Superfluid dynamics of the fractional quantum Hall state

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(Received 25 January 1990)

The fractional quantum Hall effect is due to a novel state of matter with properties reminiscent of superfluidity. I derive the Euler equation and the quantum constraint that together determine the dynamics of the superfluid. I then use these equations to explain the incompressibility of the liquid, to describe the magnetophonon excitations, and to study the forces acting on vortices whose pinning is responsible for the plateaus in the Hall resistance.

I. INTRODUCTION

A neutral substance like ^{4}He owes the existence of its superfluid phase to subtle quantum-mechanical effects. Its bulk properties, however, are still those of a classical fluid: it may be poured, shaken, and stirred like any other liquid and its magical superflow properties are simply a consequence of the fact that the superflow is irrotational, i.e., to d'Alembert's paradox. Below the critical velocity for creating phonons dissipation is due to energy absorbed in the manufacture of quantized vortex lines which behave almost classically.

Superconductors behave similarly[1] except the phonons, being turned into plasmons by the electric charge of the condensate, are even harder to create and all resistive dissipation comes from the depinning and motion of Abrikosov vortices.[2] In the Bardeen-Stephens theory,[3] the magnetic field of a moving vortex induces an electric field which drives a normal, dissipative, current through the vortex core. The forces causing the vortex to move are best understood via a fluid-dynamics analysis:[4] The superconducting condensate behaves as charged superfluid and the Abrikosov vortex has an inner core, of radius approximately equal to the coherence length, about which there is quantized circulation. This core vortex is surrounded by a screening antivortex (whose radius is approximately equal to the the magnetic penetration depth) with vorticity proportional to the magnetic field and total circulation equal and opposite to that of the core; there is thus no circulation at large distances. The force felt by a stationary core vortex embedded in a bulk flow is a *Magnus* force due to the interaction of the circulation and the uniform flow. The Magnus pressure on the core is in turn balanced by the Lorentz force on the current through the surrounding antivortex.[4]

The novel state of matter found in a two-dimensional electron gas (2D EG) at low-temperature and high magnetic fields[5] has many properties in common with superfluid phases,[6,7] including long-range order, no dissipation, and vortex soliton excitations.[8] The vortices contain the deviations of the charge density from uniform, rational fraction, filling of the Landau levels and, presumably,[9–11] it is the pinning of the vortices by impurities that yields plateaus in the Hall conductance. Since this "two-fluid picture" of the fractional quantum Hall effect (FQHE) depends crucially on vortex pinning it would be useful to have the same kind of intuitive picture of the forces on a vortex that we have for the other superfluids. It is the intention of this paper to provide that picture. In addition to plateau formation FQHE vortices have another essentially quantum role: when they are free to move they behave as solitonic quasiparticles with fractional charge and, it is believed, fractional statistics. They may in their turn undergo Bose condensation and give rise to the hierarchy of FQHE states.[12–14]

The flow properties of superfluids are determined by classical Euler equations—with an additional constraint on the vorticity which embodies the quantum mechanics. These Euler equations, being an expression of the laws of conservation of mass and momentum, have greater validity than any particular model used to derive them and can be applied over a wide range of conditions. In the FQHE case we could start from either the microscopic wave function of Laughlin[10] or the more phenomenological Landau-Ginsburg approaches based on Chern-Simons (CS) Lagrangians[6–8] and should arrive at the same equations. In this paper I will take the second route and motivate the Euler equations for the FQHE state from the mean-field Landau-Ginsburg model which I will review in Sec. II. In Sec. III I obtain the Euler equations by the methods used for conventional superfluids and, in Sec. IV, I examine the consequences of the fluid dynamics equations for the properties of the FQHE and more general anyons in a magnetic field.

II. MEAN-FIELD THEORY OF THE FRACTIONAL QUANTUM HALL EFFECT

The need for a Landau-Ginsburg picture of the FQHE was stressed by Girvin in the summary of Ref. 5 and he there indicated the necessary ingredients including a Chern-Simons term in the action. A more complete model, based on a mean-field treatment of the Chern-Simons Lagrangian, was introduced by Zhang, Hansen and Kivelson[6] and by Read.[7] In the approach of Zhang *et al.*[6] one begins with a path integral

$$Z=\int d[\phi]d[\phi^*]d[a_\mu]\exp i \times \int d^3x \left[\phi^* H\phi+\frac{i}{4\Theta_0}\epsilon^{\mu\nu\sigma}a_\mu\partial_\nu a_\sigma\right] \quad (2.1)$$

involving a *commuting* ϕ field representing the electrons, and a "statistics" field a_μ. H is an action for a nonrelativistic particle with effective mass m^*, e.g.,

$$H=\frac{1}{2m^*}(\partial_i-ia_i-iA_i)^2-i(\partial_t-ia_0-iA_0) +\lambda\left[|\phi|^2-\frac{\mu_0}{2\lambda}\right]^2 \quad (2.2)$$

and Θ_0 is the statistics parameter[15–17] which, taking one of the values $(2n+1)\pi$, ensures that the bosonic ϕ field describes particles with Fermi statistics.

In the mean-field approximation to the path integral the CS statistics field is determined by the electron density ρ through

$$2\Theta_0\rho=2\Theta_0|\phi|^2=\nabla\times\mathbf{a}\ . \quad (2.3)$$

If the density is uniform the *curl* of the CS mean field will be constant. When the external magnetic field is also uniform the density may be such that the gauge fields in (2.2) *cancel.* This requires

$$2\Theta_0\rho=2\pi(2n+1)\rho=|B| \quad (2.4)$$

or

$$\rho=\frac{|B|}{2\pi}\frac{1}{2n+1} \quad (2.5)$$

corresponding to a lowest Landau-level filling fraction of $\nu=1/(2n+1)$. In this case the ϕ field has a smooth classical solution, ϕ=const, which will dominate the path integral at low temperature. The Fermi statistics of the electrons has been nullified by the magnetic field, allowing the electrons to Bose condense. The resultant FQHE state is a novel kind of charged superfluid. It is easy to see[6] that the ground state has Hall conductivity

$$\sigma_{xy}=\frac{1}{2n+1}\frac{e^2}{h} \quad (2.6)$$

and has an energy gap resulting in dissipationless flow with $\sigma_{xx}=0$.

The solution appears to depend on the choice of n in the statistics parameter, a choice that should effect no physics, but it is best to regard the picture as being *simplest* for our choice of n rather than depending on it.

This theory is very appealing—but there is some sleight of hand in the derivation implicit in writing down (2.1) as if it were obvious. The "effective mass," m^*, should not be thought of as being the effective mass of the electrons in the 2D EG. As pointed out in Ref. 6, the magnetic field is so large that it has suppressed all the zero-point motion of the electrons and nowhere in the wave functions does any mass m occur. Reference 7 provides a less intuitive derivation of the model but shows clearly that the coefficient of the condensate stiffness, $1/2m^*$, is really a parameter depending on the Coulomb repulsion between the electrons [but see the discussion after Eq. (4.6)].

Including short-wavelength ϕ-field fluctuations in the calculation of the low-energy effective action will induce terms like $(\nabla\times\mathbf{a})^2$ and $\dot{a}^2$. The former has the effect of smearing out the δ function source each particle provides for the statistics field while preserving the essential flux-density relationship. The latter will modify higher frequency motion of the system but should not effect the slower motions that are our primary interests.

In addition to the uniform ground state it is clear by analogy with Abrikosov vortex lines in a conventional superconductor that there will be localized vortex solutions. These will have $\oint(a+A)_\mu dx^\mu=\pm 2\pi$. Since the currents in the thin sample are tiny they will not effect the distribution of the external $|B|$ field and this integral is really $\oint \delta a_\mu dx^\mu$. So, from (2.3), the vortices have a deviation from uniform charge with

$$q=\int \delta\rho d^2x=\pm\frac{1}{2n+1}\ . \quad (2.7)$$

There are two distinct kinds of vortex: the quasihole vortices have $\rho=|\phi|^2$ reduced on average, while the quasiparticle excitations will have it enhanced. In both cases ϕ winds through 2π as we circle the vortex and for finite energy ϕ must vanish at the center: so the hole and particle solutions will have quite distinct charge profiles and cannot be simply related. The Laughlin wave-function picture[4] has excitations with similar properties except there is no vanishing of the charge profile in the center of the quasiparticle.

These excitations cost energy and this energy gap is part of the origin of the stability of the odd-denominator filling-fraction states. Presumably, they are also the origin of plateau formation: any mismatch between the electron density and the magnetic-field strength will lead to the excess or deficit of charge being used to form vortices. If these vortices are pinned by impurities the charge sequestered by them will be immobile and only the "superfluid fraction" will flow and contribute to the observed FQHE. There is therefore a kind of two-fluid model for the FQHE.

Vortices are also anyons with true fractional statistics; the combination of fractional charge and 2π flux gives them a statistics parameter $\Theta_1/\pi=2p_1+\pi/\Theta_0$ where p_1 is an arbitrary integer. These excitations themselves will condense when their density is correctly chosen and there will be vortices in *their* condensate which have statistics parameter $\Theta_2/\pi=2p_2+\pi/\Theta_1$, etc. In this way we see the explanation for the heircharchy of FQHE states with a continued fraction of allowed condensate densities.[12–14]

III. EULER EQUATIONS

In this section the external magnetic field will be uniform unless variations are explicitly introduced. Bold symbols such as $\mathbf{j}$ will denote two component vectors in the plane of the 2D EG. Vectors in the perpendicular direction such as the B field and the vorticity ω will not be bold—their appearance in vector products such as $\mathbf{v}\times\omega$ should not cause confusion. I will use the symbol

$\mathcal{A}_i$ for the combination, $a_i + A_i$, of the statistics and electromagnetic fields. With this notation the equation of motion for ϕ is

$$i(\partial_t - i\mathcal{A}_0)\phi = -\frac{1}{2m^*}(\partial_i - i\mathcal{A}_i)^2\phi + 2\lambda\phi|\phi|^2 - \mu_0\phi \; . \tag{3.1}$$

The $\mathcal{A}_i$ field is determined by the density, current, and external electromagnetic field by the equations

$$\begin{aligned}(\partial_1\mathcal{A}_2 - \partial_2\mathcal{A}_i) &= -2\Theta_0\rho + B \; ,\\ (\partial_2\mathcal{A}_0 - \partial_0\mathcal{A}_2) &= -2\Theta_0 j_1 + E_2 \; ,\\ (\partial_0\mathcal{A}_1 - \partial_1\mathcal{A}_0) &= -2\Theta_0 j_2 - E_1 \; .\end{aligned} \tag{3.2}$$

These are consistent with current conservation provided

$$B + \partial_1 E_2 - \partial_2 E_1 = 0 \; , \tag{3.3}$$

i.e., if the Maxwell equation $B + \nabla\times\mathbf{E} = 0$ is satisfied.

To obtain the Euler equations we write $\phi = \sqrt{\rho}e^{i\theta}$ and, for the moment, assume ρ to be sufficiently slowly varying that we can ignore its derivatives. Equation (3.1) becomes

$$-(\dot\theta - \mathcal{A}_0) = \tfrac{1}{2}m^*(\partial_i\theta - \mathcal{A}_i)^2 - \frac{i}{2m^*}\partial_i(\partial_i\theta - \mathcal{A}_i) + \mu \; . \tag{3.4}$$

The nonlinear terms are bundled into the variable $\mu(x)$. In ordinary hydrodynamics $\mu(x)$ would be the specific enthalpy but here it will be interpreted as a local chemical potential. At $T=0$ the specific enthalpy and the chemical potential coincide, and in the hydrodynamics of superfluids it is the chemical potential that appears in the Bernoulli equation for the superfluid fraction. Defining a flow velocity field by

$$v_i = \frac{1}{m^*}(\partial_i\theta - \mathcal{A}_i) \; , \tag{3.5}$$

we see that the imaginary part of (3.4) asserts that $\nabla\cdot\mathbf{v} = 0$, while taking the gradient of the real part yields

$$\begin{aligned} m^*\partial_t v_1 &= +2\Theta_0\rho v_2 + E_1 - \partial_1\left[\frac{1}{2m^*}v^2 + \mu\right] ,\\ m^*\partial_t v_2 &= -2\Theta_0\rho v_2 + E_2 - \partial_2\left[\frac{1}{2m^*}v^2 + \mu\right] . \end{aligned} \tag{3.6}$$

We can recast these equations in a more familiar form by noting that the vorticity is given by

$$\omega = \partial_1 v_2 - \partial_2 v_1 = -\frac{1}{m^*}(\partial_1\mathcal{A}_2 - \partial_2\mathcal{A}_1) = \frac{2\Theta_0}{m^*}\rho - \frac{1}{m^*}B \; , \tag{3.7}$$

i.e.,

$$m^*\omega + B = 2\Theta_0\rho \; . \tag{3.8}$$

While Eq. (3.8) is an almost trivial extension of the $m^*\omega + e^*B = 0$ relation for a superconductor, it is this vorticity relation which is responsible for the incompressibility of the 2D EG: Any local change in density implies a net vorticity, and in two dimensions an isolated vortex has logarithmically divergent kinetic energy.

By writing $\mathbf{j} = \rho\mathbf{v}$, and replacing $2\Theta_0\rho$ by $m^*\omega + B$ we reexpress (3.7) as

$$m^*[\dot{\mathbf{v}} - (\mathbf{v}\times\omega)] = \mathbf{E} + \mathbf{v}\times B - \nabla(\tfrac{1}{2}m^*v^2 + \mu) \; , \tag{3.9}$$

which is a form of the Euler equation for a fluid of particles of mass m^* and unit charge. By comparison with ordinary fluid dynamics we see that μ is related to the pressure P by $\nabla P = \rho\nabla\mu$. Equations (3.8) and (3.9) are the principle results of this section.

The equation of motion (3.9) is compatable with the quantum vorticity condition (3.8) since taking the *curl* of (3.9) and using $\nabla\times\mathbf{E} + B = 0$, gives

$$\partial_t(m^*\omega + B) + \nabla\cdot\mathbf{v}(m^*\omega + B) = 0 \tag{3.10}$$

This equation shows that even in the absence of quantum mechanics a perfect two-dimensional charged fluid can only gain or lose vorticity via a change in the magnetic field. The anyonic statistics field ties the combination $(m^*\omega + B)$ to the density ρ and then (3.10) becomes the charge conservation equation

$$2\Theta_0[\dot\rho + \nabla\cdot(\rho\mathbf{v})] = 0 \; . \tag{3.11}$$

A consequence of (3.11) is the equations we obtained by assuming ρ essentially constant are actually more general and may be consistently used in situations where the density varies. As is the case of ordinary superfluids the keeping track of density variation terms will add a "quantum pressure" term into $\mu(x)$.[1]

IV. WAVES AND VORTICES

The simplest solutions to Eqs. (3.8) and (3.9) are obviously the steady uniform Hall flows where, e.g., $2\Theta_0\rho v_1 = E_2$. Next easiest to study are the density waves obtained by linearization of the equations of motion. For these we try

$$\begin{aligned} v_1 &= A_1\cos(kx - \Omega t) \; ,\\ v_2 &= A_2\sin(kx - \Omega t) \end{aligned} \tag{4.1}$$

corresponding to motion in ellipses, reminiscent of gravity waves in water. With this *ansatz* we find from, $\dot\rho + \nabla\cdot\rho\mathbf{v}$, that

$$\rho = \rho_0\left[1 + A_1\frac{k}{\Omega}\cos(kx - \Omega t)\right] . \tag{4.2}$$

This is consistent with the constraint (3.8) provided

$$m^*A_2k\cos(kx - \Omega t) + B = 2\Theta_0\rho\left[1 + A_1\frac{k}{\Omega}\cos(kx - \Omega t)\right] . \tag{4.3}$$

Since $B = 2\Theta_0\rho_0$, this is equivalently

$$m^*\frac{\Omega}{B} = \frac{A_1}{A_2} \; . \tag{4.4}$$

If we now substitute (4.1) into either of equations (3.6) and use $\mu=2\lambda\rho$, we find the dispersion relation

$$(m^*)^2\Omega^2=B^2+2\lambda\rho_0 k^2 . \tag{4.5}$$

In this fluid-flow picture of the magnetophonon modes we see that it is the intrinsic vorticity of the 2D EG which, because its long-range influence is governed by the same equations as the 2D Coulomb interaction, plays a role identical to that of the charge in superconducting condensate and opens a gap in the phonon dispersion relation. In this analysis I have ignored the genuine **E** field generated by the nonuniform density—but because it can escape from the plane of the 2D EG it cannot generate a plasmon gap.

The trajectories of the "particles" in the density wave become circular at long wavelength and evolve into conventional longitudinal sound waves as the wavelength decreases. In the $k=0$ mode the whole system of charge moves together in circles and this motion raises some interesting questions. According to Kohn's theorem[18] there is a bulk mode where the particles orbit as a rigid mode at the free particle cyclotron frequency,

$$\Omega_c=B/m , \tag{4.6}$$

independent of interactions. If the present $k=0$ mode is identified with this "Kohn theorem" mode then, the remarks in Sec. II notwithstanding, m^* *is* the effective mass of the individual electrons. This mode, however, is the zero-momentum limit of the *magnetoplasmon* or inter-Landau level branch of excitations. These, in the absence of interactions or at large k, evolve into *excitons* with a particle in the $n=1$ Landau level and a hole in the $n=0$ level. We are interested in the lower energy *intra*-Landau level *magnetophonon* modes and do not wish to make this identification—so we will remain with the interpretation of m^* as a Coulomb derived effective mass and its effects should be regarded only as an "analogue" of inertia.

Because I ignored the quantum pressure in deriving the quasihydrodynamic approximation we do not see any of the larger k phenomena such as the k^4 term, characteristic of the weakly interacting Bose gas model of a superfluid–but then the pressure density relationship in the real system is presumably not the same as in the weakly interacting system. Also, unless we rather artificially take λ negative, we see no sign the magnetoroton dip[19] which occurs at the reciprocal of the mean interparticle spacing and presages the low-density collapse of the FQHE ground state into a Wigner crystal. At the largest momenta the magnetophonon density-wave picture is expected to break down entirely and the excitations are expected, by analogy with the magnetoplasmons, to become vortex antivortex *quasiexciton* pairs.[19] The two vortices in a pair will be separated by a distance proportional to k and will move, each with the flow velocity induced by the other, in parallel rather like a two-dimensional smoke ring. This change of interpretation cannot possibly be described by any quasiclassical model. Despite these deficiencies I think there is some merit in this picture of the phononlike elementary excitation.

To investigate vortex motion we must extend the quantum constraint (3.8) to take into account point defects in the ϕ field. Such point vortices modify (3.8) to

$$m^*\omega(\mathbf{r})+B(\mathbf{r})=2\Theta_0\rho(\mathbf{r})+\sum \kappa_i\delta^2(\mathbf{r}-\mathbf{r}_i) , \tag{3.8a}$$

where $\kappa_i=\pm 2\pi$. To produce a low-energy configuration, each of these point vortices is surrounded by an oppositely oriented antivortex and its associated charge density. It is this composite object that is the analog of the quasiparticle and quasihole excitations of the Laughlin ground state.

The forces acting on a vortex held stationary in a steady uniform flow are a combination of electromagnetic forces, hydrodynamic Magnus forces, and the external force provided by the pinning center. To analyze them it is useful to use a version of Bernoulli's theorem that follows most directly from (3.6): assume a steady flow so $\nabla\cdot\rho\mathbf{v}=0$, then we can introduce a stream function χ with

$$\begin{aligned}\rho v_1&=\partial_2\chi ,\\ \rho v_2&=-\partial_1\chi .\end{aligned} \tag{4.7}$$

With $\dot{\mathbf{v}}=0$ and $\mathbf{E}=-\nabla V$, (3.6) can be now be written

$$\nabla(\mu+2\Theta_0\chi+V+\tfrac{1}{2}m^*v^2)=0 \tag{4.8}$$

so the combination

$$\mu_0=\mu+2\Theta_0\chi+V+\tfrac{1}{2}m^*v^2 \tag{4.9}$$

is constant everywhere—not just along streamlines.

The stream function is also useful for evaluating the total Lorentz force $\mathbf{F}_m$ due to a uniform magnetic field acting on an arbitrary steady flow in a region Ω. We can use Stokes theorem and find

$$\mathbf{F}_m=\int_\Omega \mathbf{j}\times B\, d^2x=-B\int_{\partial\Omega}\chi\mathbf{n}dS , \tag{4.10}$$

where $\mathbf{n}$ is the outward normal. The force depends only on asymptotic properties of the flow.

With this information we can evaluate the total force $\mathbf{F}$ on a region Ω, containing a stationary vortex. I will assume that we are in a region where the density takes its asymptotic value $\rho_\infty=B/2\Theta_0$. The force is the sum of the body force and the external pressure

$$\mathbf{F}=\int_\Omega(\mathbf{j}\times B+\rho\mathbf{E})d^2x-\int_{\partial\Omega}P\mathbf{n}dS . \tag{4.11}$$

Using the Bernoulli theorem (4.9) and the relation $\nabla P=\rho\nabla\mu$ to find P along the boundary, we find

$$\mathbf{F}=\int_\Omega \mathbf{E}\rho d^2x+\int_{\partial\Omega}\left[-B\chi+\rho_\infty(2\Theta_0\chi+V+\tfrac{1}{2}m^*v^2)\right]\mathbf{n}dS . \tag{4.12}$$

Since $m^*\omega+B=2\Theta_0\rho$ we see that force reduces to

$$\mathbf{F}=\int_\Omega \mathbf{E}(\rho-\rho_\infty)d^2x+\int_{\partial\Omega}(\rho_\infty\tfrac{1}{2}m^*v^2)\mathbf{n}dS . \tag{4.13}$$

At large distances the flow is uniform and without circulation. The integral of v^2 is zero and so

$$\mathbf{F}_\infty = \int_\Omega \mathbf{E}(\rho - \rho_\infty) d^2x \ . \tag{4.14}$$

Since there is no net flux of momentum across the boundary the calculated force $\mathbf{F}$ must be balanced by a pinning force on the vortex.

When the $\mathbf{E}$ field is constant the force (4.14) is just the electric buoyancy force on the vortex due to the deviation of the charge density from that of the surrounding fluid. The magnetic forces cancel. The derivation of (4.14) provides a rather convoluted route to the discovery that the force on an isolated stationary vortex is $\mathbf{E}$ times its charge q—but leads naturally into a discussion of *where* on the vortex does the pinning force act. A reasonable assumption is that there are additional conservative forces, due to charged impurities or other inhomogeneities in the interface where the 2D EG resides, acting on the individual electrons. Since such forces can be written as $\mathbf{F}_{\text{imp}} = -\rho \nabla W$, I can, without loss of generality, add them into the $\mathbf{E}$ field and, after including them, the integral (4.14) will be zero. I will now argue that such forces *cannot* hold a vortex stationary in a background flow.

Firstly we establish another momentum conservation result. Suppose that we perform the computation of the net force on a region around an *arbitrary* contour but this time use

$$\int_{\partial\Omega} P\mathbf{n}\, dS = \int_{\partial\Omega} \rho\mu\mathbf{n}\, dS - \int_\Omega \mu \nabla\rho d^2x \ , \tag{4.15}$$

i.e., use the Bernoulli result throughout the region and not just on the boundary. We find after various applications of Eqs. (3.8), (4.9), the identity

$$\mathbf{v}\cdot\nabla\mathbf{v} = \nabla \tfrac{1}{2} v^2 - \mathbf{v}\times\boldsymbol{\omega} \tag{4.16}$$

and $\nabla\cdot\rho\mathbf{v} = 0$, etc., that

$$\mathbf{F} = \int_{\partial\Omega} m^* \rho \mathbf{v}(\mathbf{v}\cdot\mathbf{n}) dS \tag{4.17}$$

so that the total force, including pinning forces, on any region between two streamlines must vanish.

Suppose now the velocity field near the vortex comprises three parts: the flow at infinity, $\mathbf{v}_\infty$ equal to the Hall flow $\propto E_\infty / B$, the vortex flow itself, $\mathbf{v}_{\text{rot}}$, and the backflow $\mathbf{v}_b$. Close to, but not in, the core of a quasiparticle vortex the streamlines are closed and in a region of slowly varying density so we can write the force on the region within the streamline as

$$\mathbf{F} = \int_\Omega \rho \mathbf{E} d^2x + \int_{\partial\Omega} \tfrac{1}{2} m^* \rho \mathbf{v}^2 \mathbf{n} dS \ . \tag{4.18}$$

The integral of v^2 yields the Magnus force equal to the circulation within the contour times the flow past the core (see the Appendix). Because of the relationship between the charge of the vortex q and its circulation given by Eq. (2.7) and because of the relation between the flow $\mathbf{v}_\infty$ and the electric field, this force is $\mathbf{E}q$. By Eq. (4.17) the force is independent of the streamline we use: If we evaluate the integral about a closed streamline further out we would find a smaller Magnus force, because the circulation is reduced by the enclosed part of the antivortex, but the reduction in the Magnus force is exactly compensated by the extra electric body force on the enclosed fluid due to the *uniform* part of the electric field. We see that the whole of the pinning force $\mathbf{E}q$ must be borne by the vortex core. This is exactly what happens in a type-II superconductor[4] where it is quite reasonable that the normal core sustains the force. In the present case the core is *hollow*: There are no electrons there to be acted on by the pinning force—the Magnus force must therefore be zero and the vortex has no recourse but to follow the flow.

Is this a disaster for pinning? Not really—since in listing the flow components near the vortex I have omitted the flow $\mathbf{v}_{\text{imp}}$ produced by the pinning force itself. Let us consider the nature of the flow near a charged impurity in the absence of the vortex. As an approximation imagine that the electric field produced by the impurity varies slowly enough that we can use the uniform Hall current formula $\mathbf{v}_{\text{imp}} \times \mathbf{B} = -\mathbf{E}_{\text{imp}}$. We see that there is a vorticity given by

$$\omega = \frac{1}{B} \nabla\cdot\mathbf{E} \tag{4.19}$$

implying an induced $\delta\rho$ proportional to $\nabla\cdot\mathbf{E}$. Since the electric field of the impurity is not confined to the plane of the 2D EG, the $\mathbf{E}$ field falls off more rapidly than the solutions of $\nabla^2 V = 0$ and the in-plane flux of the E field falls rapidly to zero as we move away. This means that although there will be an induced charge *at* the impurity this charge will come from nearby and there is no *net* charge pulled in by the impurity—this is an example of the incompressibility of the FQHE ground state.

Near the impurity there will be closed streamlines and any particular line will remain closed up to a maximum background current. Since the induced charge is not simply related to the impurity charge it will be energetically favorable for vortices to accumulate near the impurity. Their cores will move with the local flow field but they will orbit the impurity until the background flow is strong enough that their streamline is no longer closed. In particular, there will be a stable point at which the flow velocity is zero until the Hall electric field exceeds the maximum force produced by the impurity potential. A vortex can remain at rest at this point there until this happens. The energetics of this depinning are clearly identical to those of a charge q particle acted on by only the Hall field and the local pinning field. This may give one a slight sense of *nascetur mus* but it must be remembered that the actual situation is more complicated than the conclusion suggests.

V. CONCLUSION

Motivated by the Landau-Ginsburg theory I have made a quasihydrodynamical model of the FQHE. As with the derivation of hydrodynamic equations from the Landau-Ginsburg theory of conventional superfluids it seems reasonable to suppose that such equations will have greater generality than their derivation. A further merit of the classical fluid-flow paradigm is that it enables a direct application of mechanical intuition in any first attempt at understanding new phenomena—although it will in no way substitute for serious quantum-mechanical computation of the relevant parameters and constitutive relations. The fluid picture leads naturally to the effective incompressibility of the FQHE ground state and to the gap in the magneto phonon spectrum. It thus captures the essential physics.

My motivation for the examination of the fluid-flow picture of the FQHE was to seek a model for vortex pinning and unpinning. It is easy to perform a force balance analysis of the FQHE vortices by using slight modifications of the discussion in (Ref. 4), but since there is no "normal core" it seems most likely, as suggested in Sec. IV, that the vortex localization and charge sequestration occur via a slightly different mechanism than pinning in a superconductor.

In studying the vortex dynamics one must be aware that the vortices in the FQHE involve far fewer degrees of freedom and are therefore much more quantum mechanical objects than in some of the other superfluids—but it seems worthwhile to understand the classical behavior before attempting to discuss zero-point fluctuations, quantum delocalization, condensation, and consequent hierarchy of FQHE states at other rational filling factors.

ACKNOWLEDGMENTS

This work was supported by the National Science Foundation Grant Nos. NSF-DMR-88-18713 and NSF-DMR-86-12860. Financial support was also provided by the University of Illinois and by NORDITA. I thank C. J. Pethick and A. Luther for their hospitality at NORDITA. I would also like to thank Henrik Bruus for introducing me to the work on vortex pinning by the group at the H. C. Ørsted Institute and the Danish Institute of Fundamental Metrology and Tony Leggett for discussions about superfluids.

APPENDIX

For completeness I will describe here the theory of the Magnus effect in the case of incompressible, irrotational flow where it is simplest. In such a case the velocity field is described by an analytic stream function $\psi(z)$ such that

$$v_1 - iv_2 = -\frac{d\psi}{dz} . \quad \text{(A1)}$$

Suppose there is a cylinder with arbitrary shaped boundary $\partial\Omega$ about which the fluid has circulation. We can use the conventional Bernoulli theorem to write the force on the cylinder as a contour integral round the boundary streamline[20]

$$F_2 + iF_1 = -\tfrac{1}{2}\rho \int_{\partial\Omega} \left[\frac{d\psi}{dz} \right]^2 dz . \quad \text{(A2)}$$

Here ρ is the *mass density.* Close to the cylinder there will be backflow and the stream function will be a complicated function of the boundary shape—but at large distance all the complications are irrelevant and the stream function will have the simple form

$$\psi = A + Bz + C\ln z , \quad \text{(A3)}$$

where the uniform flow field **U** is given by

$$B = -(U_1 - iU_2) \quad \text{(A4)}$$

and the circulation is $\kappa = 2\pi iC$. Because of the analyticity the contour integral may be evaluated at infinity where the asymptotic data may be used to express the integrand. The force turns out to be

$$F_1 = -\rho\kappa U_2, \quad F_2 = \rho\kappa U_1 . \quad \text{(A5)}$$

There is no drag (d'Alembert's paradox) but only a lift force. In the absence of any outside force to hold the cylinder in place it will have to move with the flow.

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