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Elastic properties of a two-dimensional lattice in a weak random pinning potential: Origin of the pinning force

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The problem of an elastic system in the presence of a random pinning potential is investigated using weak-disorder perturbation theory. The occurrence of a restoring pinning force is derived from first principles. A quantitative estimate of the pinning force is obtained and agrees with the collective pinning theory of Larkin and Ovchinnikov. The renormalization of the elastic properties is calculated. Contribution from the anharmonicity of the lattice can have both possible signs whereas contribution from the pinning force always reduces the effective shear modulus.

INTRODUCTION

The pinning of an elastic system by an external potential has its origin in the breaking of translational invariance. Many physical systems exhibit this stick-slip behavior, for example, depinning of charge-density waves, Wigner crystals, Bloch walls, flux lattices in type-II superconductors, etc. Two extreme limits of this very general problem have been considered: The first one occurs when both the elastic system and the pinning potential are periodic but incommensurate as in the Frenkel-Kontorova model extensively studied by Aubry.[1] The second one occurs when the elastic system is periodic and the underlying pinning potential is random. For two physical situations, this question has been raised—namely by Lee and Rice[2] for charge-density waves and Larkin and Ovchinnikov[3] for flux pinning. The main concept in these latter cases is that the random potential breaks the lattice into "correlated volumes" that behave elastically independently and are pinned individually. The concept of "correlated volume" circumvents the problem of identifying the elastic and plastic instabilities that are responsible for the pinning force.[4] Recent computer simulations addressed the nature of these instabilities for a model system of relevance to the flux pinning problem[5] and the following picture has emerged: depending on the strength of the random potential, the lattice is either deformed purely elastically, or undergoes elastic instabilities, or is plastically deformed. In the elastic regime (which disappears as the inverse square root of the system size) there is no pinning force. Both the elastic instabilities and the plastic instabilities lead to a finite pinning force. The crossover between these two regimes decreases logarithmically with the system size. In this paper we discuss the elastic instability regime, i.e., before plastic deformation occurs. Even though this regime is absent for an infinite system, it might still be of physical relevance to systems studied experimentally. For instance, typical experimental flux-line lattices contain about 10^4 vortices.

Another interesting result shown by computer simulations in the elastic instability regime concerns the effective elastic constants of the lattice in the random potential: It has been observed that the effective shear modulus depends on the strength of the pinning potential. The effective shear modulus measured as a response to an imposed periodic shear deformation can either increase or decrease depending on the range of the vortex-vortex potential. In contrast the effective shear modulus measured as a response to imposed displacements of the boundaries of the lattice (the exact equivalent of a macroscopic shear test in solid mechanics), always decreases as the strength of the random potential increases. The latter effect produces a "softening" of the lattice that is likely to be important for the physical behavior of the flux lattice: pinning force, nucleation of dislocations, onset of diffusion, nonlinearity of the I-V characteristics.

This paper is organized as follows. In Sec. I, we introduce our model and calculate the average elastic response to a given displacement: This allows us to give an estimate for the total pinning force which is compared to the collective pinning theory by Larkin and Ovchinnikov.[3] In Sec. II, we show that the interplay between the random potential and the anharmonic elasticity of the perfect lattice leads to a renormalization of the elastic moduli. This is relevant to the situation of an imposed displacement field. In Sec. III, we address the Cauchy problem where displacements are imposed only at the boundary. In this situation, we find that the effective shear modulus always decreases in the presence of the pinning potential, in agreement with previous computer simulations.

I. AVERAGE ELASTIC RESPONSE TO A DISPLACEMENT

In this paper we consider a two-dimensional hexagonal lattice. The energy of the system is given by the interaction between vortices:

$$U_{vv}=\tfrac{1}{2}\sum_{i,j} u_{vv}(\mathbf{r}_i-\mathbf{r}_j)\ , \tag{1}$$

where u_{vv} is a two-body potential. For specific calculations, we assume a Gaussian form

$$u_{vv}(r)=A_v \exp\left[-\left[\frac{r}{R_v}\right]^2\right]. \tag{2}$$

The interaction of the lattice with the random set of pinning centers is also described by a two-body potential

$$U_{vp}=\sum_{i=1}^{N_v}\sum_{j=1}^{N_p} u_{vp}(\mathbf{r}_i-\mathbf{R}_j)\equiv\sum_{i=1}^{N_v} V(\mathbf{r}_i)\ , \tag{3}$$

where $\mathbf{r}_i$ denotes the coordinates of lattice sites and $\mathbf{R}_j$ the random positions of the otherwise identical pinning centers. Again, for specific calculations, we assume for u_{vp} the Gaussian form:

$$V(r)=-A_p \exp\left[-\left[\frac{r}{R_p}\right]^2\right]. \tag{4}$$

The method we use applies to any dimension. We restrict ourselves in this paper to the two-dimensional case in order to compare our results to computer simulations. This treatment is relevant to several physical systems, among them superconducting films.

At this stage, it should be stressed that this perturbative treatment is valid if the system size is smaller than the correlated volume introduced by Larkin and Ovchinnikov.[3] They have calculated the displacement correlation functions $g(r)$. In the case of the Gaussian potential one obtains

$$g(r)=\langle |S(r)-S(0)|^2\rangle =\tfrac{1}{8}n_p n_v A_p^2\left[\frac{1}{\mu^2}+\left[\frac{1}{\lambda+2\mu}\right]^2\right] r^2 \ln\left[\frac{L}{a}\right], \tag{5}$$

where n_p is the density of pinning centers, n_v the density of vortices, L the linear size of the system, and a the lattice spacing. When this $g(r)$ becomes of the order of R_p^2, elastic instabilities occur and perturbation theory breaks down. However, within each correlated volume V_c defined by the condition $g(r)<R_p^2$, perturbation theory remains valid. We now expand the total energy in powers of the displacement field: This corresponds to a weak-disorder limit where we assume small displacements. In this expansion, we keep terms up to fourth order in the strength of pinning potential which amounts to fourth order in displacement in the elastic energy and third-order displacements in the vortex-pinning centers energy: We Fourier transform the total energy and take the continuum limit in q space. This leads to

$$\begin{aligned}U=U_{vv}+U_{vp}=&\tfrac{1}{2}\int S^{\alpha}(q)\Phi_{\alpha\beta}S^{\beta}(-q)\frac{d^2q}{(2\pi)^2}+\tfrac{1}{6}\int q_1^{\alpha'}q_2^{\beta'}q_3^{\gamma'}S^{\alpha}(q_1)S^{\beta}(q_2)S^{\gamma}(q_3)A_{\alpha\beta\gamma}^{\alpha'\beta'\gamma'}\delta(q_1+q_2+q_3)\frac{d^2q_1d^2q_2d^2q_3}{(2\pi)^4}\\&+\tfrac{1}{12}\int q_1^{\alpha'}q_2^{\beta'}q_3^{\gamma'}q_4^{\delta'}S^{\alpha}(q_1)S^{\beta}(q_2)S^{\gamma}(q_3)S^{\delta}(q_4)B_{\alpha\beta\gamma\delta}^{\alpha'\beta'\gamma'\delta'}\delta(q_1+q_2+q_3+q_4)\frac{d^2q_1d^2q_2d^2q_3d^2q_4}{(2\pi)^6}\\&+n_v\int V'_{\alpha}(q)S^{\alpha}(-q)\frac{d^2q}{(2\pi)^2}+\frac{n_v}{2}\int V''_{\alpha\beta}(-q_1-q_2)S^{\alpha}(q_1)S^{\beta}(q_2)\frac{d^2q_1d^2q_2}{(2\pi)^4}\\&+\frac{n_v}{6}\int V'''_{\alpha\beta\gamma}(-q_1-q_2-q_3)S^{\alpha}(q_1)S^{\beta}(q_2)S^{\gamma}(q_3)\frac{d^2q_1d^2q_2d^2q_3}{(2\pi)^6}\ .\end{aligned} \tag{6}$$

The notation is as follows:

$$S^{\alpha}(q)=\int S^{\alpha}(x)e^{iqx}d^2x\ ,$$

where $S^{\alpha}(x)$ is the continuum version of the α coordinate of the displacement field $\mathbf{S}$.

$\Phi_{\alpha\beta}(q)$ is related to the Lamé coefficients[6] by

$$\Phi_{\alpha\beta}(q)=(\lambda+\mu)q^{\alpha}q^{\beta}+\mu q^2\delta^{\alpha\beta}\ . \tag{7}$$

Indeed in the case of an hexagonal lattice, the elasticity is isotropic and therefore can be described by only two elastic coefficients λ and μ. The inverse matrix of Φ is

$$\chi^{\alpha\beta}(q)=(\Phi^{-1})^{\alpha\beta}=\frac{1}{\mu q^2}\left[\delta^{\alpha\beta}-\frac{\lambda+\mu}{\lambda+2\mu}\frac{q^{\alpha}q^{\beta}}{q^2}\right]. \tag{8}$$

$A_{\alpha\beta\gamma}^{\alpha'\beta'\gamma'}$ is the tensor of third-order anharmonicity, and $B_{\alpha\beta\gamma\delta}^{\alpha'\beta'\gamma'\delta'}$ is the tensor of fourth-order anharmonicity.

In all those elastic coefficients we have taken the $q=0$ limit, which amounts to local elasticity.[7] By definition

$$V'_\alpha(q)=\int \frac{\partial}{\partial r^\alpha} V(\mathbf{r})e^{i\mathbf{q}\cdot\mathbf{r}}d^2\mathbf{r} ,$$
$$V''_{\alpha\beta}(q)=\int \frac{\partial^2}{\partial r^\alpha \partial r^\beta} V(\mathbf{r})e^{i\mathbf{q}\cdot\mathbf{r}}d^2\mathbf{r} , \tag{9}$$
$$V'''_{\alpha\beta\gamma}(q)=\int \frac{\partial^3}{\partial r^\alpha \partial r^\beta \partial r^\gamma} V(\mathbf{r})e^{i\mathbf{q}\cdot\mathbf{r}}d^2\mathbf{r} .$$

In order to calculate the average elastic coefficients, we first have to solve the equation of equilibrium of the lattice in the presence of the random potential, then to take the second derivative of the total energy around this new equilibrium configuration. Finally, we average over the random positions of pinning centers.

By definition, the elastic matrix $\Phi^{\alpha\beta}_{\text{dis}}(q)$ of the disordered system is obtained from

$$\frac{\partial^2 U}{\partial S^\alpha(q)\partial S^\beta(q')}=\frac{1}{4\pi^2}\Phi^{\alpha\beta}_{\text{dis}}(q)\delta(q+q') . \tag{10}$$

From Eq. (6) we get

$$\Phi^{\alpha\beta}_{\text{dis}}(q)\delta(q+q')=\Phi^{\alpha\beta}(q)\delta(q+q')+\frac{1}{(2\pi)^2}q^{\alpha'}q'^{\beta'}(-q-q')^{\gamma'}A^{\alpha'\beta'\gamma'}_{\alpha\beta\gamma}S^\gamma(-q-q')$$
$$+\int q^{\alpha'}q'^{\beta'}k^{\gamma'}(-q-q'-k)^{\delta'}B^{\alpha'\beta'\gamma'\delta'}_{\alpha\beta\gamma\delta}S^\gamma(k)S^\delta(-q-q'-k)\frac{d^2k}{(2\pi)^4}$$
$$+\frac{n_v}{(2\pi)^2}V''_{\alpha\beta}(-q-q')+n_v\int V'''_{\alpha\beta\gamma}(-q-q'-k)S^\gamma(k)\frac{d^2k}{(2\pi)^4} , \tag{11}$$

where **S** is the new equilibrium configuration.

In the weak-disorder limit, the first corrections to $\Phi^{\alpha\beta}$ will come from averaging terms which are quadratic in V. As a result we need to expand **S** to second order in V. However the second-order contribution to **S** is involved only in the AS term coming from third-order anharmonicity. After averaging, this term vanishes as we show in detail in Appendix A. Hence we need to expand **S** only to first order in V, namely

$$S^\alpha(q)=-n_v\chi^{\alpha\beta}(q)V'_\beta(q) . \tag{12}$$

From Eqs. (10) and (11), we obtain the average elastic matrix,

$$\langle \Phi^{\alpha\beta}_{\text{dis}}(q)\rangle\delta(q+q')=\Phi^{\alpha\beta}(q)\delta(q+q')$$
$$+B^{\alpha'\beta'\gamma'\delta'}_{\alpha\beta\gamma\delta}q^{\alpha'}q'^{\beta'}n_v^2\int k^{\gamma'}(-q-q'-k)^{\delta'}\chi^{\gamma\lambda}(k)\chi^{\delta\mu}(-q-q'-k)\langle V'_\lambda(k)V'_\mu(-q-q'-k)\rangle\frac{d^2k}{(2\pi)^4}$$
$$-n_v^2\int \chi^{\gamma\delta}(k)\langle V'''_{\alpha\beta\gamma}(-q-q'-k)V'_\delta(k)\rangle\frac{d^2k}{(2\pi)^4} , \tag{13}$$

where $\langle\ \rangle$ stands for the averaging over the random positions of the pinning centers.

In order to proceed, the form of the correlations of the random potential is needed. They will be very different depending on the range of the pinning potential. If this range is long compared to the lattice spacing, the global effect is to couple the pinning centers to compressions or dilations of the lattice. By contrast, if the range of the pinning potential is small compared to the lattice spacing, shear deformations at the scale of the lattice spacing will take place. The explicit calculation for both cases is described in Appendix B. To treat the case of short-range potential, the correlation function is truncated so that forces seen by two different sites are completely uncorrelated. The corrections to $\Phi^{\alpha\beta}_{\text{dis}}(q)$ are of two types [Eq. (13)].

(1) The terms coming from anharmonicity of the lattice vanishes for $q=0$: they amount to the renormalization of the elastic constants, which will be discussed in Sec. II.

(2) The terms coming from the anharmonicity of the pinning potential gives a finite contribution for $q=0$, which shows explicitly the breaking of translation invariance, and the appearance of a restoring force given by

$$F^\alpha=\phi^{\alpha\beta}_{\text{dis}}(q=0)S^\beta(q=0)\equiv\alpha_L S^\alpha . \tag{14}$$

α_L defined by Eq. (14) is the so-called "Labush coefficient."[8] In the special case of the Gaussian potential (see Appendix B), one gets

$$\alpha_L=\frac{\lambda+3\mu}{2\mu(\lambda+2\mu)}\frac{n_p A_p^2 n_v}{R_p^2}\ln\left|\frac{L}{a}\right| . \tag{15}$$

The large distance cutoff L is defined as $L=\min(V^{1/2};V_c^{1/2})$, V being the system size and V_c a correlated volume. From Eq. (15) one can derive an estimate for the pinning force F_p assuming that the maximum restoring force is obtained when the correlated volume has been translated by a distance of the order of

R_p at which point elastic instabilities occur. Thus the pinning force per unit surface is

$$F_p=\frac{\lambda+3\mu}{2\mu(\lambda+2\mu)}\frac{n_p A_p^2 n_v}{R_p}\ln\left[\frac{L}{a}\right] . \tag{16}$$

Noticing that

$$\frac{\lambda+3\mu}{2\mu(\lambda+2\mu)}=\frac{1}{2}\left[\frac{1}{C_{11}}+\frac{1}{C_{66}}\right] \tag{17}$$

and if moreover $C_{11} \gg C_{66}$, one finds

$$F_p \approx \frac{1}{2C_{66}}\frac{n_v n_p A_p^2}{R_p}\ln\left[\frac{L}{a}\right] . \tag{18}$$

This has the same dependencies on the model parameters as in the two-dimensional Larkin-Ovchinnikov theory[3,9] except for the exponent of the logarithmic dependence. This similarity may seem surprising since the two approaches are different. Our method is a perturbation theory in the limit of weak disorder. In this the Labush coefficient appears as the fundamental quantity and follows directly from the calculation. In comparison Larkin and Ovchinnikov obtain the pinning force directly from the statistics of individual random forces over a correlated volume. It is worth noticing that the Labush coefficient is related to the nonharmonic nature of the vortex-pinning centers potential (through the presence of V'''): such anharmonicity allows for the existence of elastic instabilities which is a necessary condition for a nonzero pinning force.[10] Moreover it appears that, provided that this anharmonicity is present, there is a nonzero pinning force for arbitrary small values of the strength A_p of the pinning centers. The Labush coefficient we have derived is of the order of magnitude of the slope of the force displacement curve found in computer simulations.[11]

However, it should be pointed out that our expression for the Labush parameter would lead to an A_p^2 dependence for the pinning force in any dimension. By contrast, in three dimensions for instance, Larkin and Ovchinnikov predict an A_p^4 dependence. We will come back to this point in Sec. III.

II. RENORMALIZATION OF THE ELASTIC MODULI

The q^2 corrections to the elastic matrix has, according to Eq. (13) (for the special case of the Gaussian pinning potential) the following form:

$$\Delta\phi^{\alpha\beta}(q)=2\pi^3 n_p n_v A_p^2 q^{\alpha'} q^{\beta'} B_{\alpha\beta\gamma\delta}^{\alpha'\beta'\gamma'\delta'} J_{\gamma\delta\gamma'\delta'} , \tag{19}$$

where

$$J_{\gamma\delta\gamma'\delta'}=\frac{1}{2\mu^2}\frac{1}{(2\pi)^3}\ln\left[\frac{L}{a}\right]\left[\delta_{\gamma\delta}\delta_{\gamma'\delta'}-\frac{1}{4}\frac{(\lambda+\mu)(\lambda+3\mu)}{(\lambda+2\mu)^2}(\delta_{\gamma\delta}\delta_{\gamma'\delta'}+\delta_{\gamma\delta'}\delta_{\gamma'\delta}+\delta_{\gamma\gamma'}\delta_{\delta\delta'})\right] . \tag{20}$$

In the limit $\lambda \gg \mu$ this can be rewritten

$$\Delta\phi^{\alpha\beta}(q)=\frac{1}{8\mu^2}n_v n_p A_p^2\ln\left[\frac{L}{a}\right]\Lambda_{\alpha\beta}^{\alpha'\beta'}q^{\alpha'}q^{\beta'} \tag{21}$$

with

$$\Lambda_{\alpha\beta}^{\alpha'\beta'}=B_{\alpha\beta\gamma\delta}^{\alpha'\beta'\gamma'\delta'}[\delta_{\gamma\delta}\delta_{\gamma'\delta'}-\tfrac{1}{4}(\delta_{\gamma\delta}\delta_{\gamma'\delta'}+\delta_{\gamma\delta'}\delta_{\gamma'\delta}+\delta_{\gamma\gamma'}\delta_{\delta\delta'})] . \tag{22}$$

For the case of the hexagonal lattice, it can be shown that the renormalized elasticity remains isotropic (i.e., $2\Lambda_{xy}^{xy}=\Lambda_{xx}^{xx}-\Lambda_{xx}^{yy}$). Therefore, it is meaningful to introduce renormalized Lamé coefficients λ and μ (see Appendix C),

$$\Delta\mu=\Lambda_{xx}^{yy}\tfrac{1}{8}n_p A_p^2\frac{1}{\mu^2}\ln\left[\frac{L}{a}\right] , \tag{23}$$

$$\Delta\lambda=(2\Lambda_{xy}^{xy}-\Lambda_{xx}^{yy})\tfrac{1}{8}n_p A_p^2\frac{1}{\mu^2}\ln\left[\frac{L}{a}\right] . \tag{24}$$

The tensor $\Lambda_{\alpha\beta}^{\alpha'\beta'}$ is calculated (see Appendix C) for an hexagonal lattice. With the approximation of keeping nearest-neighbor interactions, one finds

$$\Delta\mu=\tfrac{1}{8}n_p A_p^2 n_v\frac{1}{\mu^2}\ln\left[\frac{L}{a}\right](-\tfrac{45}{64}au_{vv}'+\tfrac{45}{64}a^2u_{vv}''+\tfrac{9}{32}a^3u_{vv}'''+\tfrac{9}{8}a^4u_{vv}'''') , \tag{25}$$

$$\Delta\lambda=\tfrac{1}{8}n_p A_p^2 n_v\frac{1}{\mu^2}\ln\left[\frac{L}{a}\right](\tfrac{123}{64}au_{vv}'-\tfrac{123}{64}a^2u_{vv}''-\tfrac{3}{32}a^3u_{vv}'''+\tfrac{9}{8}a^4u_{vv}'''') , \tag{26}$$

where the values of u_{vv}', u_{vv}'', u_{vv}''', and u_{vv}'''' are to be taken at the lattice spacing a. Specializing to the Gaussian vortex-vortex potential one gets

$$\Delta\mu=\tfrac{1}{8}n_p A_p^2 n_v A_v\frac{1}{\mu^2}\ln\left[\frac{L}{a}\right]\left[\frac{315}{16}-\frac{225}{4}\left[\frac{a}{R_v}\right]^2+18\left[\frac{a}{R_v}\right]^4\right]\left[\frac{a}{R_v}\right]^4\exp\left[-\left[\frac{a}{R_v}\right]^2\right] , \tag{27}$$

$$\Delta\lambda=\tfrac{1}{8}n_p A_p^2 n_v A_v \frac{1}{\mu^2}\ln\left[\frac{L}{a}\right]\left[\frac{75}{16}-\frac{213}{4}\left[\frac{a}{R_v}\right]^2+18\left[\frac{a}{R_v}\right]^4\right]\left[\frac{a}{R_v}\right]^4\exp\left[-\left[\frac{a}{R_v}\right]^2\right] . \qquad (28)$$

One striking feature of this equation is that the sign of the correction depends on the range R_v of the vortex-vortex potential, namely,

$$\Delta\mu<0 \quad \text{for } 0.63<\frac{a}{R_v}<1.65 \quad \text{and } \Delta\mu>0 \text{ otherwise ;}$$

$$\Delta\lambda<0 \quad \text{for } 0.30<\frac{a}{R_v}<1.69 \quad \text{and } \Delta\lambda>0 \text{ otherwise .}$$

The exact values of a/R_v for which the corrections $\Delta\mu$ and $\Delta\lambda$ change signs depend on the approximation used to compute $\Lambda_{\alpha\beta}^{\alpha'\beta'}$. Nevertheless the qualitative behavior remains when one includes more and more neighbors. We have observed this cross over between hardening and softening in computer simulations. One must notice that the sign of the corrections to the elastic moduli depends crucially on the shape of the vortex-vortex interaction (whether the fourth derivative can change sign or not). It should be emphasized that the case studied in this section corresponds to a prescribed displacement of all lattice points. By contrast in many physical situations, the displacement field is imposed only at boundaries. This is the case in a macroscopic shear experiment, where we impose the displacements at the boundaries of the sample. This is also the case for a dislocation where discontinuity of the displacement field is imposed along the dislocation line. It strongly suggests that we consider Cauchy-type problems in a random pinning potential.

III. ELASTIC PROPERTIES: THE CAUCHY PROBLEM

This calculation has been motivated by a numerical shear experiment.[12] Imposing the displacements in the y direction at the boundary of the sample, the increase in elastic energy ΔU_{vv} is measured and an effective shear modulus $\mu(A_p)$ is defined by

$$\Delta U_{vv}=\tfrac{1}{2}\mu(A_p)\int\left[\frac{\partial S^y}{\partial x}\right]^2 dx\,dy . \qquad (29)$$

It was found that $\mu(A_p)$ decreases with increasing pinning strength A_p (see Fig. 1). The result was only weakly dependent on the specific configuration of pinning centers. Moreover, this decrease is found for any range in the vortex-vortex potential we could reach. This suggests that the effect does not depend crucially on the anharmonicity of the vortex-vortex potential. The physical origin of this softening is the same as the pinning force—namely the nonlinear coupling to the pinning potential. For this reason we will consider only harmonic elasticity and nonlinear coupling to the pinning potential:

$$U=\tfrac{1}{2}\int\int S^\alpha(r)\Phi^{\alpha\beta}(r,r')S^\beta(r')dr\,dr' + n_v\int V'_\alpha(r)S^\alpha(r)dr+\frac{n_v}{2}\int V''_{\alpha\beta}(r)S^\alpha(r)S^\beta(r)dr + \frac{n_v}{6}\int V'''_{\alpha\beta\gamma}(r)S^\alpha(r)S^\beta(r)S^\gamma(r)dr . \qquad (30)$$

We look for a solution $S^\alpha(r)$ which minimizes U. The system is restricted to the region $0\le x\le L$. When we impose on the boundary a displacement Δ, i.e., a shear deformation Δ/L, in the absence of pinning centers the displacement field would be

$$S^\alpha(r)=\Delta\frac{x}{L}\delta^{\alpha y} .$$

In the presence of pinning centers, we look for a solution of the form

$$S^\alpha(r)=S_0^\alpha(r)+\Delta\left[\frac{x\delta^{\alpha y}}{L}+S_1^\alpha(r)\right]+\Delta^2 S_2^\alpha(r) , \qquad (31)$$

$S_0^\alpha, S_1^\alpha, S_2^\alpha$ are chosen with vanishing boundary conditions. Here S_0^α is the displacement coming from the relaxation of the lattice to the pinning potential before the shear is applied.

We solve the minimization problem by using the real-space Green's function of the Φ operator, which is defined by

$$-(\lambda+\mu)\frac{\partial^2}{\partial r_\alpha\partial r_\beta}\chi^{\beta\gamma}(r,r')-\mu\nabla_r^2\chi^{\alpha\gamma}(r,r')=\delta^{\alpha\gamma}(r-r') \qquad (32)$$

and with vanishing boundary conditions.

Then one gets

$$S^\alpha(r)=-n_v\int\chi^{\alpha\beta}(r,r')[V'_\beta(r')+V''_{\beta\gamma}(r')S^\gamma(r') + \tfrac{1}{2}V'''_{\beta\gamma\delta}(r')S^\gamma(r')S^\delta(r')] \times dr'+\Delta\frac{x}{L}\delta^{\alpha y} . \qquad (33)$$

Here we have used the fact that Φ operating on the bare shear profile $(\Delta x/L)\delta^{\alpha y}$ is zero. The elastic energy of the relaxed sheared state is

$$E=E_0+\Delta E_1+\tfrac{1}{2}\Delta^2 E_2 . \qquad (34)$$

We are interested in the term quadratic in Δ,

$$E_2=\int\int\frac{x}{L}\Phi^{yy}(r,r')\frac{x'}{L}dr\,dr' + \int\int S_1^\alpha(r)\Phi^{\alpha\beta}(r,r')S_1^\beta(r')dr\,dr' + 2\int\int S_0^\alpha(r)\Phi^{\alpha\beta}(r,r')S_2^\beta(r')dr\,dr' . \qquad (35)$$

We have used again the stationarity of the elastic energy for the linear profile to eliminate first-order terms in S_1^α.

We want to average E_2 over the disorder: The first nonzero contribution will be second order in V so that S_0, S_1, S_2 have to be expanded to first order in V. This gives

$$S_0^\alpha=-n_v\int\chi^{\alpha\beta}(r,r')V'_\beta(r')dr' , \qquad (36)$$

$$S_1^\alpha = -\frac{n_v}{L} \int \chi^{\alpha\beta}(r,r') V''_{\beta y}(r') x' dr' , \quad (37)$$

$$S_2^\alpha = -\frac{n_v}{L^2} \int \chi^{\alpha\beta}(r,r') V'''_{\beta yy}(r') (x')^2 dr' . \quad (38)$$

Therefore, after averaging one gets

$$\langle E_2 \rangle = \mu + \frac{n_v^2}{L^2} \int (G_{\alpha\beta yy} + 2H_{\alpha\beta yy}) \chi^{\alpha\beta}(r,r) x^2 dr , \quad (39)$$

where we have defined

$$\langle V''_{\alpha y}(r) V''_{\beta y}(r') \rangle = G_{\alpha\beta yy} \delta(r-r') , \quad (40)$$

$$\langle V'_{\alpha}(r) V'''_{\beta yy}(r') \rangle = H_{\alpha\beta yy} \delta(r-r') . \quad (41)$$

For any impurity potential, we have $H_{\alpha\beta yy} = -G_{\alpha\beta yy}$, so that

$$\langle E_2 \rangle = \mu - \frac{n_v^2}{L^2} \int G_{\alpha\beta yy} \chi^{\alpha\beta}(r,r) x^2 dr . \quad (42)$$

For a spherically symmetric potential $G_{\alpha\beta yy}$ and $H_{\alpha\beta yy}$ are zero for $\alpha \neq \beta$. Since the elastic energy is positive definite, the diagonal Green's function $\chi^{\alpha\alpha}(r,r)$ is positive. Therefore, for any vortex-pin potential of spherical symmetry, $\langle E_2 \rangle$ is less than μ corresponding to a softening of the lattice. One must notice that the quantity G, which governs this softening, is the same as the one that enters into the Labush coefficient.

Specializing to the Gaussian potential, we have

$$\langle E_2 \rangle = \mu - \frac{n_v^2}{L^2} \frac{\pi}{2} \frac{n_p A_p^2}{R_p^2} \int [\chi^{xx}(r,r) + 3\chi^{yy}(r,r)] x^2 dr . \quad (43)$$

From dimensional analysis it follows that the integral in Eq. (43) is proportional to L^4, the proportionality constant can be estimated in the case of periodic boundary conditions by the use of Eq. (8). One finally gets

$$\langle E_2 \rangle = \mu - \tfrac{1}{6} L^2 \frac{n_p n_v A_p^2}{R_p^2} \frac{1}{\mu} \frac{\lambda + 3\mu}{\lambda + 2\mu} \ln \left[\frac{L}{a} \right] . \quad (44)$$

Using Eq. (15) for the Labush coefficient α_L, one finds

$$\langle E_2 \rangle = \mu - \tfrac{1}{3} \alpha_L L^2 . \quad (45)$$

Although this formula gives the right trend that the elastic shear modulus decreases with increasing pinning potential, it is clear that this expression can apply only if (μ/α_L) is much larger than the system size. When $(\mu/\alpha_L) \simeq V_c$ the correlated volume [see Eq. (5)] becomes smaller than the system size, and the effective shear modulus can rather be estimated in the following way.[13] Assume that the system breaks into correlated volumes of size V_c separated by thin boundary regions of thickness a. The respective volume fractions of the boundary regions and of the correlated volumes, are f and $(1-f)$ where

$$f \sim \frac{a V_c^{1/2}}{V_c} = \frac{a}{V_c^{1/2}} . \quad (46)$$

This system is composed of stiff regions (correlated volumes) separated by soft regions (boundary layers) which are bounded to deform in a coherent way. The effective shear modulus which results is

$$\mu_{\text{eff}} = f \mu_b + (1-f)\mu ; \quad (47)$$

if $\mu_b \ll \mu$, one gets

$$\mu_{\text{eff}} \simeq \left[1 - \frac{a}{V_c^{1/2}} \right] \mu = \left[1 - a \left[\frac{\alpha_L}{\mu} \right]^{1/2} \right] \mu ; \quad (48)$$

from Eq. (15) one obtains (for $\lambda \gg \mu$),

$$\mu_{\text{eff}} \simeq \mu - \left[\frac{a}{R_p} \right] \frac{(n_p n_v)^{1/2}}{\sqrt{2}} \left[\ln \frac{L}{a} \right]^{1/2} A_p . \quad (49)$$

This expression has the linear dependence on A_p observed in computer simulations (see Fig. 1). The coefficient of A_p for the parameters considered in Fig. 1 is 4.0, which is of the right order of magnitude. This linear behavior in A_p is difficult to get in weak-disorder expansions since they produce even powers of A_p.

We want to close this section by a discussion of the behavior in three dimensions. The same procedure we have already applied leads to a correction of the elastic shear modulus of the form:

$$\langle E_2 \rangle = \mu - \text{const}\, A_p^2 L^3 . \quad (50)$$

It appears that elasticity breaks down for volumes of the size:

$$L^3 \sim \frac{\mu}{A_p^2} . \quad (51)$$

For small values of A_p this volume is much smaller than the correlated volume derived by Larkin and Ovchinnikov which behaves as $1/A_p^6$. Their derivation makes explicit use of elasticity theory [see Eq. (5)].

In three dimensions the displacement-displacement

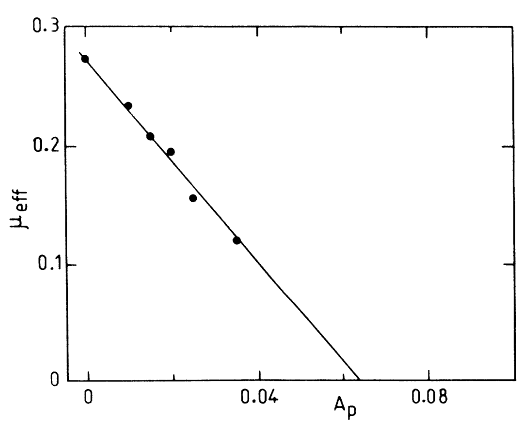

FIG. 1. The effective shear modulus measured numerically by use of Eq. (29). The theoretical ideal lattice shear modulus is $\mu = 0.27$. The system parameters are $N_v = 1020$, $R_v = 0.6$, $N_p = 438$, $R_p = 0.25$.

correlation function $g(r)$ behaves as

$$g(r) \sim A_p^2 r \ , \tag{52}$$

which leads to the $1/A_p^6$ dependence for the correlated volume. Note that this scaling gives A_p^4 dependence for the force as already mentioned at the end of Sec. I.

It has been suggested from experiments[14,15] that plastic deformations make the correlated volume much smaller than the estimate given by Larkin and Ovchinnikov. We may speculate that in three dimensions the actual correlated volume corresponds to vanishing shear modulus and occurrence of plastic deformation. Using Larkin-Ovchinnikov expression for the pinning force,

$$F_p \sim \left[\frac{n_p A_p^2}{V_c} \right]^{1/2} \tag{53}$$

and Eq. (51) for the correlated volume, one gets that the pinning force scales as A_p^2 and is thus proportional to the Labush coefficient obtained from perturbation theory.

CONCLUSIONS

In this paper we have shown how to set up a first-principles approach to pinning problems. By explicit calculations we have shown that the crucial ingredient is the nonlinearity of the vortex-pinning centers interaction. This nonlinearity is at the origin of elastic instabilities that break the system into finite correlated volumes. Inside those correlated volumes one can perform weak-disorder expansions. This treatment leads explicitly to broken translation invariance, namely a finite Labush coefficient. This leads to pinning if the size of the system is larger than the correlated volume and our estimate for the pinning force agrees with Larkin and Ovchinnikov's in two dimensions. In three dimensions our calculation suggests that the softening of the lattice by the random potential leads to breakdown of elasticity on length scales shorter than the ones estimated by Larkin and Ovchinnikov. We develop a systematic treatment of the Cauchy problem in the presence of a random pinning potential. We find that the shear modulus is always smaller than its value for the pure system. However, the explicit dependence of this softening on the pinning potential amplitude A_p shows that it cannot be described by simple perturbation theory. Instead we have shown that the numerical results of shear experiments can be described by the assumption of a mixture of elastic correlated volumes separated by soft boundary regions.

The discussion in the present paper is limited to the elastic and elastic instability regimes. Elastic instabilities occur when the correlated volume becomes smaller than the system size. Further increase in the pinning potential eventually leads to plastic deformation characterized by a finite density of dislocations. Natural extension of this work would be a treatment of dislocations in the presence of a random pinning potential.

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APPENDIX A: CONTRIBUTION OF THE THIRD ORDER ANHARMONICITY TO RENORMALIZATION OF ELASTIC CONSTANTS

The contribution of the third-order anharmonicity (i.e., $A_{\alpha\beta\gamma}^{\alpha'\beta'\gamma'}$) to the renormalized $\Phi^{\alpha\beta}$ is [see Eq. (11)]

$$q^{\alpha'} q'^{\beta'} (-q-q')^{\gamma'} A_{\alpha\beta\gamma}^{\alpha'\beta'\gamma'} S^{\gamma}(-q-q') \tag{A1}$$

to second order in V one has

$$\begin{aligned} S^{\gamma}(-q-q') = & -\chi^{\gamma\alpha}(-q-q')V'_{\alpha}(-q-q') - \tfrac{1}{2}\int \chi^{\gamma\alpha}(-q-q')\chi^{\beta\lambda}(k)\chi^{\delta\mu}(-q-q'-k)(q+q')^{\alpha'} \\ & \times k^{\beta'}(-q-q'-k)^{\delta'} V'_{\lambda}(k) V'_{\mu}(-q-q'-k) A_{\alpha\beta\delta}^{\alpha'\beta'\delta'} \frac{dk}{(2\pi)^2} \\ & + \chi^{\gamma\alpha}(-q-q')\int \chi^{\beta\delta}(k) V''_{\alpha\beta}(-q-q'-k) V'_{\delta}(k) \frac{dk}{(2\pi)^2} \ . \end{aligned} \tag{A2}$$

Averaging S over the disorder (see Appendix B) will make first-order terms in V disappear. The averaging of second-order terms will give a contribution to $\langle S(-q-q')\rangle$ proportional to $\delta(-q-q')$. But, on the other hand,

$$(-q-q')\delta(-q-q') = 0 \ . \tag{A3}$$

Therefore, the contribution to $\Phi^{\alpha\beta}$ coming from third-order anharmonicity vanishes. Consequently, in the renormalization of elastic constants the expansion of S in powers of V is needed to first order only, and only the fourth-order anharmonicity will give a nonzero contribution.

APPENDIX B: CORRELATIONS OF THE RANDOM POTENTIAL

Let us consider correlation functions of the type

$$\langle \tilde{G}(q) \tilde{H}(q') \rangle \ ,$$

where

$$G(r)=\sum_p g(r-R_p)\ , \tag{B1}$$

$$H(r)=\sum_p h(r-R_p)\ . \tag{B2}$$

From these definitions it follows that

$$\langle \tilde{G}(q)\tilde{H}(q')\rangle = \int dr\, dr' e^{iq'r'}e^{-iqr}\sum_{R_p}\sum_{R_{p'}}\langle g(r-R_p)h(r'-R_{p'})\rangle\ . \tag{B3}$$

Two possible situations can be considered.

(1) If the range of the pinning potential is much larger than the intervortex spacing, the pinning potential can be considered as smooth and correlations on different sites of the lattice can exist. In this case, one obtains,

$$\langle \tilde{G}(q)\tilde{H}(q')\rangle_{\rm LRP}=n_p(2\pi)^2\delta(q+q')\tilde{g}(q)\tilde{h}(q')\ , \tag{B4}$$

where $\tilde{g}$ and $\tilde{h}$ denote the Fourier transforms of g and h.

(2) If, as in our case, the range of the pinning potential is much smaller than the intervortex spacing, the potential can only be correlated on identical lattice sites. Then we have

$$\langle \tilde{G}(q)\tilde{H}(q')\rangle_{\rm SRP} = \int dr\, dr' e^{-iqr}e^{-q'r'} \times \sum_{R_p}\sum_{R_{p'}}\frac{\delta(r-r')}{n_v}\langle g(r-R_p)h(r'-R_{p'})\rangle\ , \tag{B5}$$

which leads to

$$\langle \tilde{G}(q)\tilde{H}(q')\rangle_{\rm SRP} = \frac{1}{n_v}n_p(2\pi)^2\delta(q+q')\int \frac{dk}{(2\pi)^2}\tilde{g}(k)\tilde{h}(-k)\ . \tag{B6}$$

APPENDIX C: CALCULATION OF THE FOURTH-ORDER ANHARMONIC COEFFICIENTS $B_{\alpha\beta\gamma\delta}^{\alpha'\beta'\gamma'\delta'}$

We have to expand the elastic energy to fourth order in the displacements.

$$E^{\rm elastic}=\tfrac{1}{2}\sum_i\sum_\tau u_{vv}(\mathbf{S}_i-\mathbf{S}_{i+\tau}+\tau)\ . \tag{C1}$$

The first summation is on the lattice sites i, whose displacement is S_i, the second summation is on the lattice vectors τ.

Let us put $\tau=\tau\hat{\mathbf{n}}$ ($\hat{\mathbf{n}}$ being of length 1). The fourth-order term in $E^{\rm elastic}$ is

$$\begin{aligned}E^{(4)}=&\tfrac{1}{2}\sum_\tau \frac{f(\tau)}{\tau^4}\sum_i\sum_{\hat{\mathbf{n}}}(\mathbf{S}_i-\mathbf{S}_{i+\tau})^4\\ &+\tfrac{1}{2}\sum_\tau\frac{g(\tau)}{\tau^4}\sum_i\sum_{\hat{\mathbf{n}}}(\mathbf{S}_i-\mathbf{S}_{i+\tau})^2\cdot[\hat{\mathbf{n}}\cdot(\mathbf{S}_i-\mathbf{S}_{i+\tau})]^2\\ &+\tfrac{1}{2}\sum_\tau\frac{h(\tau)}{\tau^4}\sum_i\sum_{\hat{\mathbf{n}}}[\hat{\mathbf{n}}\cdot(\mathbf{S}_i-\mathbf{S}_{i+\tau})]^4\end{aligned} \tag{C2}$$

with

$$f(\tau)=\left[-\frac{1}{8}\frac{u'_{vv}(\tau)}{\tau^3}+\frac{1}{8}\frac{u''_{vv}(\tau)}{\tau^2}\right]\tau^4\ , \tag{C3}$$

$$g(\tau)=\left[+\frac{3}{4}\frac{u'_{vv}(\tau)}{\tau^3}-\frac{3}{4}\frac{u''_{vv}(\tau)}{\tau^2}+\frac{1}{4}\frac{u'''_{vv}(\tau)}{\tau}\right]\tau^4\ , \tag{C4}$$

$$h(\tau)=\left[-\frac{5}{8}\frac{u'_{vv}(\tau)}{\tau^3}+\frac{5}{8}\frac{u''_{vv}(\tau)}{\tau^2} -\frac{1}{4}\frac{u'''_{vv}(\tau)}{\tau}+u''''_{vv}(\tau)\right]\tau^4\ . \tag{C5}$$

In Fourier space, in the continuum limit and for $q\tau\ll 1$ one finally gets

$$\begin{aligned}B_{\alpha\beta\gamma\delta}^{\alpha'\beta'\gamma'\delta'}=&\sum_\tau 2f(\tau)T_{(\tau)}^{\alpha'\beta'\gamma'\delta'}(\delta_{\alpha\beta}\delta_{\gamma\delta}+\delta_{\alpha\gamma}S_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma})+\sum_\tau g(\tau)(T_{(\tau)}^{\alpha'\beta'\gamma'\delta'\alpha\beta}\delta_{\gamma\delta}\\ &+T_{(\tau)}^{\alpha'\beta'\gamma'\delta'\alpha\gamma}\delta_{\beta\delta}+T_{(\tau)}^{\alpha'\beta'\gamma'\delta'\alpha\delta}\delta_{\beta\gamma}+T_{(\tau)}^{\alpha'\beta'\gamma'\delta'\beta\gamma}\delta_{\alpha\delta}+T_{(\tau)}^{\alpha'\beta'\gamma'\delta'\beta\delta}\delta_{\alpha\gamma}+T_{(\tau)}^{\alpha'\beta'\gamma'\delta'\gamma\delta}\delta_{\alpha\beta})+\sum_\tau h(\tau)T_{(\tau)}^{\alpha'\beta'\gamma'\delta'\alpha\beta\gamma\delta}\ ,\end{aligned} \tag{C6}$$

where the $T_{(\tau)}$ tensors are defined as follows:

$$T_{(\tau)}^{\alpha'\beta'\gamma'\delta'}=\sum_{\hat{\mathbf{n}}} n^{\alpha'}n^{\beta'}n^{\gamma'}n^{\delta'}\ , \tag{C7}$$

$$T_{(\tau)}^{\alpha'\beta'\gamma'\delta'\alpha\beta}=\sum_{\hat{\mathbf{n}}} n^{\alpha'}n^{\beta'}n^{\gamma'}n^{\delta'}n^{\alpha}n^{\beta}\ , \tag{C8}$$

$$T_{(\tau)}^{\alpha'\beta'\gamma'\delta'\alpha\beta\gamma\delta}=\sum_{\hat{\mathbf{n}}} n^{\alpha'}n^{\beta'}n^{\gamma'}n^{\delta'}n^{\alpha}n^{\beta}n^{\gamma}n^{\delta}\ , \tag{C9}$$

where the summtion is to be taken on the unit vectors $\hat{\mathbf{n}}$ linking a given site to its neighbors which are at a distance τ.

For a two-dimensional (2D) hexagonal lattice, with only nearest-neighbor interactions, the tensor $\Lambda_{\alpha\beta}^{\alpha'\beta'}$ defined by

$$\Lambda_{\alpha\beta}^{\alpha'\beta'}=B_{\alpha\beta\gamma\delta}^{\alpha'\beta'\gamma'\delta'}[\delta_{\gamma\delta}\delta_{\gamma'\delta'}-\tfrac{1}{4}(\delta_{\gamma\delta}\delta_{\gamma'\delta'}+\delta_{\gamma\delta'}\delta_{\gamma'\delta}+\delta_{\gamma\gamma'}\delta_{\delta\delta'})] \tag{C10}$$

is given by

$$\Lambda_{xx}^{xx}=\tfrac{21}{2}f(a)+\tfrac{21}{4}g(a)+\tfrac{27}{3}h(a)\ , \tag{C11}$$

$$\Lambda_{xx}^{yy}=\tfrac{27}{2}f(a)+\tfrac{9}{4}g(a)+\tfrac{9}{8}h(a)\ , \tag{C12}$$

$$\Lambda_{xy}^{xy}=-\tfrac{3}{8}f(a)+\tfrac{3}{2}g(a)+\tfrac{9}{8}h(a)\ . \tag{C13}$$

It can be checked that elasticity remains isotropic (i.e., $2\Lambda_{xy}^{xy}=\Lambda_{xx}^{xx}-\Lambda_{xx}^{yy}$).

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