$ emanating from the north pole and terminating at the south pole and which must be present to ensure that all closed loops give the correct statistical phase. M is an integer chosen to preserve the azimuthal symmetry which we use to block diagonalize the Hamiltonian matrix.

III. SIGNATURES OF SUPERFLUIDITY

The main purpose of this study is to investigate the correlations due to statistics in systems of semions and, in particular, to investigate the possible pairing of semions. The questions we focus on are (i) is the statistical interaction alone enough to pair semions and (ii) if so, do these pairs form a coherent state and hence a superfluid? We discuss the following four signatures of a superfluid state consisting of pairs of particles in a finite system.

(1) *Pairing Energy*: As a first check on whether there are correlations which tend to pair the particles, one can calculate the pairing energy defined as

$$\Delta_2(N)=E(N+2)+E(N)-2E(N+1) , \quad (3.1)$$

where $E(N)$ is the total ground-state energy of a system of N particles. If the particles are paired one finds that $\Delta_2(N)<0$ for even N and $\Delta_2(N)>0$ for odd N. In the thermodynamic limit, $\Delta(N_{\rm odd})=-\Delta(N_{\rm even})$.

While a negative $\Delta_2(N)$ is a necessary condition for particles to form pairs, it is not sufficient for those pairs may cluster further into still larger clusters. Therefore, one also needs to demonstrate that the effective interactions between pairs are repulsive. This can be done by calculating the quadrupling energy defined as

$$\Delta_4(N)=E(N+4)+E(N)-2E(N+2) = \Delta_2(N+2)+\Delta_2(N)+2\Delta_2(N+1) . \quad (3.2)$$

For repulsive effective interaction between pairs $\Delta_4(N)$ would be positive for a paired state and no clustering should occur.

(2) *Periodicity of* $E(\phi)$: Assuming that the dimension of a plaquette is larger than the coherence length, one can pass a thin solenoid through the north and south poles of the sphere and vary the flux through the solenoid continuously. Then, in the thermodynamic limit all physical quantities including the energy $E(\phi)$ must be periodic in the flux ϕ with period ϕ_0. However, if the particles are paired as are the electrons in a BCS superconductor, then the energy will be periodic with period $\phi_0/2$—i.e., the flux quantum for a paired state is $\phi_0/2$. The periodicity in $E(\phi)$ is a measure of the charge of the carriers. In a finite system the energy will not be precisely periodic. However, if pairing exists the energy will exhibit minima at integer multiples of $\phi_0/2$ and the difference in energy between the minima at odd-integer multiples and the minima at even-integer multiples will vanish in the thermodynamic limit.

(3) *Flux Quantization*: If a coherent, paired state exists with off-diagonal long-range order (ODLRO), then the flux through the solenoid is quantized in integer multiples of $\phi_0/2$. Thus, in the thermodynamic limit the energy barrier between two adjacent minima in $E(\phi)$ is infinite. In a finite system this energy barrier $[E_{\rm peak}=E_{\rm max}(\phi)-E_{\rm min}(\phi)]$ scales as the number of particles N if there is ODLRO.

(4) *Critical Field*: If there is ODLRO flux quantization will persist in the presence of a small, uniform magnetic field but will be destroyed by a sufficiently strong field. For the case of semions, even in the absence of an external magnetic field the system does not have time-reversal symmetry. Therefore, the critical field which destroys the ODLRO will, in general, be different for the two different orientations of the field.

The first two signatures are indications of pairing while the last two are indications of coherence or ODLRO. In order to distinguish between those signatures resulting from finite-size effects and those that will survive in the thermodynamic limit, we simultaneously search for all four signatures described above and check for consistency. These signatures have the advantage that they only depend on energies and not on wave functions. The singular gauge transformation that allows one to treat anyons as bosons plus flux tubes does not leave the wave functions invariant and, in particular, does not leave the one- or two-body density matrices invariant. This makes it difficult to study ODLRO directly in the density matrices.

In order to check that the systems we can study numerically are large enough to give rise to clear signatures of ODLRO, we first study the case of fermions with attractive interactions. We use the Hamiltonian given by Eqs. (2.3) and (2.7) with $\alpha=1$ and with the interaction V negative. Since we are considering spinless fermions for sufficiently strong attractive interactions the fermions are expected to pair and form a triplet superconductor.

Figure 1 shows the results for six fermions on an icosahedron (12 sites). With no nearest-neighbor interaction V there is no indication of a second minimum in $E(\phi)$ at $\phi_0/2$ [Fig. 1(a)]. With a sufficiently strong attractive interaction $V\geq V_0$, a second minimum develops at $\phi_0/2$ [Fig. 1(b)]. These two minima persist in a small magnetic field [Fig. 1(c)] and the second minimum is destroyed by a field of $B\geq 5\phi_0$ [Fig. 1(d)] as expected. Figure 2 presents similar results for eight fermions. In this case, the pairing is very stable in the sense that it persists up to the largest magnetic field one can apply to the system which is 10 ϕ_0 [see Fig. 2(c)]. On a lattice the ground-state energy is a periodic function of external

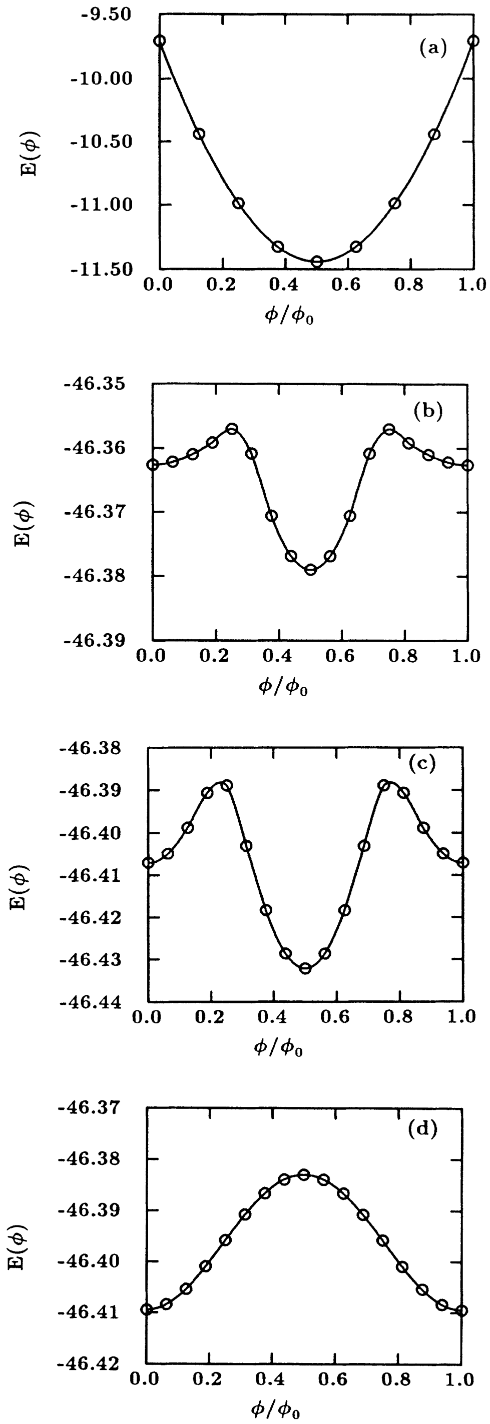

FIG. 1. Ground-state energy as a function of flux for six fermions on an icosahedral lattice: (a) $V=0$, $B=0$; (b) $V=-15$, $B=0$; (c) $V=-15$, $B=\phi_0$; (d) $V=-15$, $B=5\phi_0$. The curves shown in (b) and (c) display the two minima at $\phi=0$ and $\phi_0/2$ expected for a charge $2e$ superconductor.

magnetic field with a period of one flux quantum per plaquette. Furthermore, the time-reversal symmetry which fermions obey implies that $E(B)=E(-B)$ and hence, the maximum field is a half flux quantum per plaquette. It should be mentioned that in some cases pairing of fer-

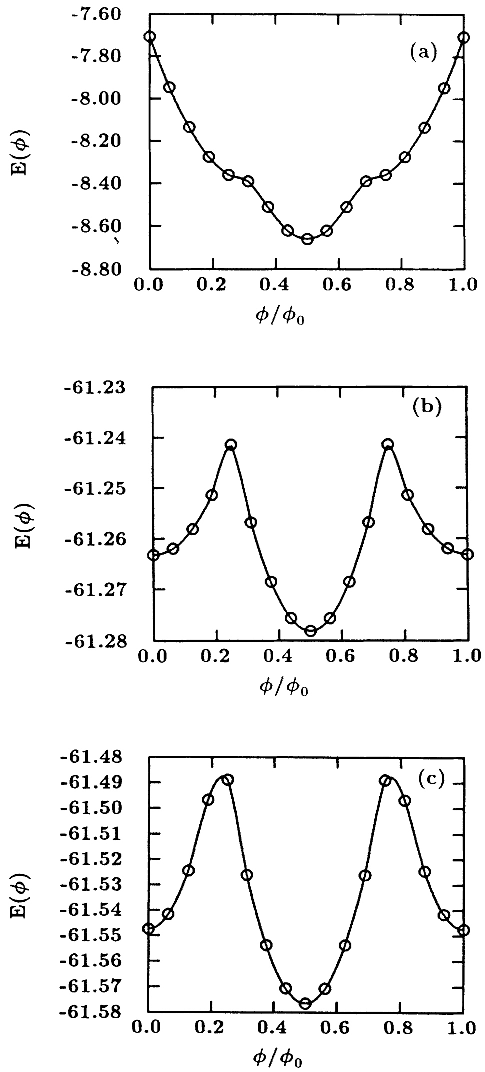

FIG. 2. Ground-state energy of eight fermions on an icosahedron: (a) $V=0$, $B=0$; (b) $V=-15$, $B=0$; (c) $V=-15$, $B=10\phi_0$.

mions was not observed. The reason is that one typically needs large attractive interactions to pair spinless fermions. If the interaction V is increased sufficiently, the fermions will form a cluster rather than a coherent paired state and other minima in the energy develop.

We have also studied hard-core bosons since this is one of the simplest systems that displays ODLRO. Figure 3 shows the total ground-state energies as functions of flux ϕ for different numbers of particles on a dodecahedral lattice. These are charge e superfluids and hence $E(\phi)$ has periodicity ϕ_0 as expected. The most noticeable feature in Fig. 3 is that the energy barrier E_{peak} scales as the number of particles N. This energy barrier is plotted in Fig. 4 as a function of N for bosons on both dodecahedral and icosahedral lattices. For the lattices considered, there is particle-hole symmetry for boson systems. Therefore, for the dodecahedral lattice, $E_{\text{peak}}(N)$ is only plotted up to half filling. Figure 4 clearly shows scaling

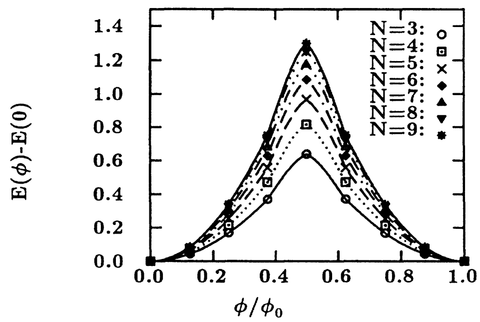

FIG. 3. Ground-state energy as a function of flux for bosons on a dodecahedron. The curves have been shifted so as to coincide at zero flux.

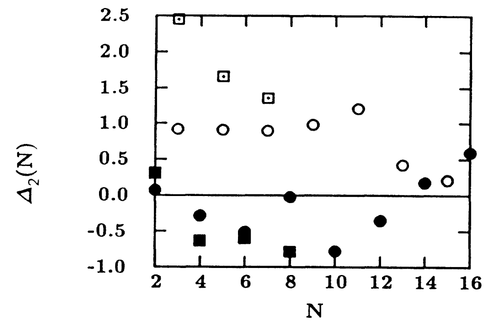

FIG. 5. Pairing energy as a function of particle number for semions on both lattices. The circles are for the dodecahedron and the squares are for the icosahedron. The open circles and squares are for odd N and solid ones are for even N. Note that this energy is negative for even N and positive for odd N if the particles are paired.

behavior except at half filling. This scaling behavior is a clear signature of ODLRO. The deviation from scaling at half filling is attributed to lattice effects.[13]

The above results for fermions and bosons show that the lattices considered are large enough to display clear signatures of ODLRO.

IV. PAIRING OF SEMIONS

Next we consider semion systems. We have studied many semion systems on icosahedral and dodecahedral lattices. Here, we present the numerical results for free semion systems.

We first consider the semion pairing energy $\Delta_2(N)$ as defined in Eq. (3.1). In Fig. 5, the pairing energy is plotted as a function of particle number N, where the circles are data on a dodecahedron and the squares are those on an icosahedron. The open cirlces and squares are those for even N while the solid ones are those for odd N. It can be clearly seen from Fig. 5 that there is a distinct correlation in the pairing energy. Namely, $\Delta_2(N)$ is negative for all even N, except for those systems with a very small Hilbert space, and is positive for all odd N, as expected for paired states. This correlation in the pairing energy is strong evidence that there are effective attractive interactions between free semions. In order to make sure that those attractive interactions will not lead to clustering of semions, we also calculate the quadrupling energy $\Delta_4(N)$ as defined in Eq. (3.3). For the cases of pairing states (even-number semion systems), we find that all the quadrupling energies are positive indicating that the effective interactions between semion pairs are repulsive. Therefore, we can conclude that the fractional statistics provides a pairing mechanism for the free-semion systems and it is conceivable that this pairing mechanism will lead to a condensate at low temperatures.

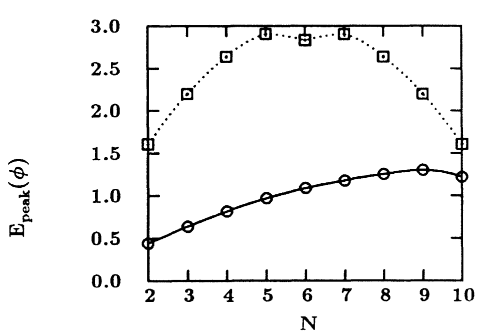

FIG. 4. Energy barrier in $E(\phi)$ as a function of particle number for bosons. The circles are those on a dodecahedral lattice and squares are those on an icosahedral lattice. Due to particle-hole symmetry, only points up to half filling are shown here for the dodecahedral lattice. This barrier is proportional to N in a system with ODLRO. Lattice effects are responsible for the deviation from linear behavior near half filling.

We have also studied flux quantization for semion systems. In Figs. 6–8, we present the results for the two largest systems, excluding the cases of half filling, on the dodecahedron (20 sites) and the icosahedron (12 sites). Figure 6 shows the energy as a function of flux (in units of ϕ_0) for 12 free semions on a dodecahedral lattice. Figure 6(a) clearly shows a second minimum in the energy at $\phi_0/2$, a signature of pairing. Figure 6(b) shows the same system in the presence of a uniform external magnetic field with total flux equal to one flux quantum. In this case, the two energy minima persist. However, when the magnetic field is increased to 3 flux quanta only one minimum remains [see Fig. 6(c)]. The critical field depends on the orientation of the field. If the field orientation is reversed, the critical field changes [see Fig. 6(d)]. This signature of pairing is stable with respect to small perturbations in the system. Figure 7 shows that the pairing features persist in the presence of small repulsive or attractive interactions between semions. Figure 8 displays the results for four free semions on an icosahedral lattice which also display signatures of ODLRO.

Curves similar to those shown in Figs. 6–8 were found for all cases of even numbers of semions with $N \geq 4$ except for the two cases of half filling ($N=6$ on the icosahedron and $N=10$ on the dodecahedron) and for $N=8$ and $N=14$ on the dodecahedron. At half filling,

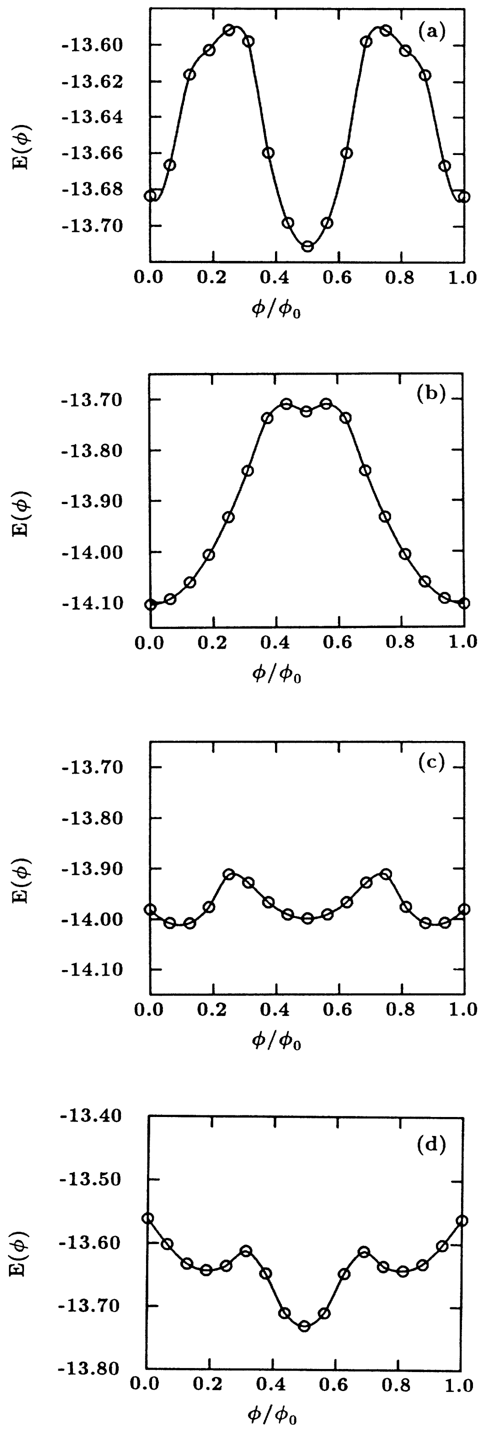

FIG. 6. Ground-state energy as a function of flux for 12 semions on a dodecahedral lattice: (a) $V=0$, $B=0$; (b) $V=0$, $B=\phi_0$; (c) $V=0$, $B=3\phi_0$; (d) $V=0$, $B=-\phi_0$.

additional minima were observed. These results are consistent with those found by Canright *et al.*[8] and with analytic results of Fradkin[13] which suggest a competing quantum Hall state at half filling. We note that this absence of pairing at half filling accounts for the small pairing energy for $N=8$ on the dodecahedron (see Fig. 5).

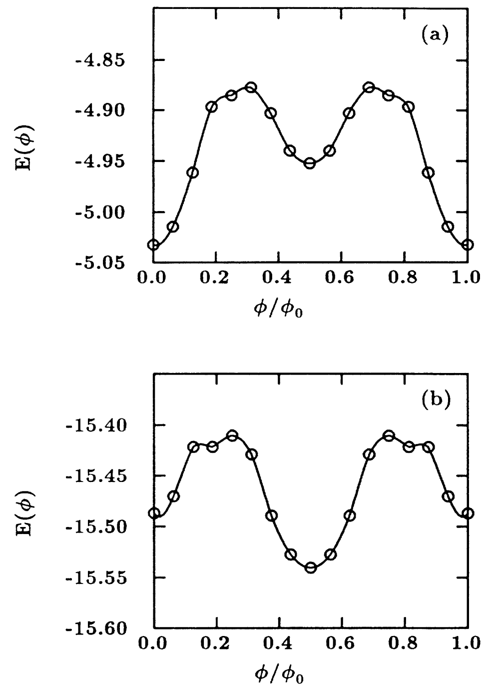

FIG. 7. Ground-state energy as a function of flux for 12 semions on a dodecahedral lattice: (a) $V=1$, $B=0$; (b) $V=-0.2$, $B=0$.

The absence of a clear signature of pairing for the cases of $N=8,14$ suggests significant finite-size effects. We note that semion pairing was also not observed in Ref. 8 for the case of 14 semions on 20 sites, suggesting that lattice commensuration effects may also be playing a role.

For odd numbers of semions no pairing was observed. Figure 9, for example, shows the energy flux curve for five semions on the icosahedron and 11 semions on the dodecahedron. This is a finite-size effect which one would expect to disappear in sufficiently large systems. However, it clearly points out the need to look for multiple signatures of pairing and ideally to study the signatures as a function of system size.

In all of the curves presented, the boundary conditions for even numbers of semions were treated by localizing an extra "semion" at the north pole. The observed second minimum is deeper in this case than it is when a small external field is added. One reason for this is that for a finite system a field with total flux $\phi_0/2$ is not a very small field, and hence it may weaken the coherence in the system.

An important symmetry emerges from our calculations. Namely, the semion ground state energy as function of flux ϕ always satisfies

$$E(\phi)=E(-\phi) . \qquad (4.1)$$

In particular, $\phi=0$ is always a energy minimum for all the paired semion systems. This symmetry would be required for any systems with time-reversal symmetry (T)—such as boson and fermion systems. However, T is

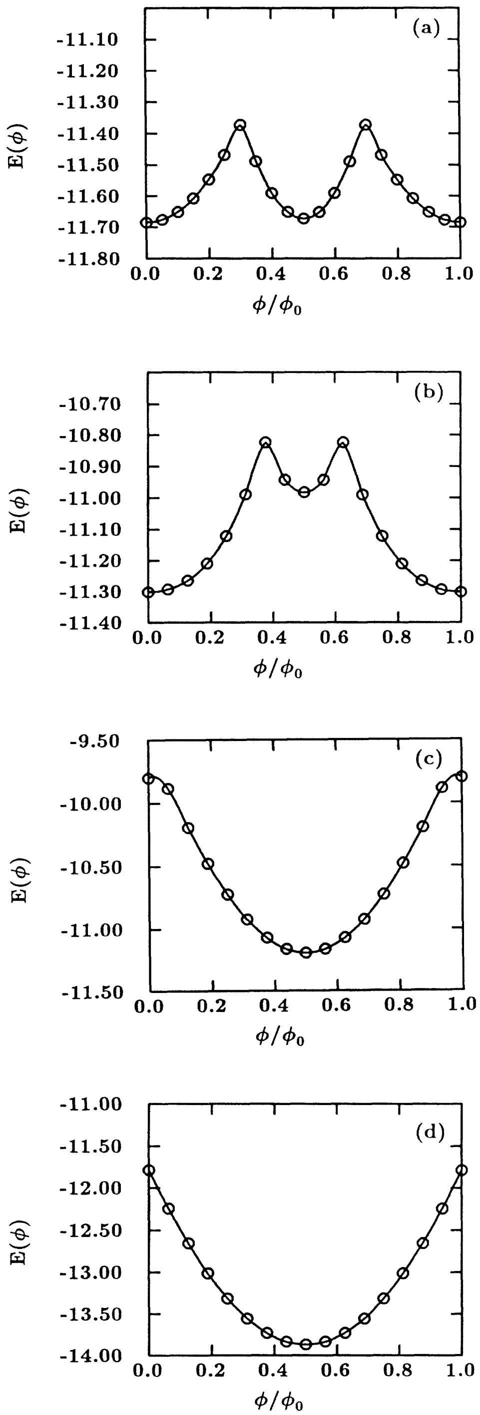

FIG. 8. Ground-state energy as a function of flux for 4 semions on an icosahedral lattice: (a) $V=0$, $B=0$; (b) $V=0$, $B=\phi_0$; (c) $V=0$, $B=2\phi_0$; (d) $V=0$, $B=-\phi_0$.

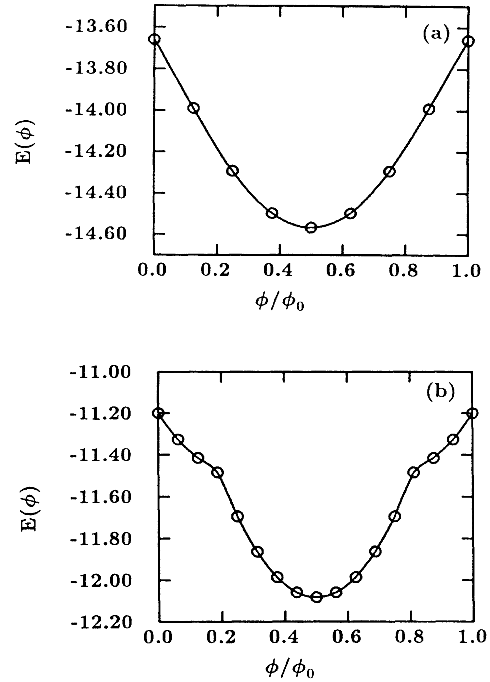

FIG. 9. Ground-state energy as a function of flux for (a) 11 semions on a dodecahedral lattice; (b) 5 semions on an icosahedral lattice.

broken in anyon systems and it is not obvious that Eq. (4.1) should be satisfied.[14] From their numerical results Canright, Girvin, and Brass[8] concluded that the minima in $E(\phi)$ are located at odd-integer multiples of $\frac{1}{4}\phi_0$, and hence $E(\phi)\neq E(-\phi)$. They attributed this to broken time-reversal symmetry. Since the positions of the energy minima are experimentally measurable, a shift away from $\phi=0$ could be one signature of anyon superconductivity. If the energy minimum is at $\phi\neq 0$, then in the flux quantization experiment one would observe a sudden appearance of a nonzero net flux when the system is cooled below T_c under zero external field. Such a phenomenon, if it occurred, would be quite strange indeed. Our results show that $\phi=0$ is a energy minimum for the spherical geometry and that the above mentioned anomaly does not appear.

The position of the minimum of the $E(\phi)$ curve is dependent on the boundary conditions applied. For anyon systems with cylindrical topology the phases associated with all closed loops, except for two on the open edges, are well defined. Let the phases associated with the two-edge loops be ϕ_1 and ϕ_2 respectively. Then the fractional statistics require that

$$\phi_1-\phi_2=2\pi\alpha(N-1) \ . \qquad (4.2)$$

Therefore, one of the phases, say ϕ_1, is still undefined. The choice of this phase will directly affect the position of the minimum of the $E(\phi)$ curve.

A natural way to choose the boundary conditions (in this case ϕ_1) is that they should respect the symmetries of the inifinte system. The Hamiltonian (2.1) possesses *TP* symmetry but *T* and *P* are broken individually. There is also two-dimensional spatial inversion symmetry, i.e., $x\rightarrow -x$ and $y\rightarrow -y$. For the case of a cylinder this inversion symmetry requires that

$$\phi_1 = -\phi_2 = \pi\alpha(N-1) , \tag{4.3}$$

when the external flux is zero. Eq. (4.3) tells us that for semions ($\alpha=\frac{1}{2}$), one can choose $\phi_1=0$ mod(π) only when the number of semions is odd. For an even number of semions, which is the case of interest, one must choose $\phi_1=(\pi/2)$ mod(π). This is exactly the shift which Canright *et al.* have observed in Ref. (8), since they set $\phi_1=0$ for even N. Therefore, for anyon systems with cylindrical topology the ground-state energy $E(\phi)$ is always an even function of ϕ if inversion symmetry is imposed.

On a spherical lattice with zero test flux there are no edges and the phases associated with any closed loops are well defined. Thus, there are no ambiguities in the origin of $E(\phi)$, and we have found that $E(\phi)$ is always a even function of ϕ. Although the boundary problem does not occur in the spherical topology, there is a closely related phenomena—namely, the constraint on the allowed statistics.

For the disk geometry discussed in Ref. (8), there is no obvious symmetry that would require that the boundary condition satisfies Eq. (4.3). However, we believe that the shift in the minimum of $E(\phi)$ would not survive in the thermodynamic limit. First, the shift is nonzero only for an even number of semions. Second, one can eliminate the shift by adding an external magnetic field with total flux $\frac{1}{2}\phi_0$. In the thermodynamic limit such a field is negligible.

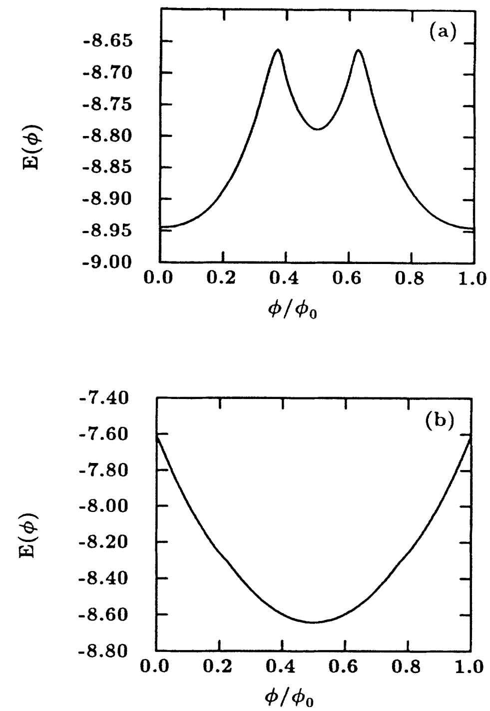

FIG. 10. Level crossings mimicking pairing features for two filled Landau levels. The case of eight fermions on an icosahedral lattice is shown: (a) $V=0$, $B=2\phi_0$; (b) $V=0$, $B=\phi_0$. Note the sensitivity to magnetic field in this case.

V. CONCLUSIONS AND DISCUSSIONS

We have studied semions on finite lattices with spherical topology by numerically diagonalizing the Hamiltonian matrix focusing on the possible superfluid correlations in such systems. The cases of fermions with attractive interactions and hard-core bosons were studied for the purposes of testing our numerical procedures and confirming that the lattices under study are large enough to display various signatures of a superfluid state. (Note that comparing the results for fermions treated as bosons plus flux tubes with the sum of single-particle energy levels provides a nontrivial check of the numerical calculations.) In order to distinguish genuine features of superfluidity from artifacts due to finite system effects, we believe it is necessary to study several different signatures simultaneously in different size systems. Therefore, we have studied pairing energies, flux quantization, and critical fields in semion systems and have checked the consistency among these signatures. From our numerical results we draw two conclusions:

(1) There is a clear correlation in the pairing energy, i.e., there are effective pairing interactions between semions and the effective interactions between pairs are repulsive. Furthermore, flux-quantization calculations also reveal that the charge carriers in the systems have charge $2e$, further evidence that semions pair due to their statistical correlations.

(2) The pairing features persist in small external magnetic fields but disappear when the external field exceeds a critical value. This indicates that the semion pairs form

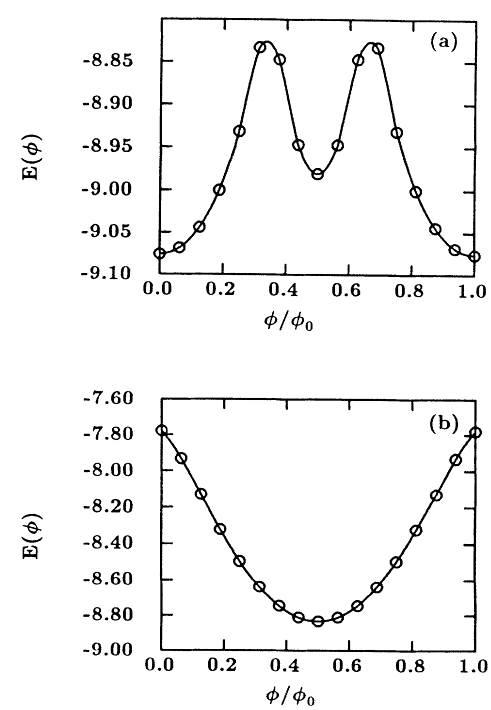

FIG. 11. Ground-state energy as function of flux for eight anyons with $\alpha=\frac{5}{7}$. In the mean-field approximation this corresponds to the case of two filled Landau levels: (a) $V=0$, $B=0$; (b) $V=0$, $B=\phi_0$.

a coherent state, and hence a superfluid.

The coexistence of several signatures of a coherent pairing state which we have discovered in our numerical investigations strongly supports the hypothesis that semions pair and form a superfluid due to the effective statistical interactions between them. Thus, it seems likely that in the thermodynamic limit charged semion systems will form a superconducting state at low temperatures. Such a superconductor is expected to exhibit various exotic properties. For example, because the time-reversal symmetry is broken in a system with fractional statistics the magnetization curve for a semion superconductor will generally depend on the orientation of the external field. On the other hand, we have also argued that in flux quantization measurements the ground-state energy of a semion gas $E(\phi)$ is an even function of ϕ, as is the case for an ordinary BCS superconductor.

Due to the limitation of computer resources, the largest system we could study on a Cray-MP/24 was ten particles on a 20-site lattice. Finite-size effects are still relatively large for such a system as evidenced by the fact that signatures of pairing are not seen in any systems with odd numbers of particles. Therefore, it is necessary to study several signatures simultaneously. For example, the discrete energy levels in a finite system undergo level crossings which accidently mimic pairing in the flux quantization. We have found that the possibility of such undesirable resemblances is larger than one might expect. In fact, we have found that for free fermions in a uniform magnetic field, level crossings always lead to accidental pairing features when exactly two Landau levels are filled. An example of this is shown in Fig. 10. Figure 10(a) shows eight free fermions in a uniform magnetic field with total flux equal to 2 flux quanta, which corresponds to two filled Landau levels. It shows an accidental pairing feature. However, this feature does not survive if the magnetic field is changed slightly as shown in Fig. 10(b). We have also calculated the Hofstadter spectrum for both icosahedral and dodecahedral lattices, and have found that for all cases of two-filled Landau level, the energy flux curves always show accidental pairing features due to level crossing. This illustrates the importance of consistency checks among different signatures in order to identify the genuine features of a superfluid state.

Other systems may also mimic the flux quantization curves identified with a paired state. Figure 11 shows the case of eight anyons with $\alpha=\frac{5}{7}$. Figure 10 is the mean-field counterpart of this case. If the anyons are treated in the mean-field approximation by replacing the flux tubes with a uniform magnetic field, the ground state corresponds to two filled Landau levels. Figure 11 is very similar to Fig. 10, indicating that mean-field theory may be a good approximation for anyon systems provided an integer number of Landau levels are filled.

Although we have observed consistent signatures of a superfluid state in semion systems, we do not have enough data points to test the scaling behavior in flux quantization in either the semion or the fermion systems. Although scaling behavior was seen clearly in the case of bosons, it appears that one would need to study even larger systems in order to study scaling behavior in systems involving pairs of particles.

ACKNOWLEDGMENTS

The authors would like to thank A. J. Berlinsky, G. Canright, S. M. Girvin, A.-M. S. Tremblay, and X. G. Wen for useful conversations. This research was supported by the Natural Sciences and Engineering Research Council of Canada and by the High-Temperature Superconductivity Consortium of the Ontario Center for Materials Research. In addition, one of the authors (C. K.) acknowledges support from the Alfred P. Sloan Foundation.

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