# Effects of quasiparticle dissipation on quantum fluctuations in granular superconductors

R. S. Fishman
*Department of Physics, North Dakota State University, Fargo, North Dakota 58105-5566*
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Quasiparticle dissipation in a granular superconductor is modeled by an effective nearest-neighbor capacitance $\Delta C$ between the grains of a superconducting array. Using an expansion in $1/z$, where $z$ is the number of nearest neighbors in the array, I study the effects of quasiparticle dissipation on the transition temperature and short-range order of a granular superconductor. In agreement with experimental results, quasiparticle dissipation suppresses the quantum fluctuations in a superconducting array. If the self-capacitance of a grain is $C_0$, then both the long-range and the short-range order of the array are enhanced as the ratio $\lambda = C_0/z\Delta C$ decreases. In disagreement with other work, the transition temperature is not reentrant for any value of $\lambda$. The results of this formalism, which consistently treats quantum fluctuations to first order in $1/z$, should be valid in three-dimensional materials.

## I. INTRODUCTION

Recent experiments[1-3] have revealed that dissipative effects are vital to an understanding of granular superconductors. In particular, experiments indicate that a two-dimensional granular film with very small grains[1,2] becomes superconducting when its normal-state resistance drops below the quantum resistance $R_Q = h/(2e)^2 = 6.5$ kΩ. This discovery disagreed with prior theoretical predictions[4-9] that superconductivity is impossible when the grain diameter is smaller than a critical value. Subsequent work[10,11] established that the tunneling of normal electrons between grains creates an effective intergrain capacitance. When this new capacitive term is added to the model Hamiltonian for array of superconducting grains,[12-15] a resistive threshold close to $R_Q$ is indeed produced.

So the model Hamiltonian of a granular superconductor contains two capacitive terms. The first term is the electrostatic charging energy,[16] which originates from the finite capacitance of each grain. The second term arises from the virtual tunneling of quasiparticles[10,11] between neighboring grains. Larkin and Ovchinnikov[10] demonstrated that quasiparticle tunneling generates the nearest-neighbor capacitance[11]

$$\Delta C = \frac{3\pi}{32}\frac{\sigma_N}{\Delta(0)} , \tag{1}$$

where $\hbar = 1$, $\Delta(0)$ is the zero-temperature gap on each grain, and $\sigma_N$ is the normal-state conductance of a junction. As the normal-state resistance of the film decreases, $\Delta C$ increases. In the limit of very small grains or very large $\sigma_N$, the self-capacitance of a grain can be neglected compared to the capacitance provided by quasiparticle dissipation. If, on the other hand, the normal-state conductance is very small or the grains are very large, only the electrostatic charging energy contributes to the Hamiltonian.

In the latter case, $\Delta C = 0$ and the model Hamiltonian[4] for an array of superconducting grains is

$$H' = 2U_0 \sum_i n_i^2 - J \sum_{\langle i,j \rangle} \cos(\phi_i - \phi_j) , \tag{2}$$

where the first sum runs over all lattice sites and the second over all nearest neighbors. The operator $\phi_i$ is the phase of the superconducting order parameter on the $i$th grain, while $n_i$ is the operator for the excess number of Cooper pairs on this grain. The commutation relations between $n_i$ and $\phi_j$,

$$[n_i, \phi_j] = -i\delta_{ij} , \tag{3}$$

signify that the order parameter phase and Cooper-pair number cannot be specified simultaneously on the same grain.

The Hamiltonian of Eq. (2) contains two, competing energies. The Josephson energy, proportional to $J > 0$, favors the phase-coherent state with the identical phase on every grain. The constant $J$ is proportional to the probability for a Cooper pair to tunnel between neighboring grains. In terms of the normal-state conductance of a junction, the zero-temperature value for $J$ is[17]

$$J = \frac{2\pi}{8e^2}\Delta(0)\sigma_N . \tag{4}$$

The first term in Eq. (2) is the capacitive energy[16] proportional to $U_0 = e^2/C_0$, where $C_0$ is the self-capacitance of a grain. Since $q_i = 2en_i$ is the excess charge on the $i$th grain,

$$2U_0 \sum_i n_i^2 = \frac{1}{2C_0}\sum_i q_i^2 \tag{5}$$

is just the electrostatic energy for a distribution of excess charge in the lattice. Unlike the Josephson energy, the charging energy of Eq. (5) is minimized when every grain is charge neutral. Due to the uncertainty relations of Eq. (3), the charging and Josephson energies compete: The Josephson energy favors the phase-coherent state with large charge fluctuations while the charging energy favors the charge-neutral state with large phase fluctuations.

 

The relative strength of the Josephson and charging energies is determined by the dimensionless parameter $\alpha = zJ/U_0$, which is proportional to the grain diameter $\sim C_0$. When $\alpha$ is smaller than the critical value $\alpha_c$, the charging energy dominates the Josephson energy and superconductivity becomes impossible.[4] In this regime, quantum fluctuations of the phase destroy the long-range order in the lattice and the global order parameter $M = \langle \cos\phi_1 \rangle$ vanishes at all temperatures. If $\alpha > \alpha_c$, the order parameter $M$ is nonzero and the resistivity $\rho$ vanishes when $T < T_c$, which is the superconductivity transition temperature of the array. In the normal state above $T_c$, $M = 0$ and $\rho \neq 0$ even though the superconductivity gap $\Delta(T)$ may be nonzero on each grain. Above $T_c$, the tunneling of Cooper pairs between grains is unable to overcome the quantum *and* thermal fluctuations of the phase.

The critical parameters in this theory are the transition temperature $T_c$ and the critical grain diameter $\alpha_c \sim 1$. When $T > T_c$ or $\alpha < \alpha_c$, long-range order is absent and the granular material is normal. However, experiments on granular materials tell a different story. In some *superconducting* samples,[18,19] the measured values of $\alpha$ are orders of magnitude smaller than $\alpha_c$. Theories based on Eq. (2) cannot explain why quantum fluctuations do not destroy the long-range order in these samples. Obviously, an important piece is missing from the simple Hamiltonian of Eq. (2).

This missing piece describes the tunneling of normal electrons in the material. Since the transfer of a Cooper pair from one grain to another creates a voltage difference experienced by the normal electrons, the tunneling of Cooper pairs is coupled to the tunneling of quasiparticles. To model quasiparticle dissipation, a nearest-neighbor capacitive term[12,15] is added to the charging energy of the granular Hamiltonian. This modification is performed most easily in the Lagrangian formalism. The Hamiltonian of Eq. (2) is equivalent to the Lagrangian

$$L' = \tfrac{1}{2} C_0 \sum_i V_i^2 + J \sum_{\langle i,j \rangle} \cos(\phi_i - \phi_j) , \tag{6}$$

where $V_i = \dot{\phi}_i / e$ is the voltage on the $i$th grain. Including the nearest-neighbor capacitance, the Lagrangian becomes

$$L = \tfrac{1}{2} C_0 \sum_i V_i^2 + \tfrac{1}{2} \Delta C \sum_{\langle i,j \rangle} (V_i - V_j)^2 + J \sum_{\langle i,j \rangle} \cos(\phi_i - \phi_j) , \tag{7}$$

which has been studied by a variety of methods.[12-15,20] If the grains are extremely small, as in most granular films, the self-capacitance $C_0$ can be neglected compared to the effective capacitance $\Delta C$. However, the full Lagrangian must be used to study fabricated arrays[3] which contain large grains with $C_0$ comparable to $\Delta C$.

If $C_0 \ll z\Delta C$, the critical properties of the array are determined by the dimensionless parameter $\alpha = zJ(z\Delta C)/e^2$. When the normal-state resistance is large enough that $\alpha < \alpha_c$, the charging energy dominates the Josephson energy and the array is always normal. But if the normal-state resistance is sufficiently small that $\alpha > \alpha_c$, the granular array will become superconducting below $T_c$ even though $C_0$ is negligible. So this model explains why quantum fluctuations may not destroy superconductivity in materials with very small grains.

Of course, the electrostatic energy of the array may also contain a nearest-neighbor capacitance, which is connected in parallel with the virtual capacitance of Eq. (1). The total nearest-neighbor capacitance $\Delta C$ is the sum of the geometric and virtual contributions.[21]

Ferrell and Mirhashem[15] have studied the Lagrangian of Eq. (7) with $C_0 = 0$ by expanding the critical value $\alpha_c$ in powers of $1/z$. The lowest-order term in this expansion is the mean-field (MF) result, which neglects the coupling of charge and phase fluctuations on neighboring grains. Because the first-order correction to $\alpha_c$ is positive, the coupling of fluctuations increase $\alpha_c$ from its MF value. The critical value $\alpha_c$ obtained by Ferrell and Mirhashem is close to the observed threshold in two-dimensional films. But unlike the experimental threshold, the theoretical one depends on the coordination number $z$ of the sample.

In a series of recent publications,[22-25] I have developed a new technique for the study of lattice Hamiltonians. Like the method of Ferrell and Mirhashem,[15] this formalism also uses an expansion in powers of $1/z$. Unlike their method, however, the new formalism can be applied to a wide class of lattice Hamiltonians[25] at nonzero temperature. In Refs. 22–24, this method was used to study the simplified Hamiltonian of Eq. (2). The first-order, fluctuation correction corrections to the order parameter $M$, the transition temperature $T_c$, the critical grain diameter $\alpha_c$, the free energy, and the specific heat were evaluated. At zero temperature, the results[23] for $\alpha_c$ agree with Ferrell and Mirhashem. At nonzero temperature, quantum fluctuations are responsible for peaks in the fluctuation specific heat below $T_c$ and in the short-range parameter above $T_c$.[24]

In this paper, I employ the $1/z$ expansion to study the general Lagrangian of Eq. (7). Because both the diagonal and nearest-neighbor capacitances are retained, the results of this paper apply to both granular films with very small grains and to fabricated arrays with large grains. The generalized formalism of Sec. II is a relatively straightforward extension of the discussion in Ref. 24, except that the fluctuation Hamiltonian now includes two terms instead of one.

This formalism is used in Sec. III to calculate the fluctuation correction to the transition temperature and the critical parameter $\alpha_c$. As expected, $T_c$ increases and $\alpha_c$ decreases as the ratio $\lambda = C_0/z\Delta C$ decreases. Hence, the long-range order of the lattice is enhanced by quasiparticle dissipation. The results of Sec. III disagree with a previous MF calculation,[26] which found that $T_c$ is reentrant when $\lambda$ is less than a critical value $\lambda_c$. A reentrant phase transition is characterized by two transition temperatures: one from the normal state to the superconducting state, another back to the normal state at a lower temperature. Unlike the present work, the MF calculation of Ref. 26 is inconsistent: The capacitance matrix is

inverted exactly, to all orders in $1/z$, but the transition temperature is evaluated only to zero order in $1/z$. In this paper, both the Coulomb matrix and the transition temperature are consistently expanded to first order in $1/z$.

In Sec. IV, I calculate the short-range order parameter $\langle \cos(\phi_1-\phi_2)\rangle$, where grains 1 and 2 are nearest neighbors. The fluctuation correction to the short-range order parameter contains two terms. The first term, $\Sigma_1$, is independent of $\Delta C$ and was previously evaluated in Ref. 24. The second term, $\Sigma_2/(1+\lambda)$, is the correction due to quasiparticle dissipation. Because $\Sigma_2>0$ below $T_c$, quasiparticle dissipation enhances the short-range order as well as the long-range order of a granular array in the superconductivity state. The fluctuation correction $\Sigma_2$ vanishes above $T_c$, so quasiparticle dissipation does not affect the short-range order of the normal state, at least to first order in $1/z$. Therefore, the peak in the short-range order parameter of the normal state[27] is unchanged by dissipative effects.

The relevance of this work to experimental measurements is discussed in Sec. V, which also summarizes the results of the previous sections. Finally, Appendices A and B present some of the more unwieldy results that are used in the calculation.

Before embarking on the details of this work, it is fair to question the validity of the nearest-neighbor capacitance model of Eq. (7). The effective capacitance of Eq. (1) is generated by the *virtual* excitation of quasiparticles across a Josephson junction at zero temperature. Of course, the *real* excitation of quasiparticles is forbidden at zero temperature because of the gap $\Delta(0)$ in the quasiparticle energy spectrum. At nonzero temperature, the probability for an excitation over the gap is proportional to $e^{-\Delta(0)/T}$. The model of Eq. (7) does not include these real quasiparticle excitations. Hence, the formalism of this paper cannot be used to calculate the normal-state resistivity of a granular superconductor, which depends on the normal electron current. But this formalism should provide an accurate description of the phase dynamics of a granular array. In particular, the transition temperature and the short-range order parameter should be relatively insensitive to excitations over the energy gap.

Indeed, the approximations used to construct Eq. (7) are self-consistent. The simplified Hamiltonian of Eq. (2) already neglects the fluctuations of the energy gap $\Delta(T)$. This "phase-only" approximation is valid if $T \ll T_{c0}$, where $T_{c0}$ is the bulk transition temperature of a grain below which $\Delta(T)$ becomes nonzero. Since $\Delta(0)\approx 2T_{c0}$, the proviso that $T \ll T_{c0}$ implies that the probability $\sim e^{-\Delta(0)/T}$ for the excitation of a quasiparticle above the energy gap is also extremely small. Hence, the "phase-only" approximation is consistent with the nearest-neighbor capacitance model.

## II. FORMALISM

In this section, I develop the $1/z$ expansion for the Lagrangian of Eq. (7). Because the details of the formalism have been discussed elsewhere,[24] this section concentrates on the special problems posed by the nearest-neighbor capacitance in Eq. (7). It is convenient to rewrite this Lagrangian as

$$L=\tfrac{1}{2}(C_0+z\Delta C)\sum_{i,j} A_{ij}V_iV_j+J\sum_{\langle i,j\rangle}\cos(\phi_i-\phi_j)\ . \tag{8}$$

The dimensionless capacitance matrix $\underline{A}$ is defined by

$$\underline{A}=\underline{I}-\frac{1}{z(1+\lambda)}\underline{K}\ , \tag{9}$$

where $I_{ij}=\delta_{ij}$ is the unit matrix and $K_{ij}=0$ unless grains $i$ and $j$ are nearest neighbors, in which case $K_{ij}=1$. The dimensionless parameter $\lambda$, which was previously introduced in Ref. 26, is defined by

$$\lambda=\frac{C_0}{z\Delta C}\ . \tag{10}$$

This parameter is constructed with a factor of $z$ in the denominator because the single sum in the charging energy acts over $N$ grains while the nearest-neighbor sums in both the charging and Josephson energies act over $Nz/2$ junctions. So the dimensionless parameters of the theory are constructed by multiplying both $J$ and $\Delta C$ by $z$.

Because the $1/z$ expansion is performed in the Hamiltonian formalism, I transform Eq. (8) into the granular Hamiltonian

$$H=\frac{2e^2}{C_0+z\Delta C}\sum_{i,j}B_{ij}n_in_j-J\sum_{\langle i,j\rangle}\cos(\phi_i-\phi_j)\ , \tag{11}$$

where $\underline{B}=\underline{A}^{-1}$ and the Cooper-pair number $n_i=-id/d\phi_i$ is related to the voltage operators by

$$2en_i=(C_0+z\Delta C)\sum_j A_{ij}V_j\ . \tag{12}$$

The range of the Coulomb matrix $\underline{B}$ is determined by $\lambda$. If $\lambda=\infty$, then $\underline{A}$ and $\underline{B}$ are diagonal matrices with $B_{11}=1/A_{11}$. As $\lambda\to 0$, the range of $\underline{B}$ increases: When $\lambda=0$, $B_{ij}$ falls off like $1/|\mathbf{x}_i-\mathbf{x}_j|$ in three dimensions.[26]

The Hamiltonian of Eq. (11) can be separated[22] into three parts:

$$H=H_{\rm eff}+H_1+H_2\ , \tag{13}$$

$$H_{\rm eff}=\sum_i H^i_{\rm MF}\ , \tag{14}$$

$$H^i_{\rm MF}=\frac{2e^2}{C_0+z\Delta C}B_{11}n_i^2-zJM_0\cos\phi_i\ , \tag{15}$$

$$H_1=\tfrac{1}{2}NzJM_0^2\ , \tag{16}$$

$$H_2=-J\sum_{\langle i,j\rangle}R_{ij}+\frac{2e^2}{C_0+z\Delta C}\sum_{i\neq j}B_{ij}n_in_j\ , \tag{17}$$

$$R_{ij}=(\cos\phi_i-M_0)(\cos\phi_j-M_0)+\sin\phi_i\sin\phi_j\ , \tag{18}$$

where $M_0=\langle\cos\phi_1\rangle_{\rm MF}$ is the mean-field order parameter, evaluated with the fluctuation Hamiltonian $H_2$ set to zero. Because $\langle n_i\rangle_{\rm MF}=0$, the MF expectation value of $H_2$ vanishes. To simplify the MF Hamiltonian $H_{\rm eff}$, I define the Coulomb constant

$$U=\frac{e^2}{C_0+z\Delta C}B_{11} . \tag{19}$$

The MF order parameter $M_0$ is then a function only of the dimensionless temperature $T^*=T/zJ$ and the dimensionless parameter $\alpha=zJ/U$. When $\lambda=\infty$ or $\Delta C=0$, the parameter $\alpha$ coincides with the dimensionless grain diameter defined for the simplified Hamiltonian of Eq. (2).

The MF Hamiltonian of Eqs. (14) and (15) is well known. Using only $2\pi$ periodic eigenfunctions, the transition temperature is a monotonically increasing function[6-8] of $\alpha$. When $\alpha$ is less than $\alpha_0=2$, the charging energy dominates the Josephson energy and the array is normal at all temperatures. The MF result for the dimensionless transition temperature $T_c^*=T_c/zJ$ is plotted in the solid curve of Fig. 3.

To calculate the fluctuation corrections to MF theory, I switch to the interaction representation with the order parameter $M=\langle\cos\phi_1\rangle$ defined by[28]

$$M=\frac{1}{Z}\mathrm{Tr}\left[e^{-\beta H_{\mathrm{eff}}}T_\tau\exp\left[-\int_0^\beta\hat{H}_2(\tau)d\tau\right]\cos\hat{\phi}_1(0)\right] , \tag{20}$$

$$Z=\mathrm{Tr}\left[e^{-\beta H_{\mathrm{eff}}}T_\tau\exp\left[-\int_0^\beta\hat{H}_2(\tau)d\tau\right]\right] , \tag{21}$$

where $\beta=1/T$, $T_\tau$ is the time-ordering operator, and operators in the interaction representation are defined by

$$\hat{O}(\tau)=e^{\tau H_{\mathrm{eff}}}Oe^{-\tau H_{\mathrm{eff}}} . \tag{22}$$

The $1/z$ expansion is generated from Eqs. (20) and (21) by expanding both the numerator of $M$ and the partition function $Z$ in powers of the fluctuation energy $H_2$.

When $\Delta C$ is nonzero, the fluctuation Hamiltonian must itself be expanded in powers of $1/z$. This can be demonstrated by rewriting $H_2$ as

$$H_2=-J\sum_{\langle i,j\rangle}R_{ij}+\frac{2U}{B_{11}}\sum_{i\neq j}B_{ij}n_in_j . \tag{23}$$

Inverting Eq. (9) for $\underline{A}$, I find that

$$\underline{B}=\underline{I}+\frac{1}{z(1+\lambda)}\underline{K}+\frac{1}{z^2(1+\lambda)^2}\underline{K}^2+\cdots . \tag{24}$$

So to lowest order in $1/z$,

$$\frac{B_{i\neq j}}{B_{11}}=\frac{1}{z(1+\lambda)}K_{ij} . \tag{25}$$

The higher-order terms in this expansion do not contribute to the *lattice-independent* corrections of the theory. As discussed in Ref. 24, the *lattice-dependent* corrections are numerically much smaller than the lattice-independent corrections. Hence, to order $1/z$, $H_2$ can be replaced by

$$H_2^{\mathrm{NN}}=-J\sum_{\langle i,j\rangle}R_{ij}+\frac{4U}{z(1+\lambda)}\sum_{\langle i,j\rangle}n_in_j , \tag{26}$$

where both sums now act over nearest neighbors.

Another complication when $\Delta C\neq 0$ is that $U$ depends nontrivially on $z$. Since $(\underline{K}^2)_{11}=z$, the lowest-order terms in the diagonal matrix element $B_{11}$ are

$$B_{11}=1+\frac{1}{z(1+\lambda)^2} . \tag{27}$$

So the parameter

$$U=\frac{e^2}{C_0+z\Delta C}\left[1+\frac{1}{z(1+\lambda)^2}\right] \tag{28}$$

is not just a function of $C_0$ and $z\Delta C$. When $\lambda<\infty$, the constant $U$ and the critical parameter $\alpha$ also depend explicitly on the coordination number $z$.

With the modified fluctuation Hamiltonian $H_2^{\mathrm{NN}}$, the derivation of the $1/z$ expansion is almost identical to the derivation in Ref. 24. To order $1/z$, the order parameter $M=\langle\cos\phi_1\rangle$ is again expanded as

$$M=M_0(\alpha,T^*)+\frac{1}{z}M_1(\alpha,T^*) , \tag{29}$$

where both the MF order parameter $M_0$ and the $1/z$ coefficient $M_1$ depend on $z$ only through the dimensionless parameters $T^*$ and $\alpha$. As shown in Fig. 1, the diagrammatic expansion of $M_1$ contains a single "egg" diagram and an infinite number of "sperm" diagrams with ever-growing tails. Although the same set of diagrams appeared in Ref. 24, the line coupling nearest neighbors $i$ and $j$ now represents

$$-JR_{ij}+\frac{4U}{z(1+\lambda)}n_in_j \tag{30}$$

instead of just $-JR_{ij}$.

The infinite sum for $M_1$ is again evaluated with the help of recursion relations. Summing a geometric series yields the exact result

$$M_1(\alpha,T^*)=\frac{M_1^{(2)}(\alpha,T^*)}{1-f(\alpha,T^*)} , \tag{31}$$

where

$$f(\alpha,T^*)=zJ\int_0^\beta d\tau\langle\hat{U}_1(\tau)\hat{U}(0)\rangle_{\mathrm{MF}} , \tag{32}$$

$$\hat{U}_i(\tau)=\cos\hat{\phi}_i(\tau)-M_0 . \tag{33}$$

Although the "scaling" function $f$ was previously defined in Ref. 24, the contribution $M_1^{(2)}$ of the "egg" diagram now contains two parts instead of one:

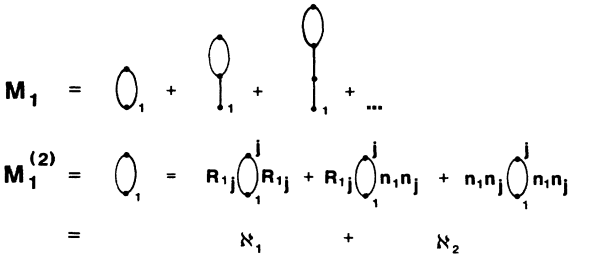


FIG. 1. The series of diagrams which contribute to $M_1$. The contributions $\aleph_1$ and $\aleph_2$ to $M_1^{(2)}$ are shown.

$$M_1^{(2)}(\alpha,T^*)=\aleph_1(\alpha,T^*)+\aleph_2(\alpha,T^*) . \quad (34)$$

The first term, $\aleph_1$, was calculated in Ref. 24 and is independent of $\Delta C$. The second term, $\aleph_2$, is the new correction due to the $n_i n_j$ terms in $H_2^{\rm NN}$. When $\lambda=\infty$, $\aleph_2=0$ and $M_1^{(2)}=\aleph_1$.

Using the definition of Eq. (33), $\aleph_1$ and $\aleph_2$ can be written compactly as

$$\aleph_1=(zJ)^2\int_0^\beta d\tau_1\int_0^{\tau_1}d\tau_2\langle\hat{R}_{12}(\tau_1)\hat{R}_{12}(\tau_2)\hat{U}_1(0)\rangle_{\rm MF} , \quad (35)$$

$$\aleph_2(\lambda)=-\frac{4(zJ)^2}{\alpha(1+\lambda)}\int_0^\beta d\tau_1\int_0^{\tau_1}d\tau_2[\langle\hat{R}_{12}(\tau_1)\hat{n}_1(\tau_2)\hat{n}_2(\tau_2)\hat{U}_1(0)\rangle_{\rm MF}+\langle\hat{n}_1(\tau_1)\hat{n}_2(\tau_1)\hat{R}_{12}(\tau_2)\hat{U}_1(0)\rangle_{\rm MF}]$$
$$+\frac{16(zJ)^2}{\alpha^2(1+\lambda)^2}\int_0^\beta d\tau_1\int_0^{\tau_1}d\tau_2\langle\hat{n}_1(\tau_1)\hat{n}_2(\tau_1)\hat{n}_1(\tau_2)\hat{n}_2(\tau_2)\hat{U}_1(0)\rangle_{\rm MF} . \quad (36)$$

As indicated in Fig. 1, the first two terms in $\aleph_2$ correspond to an "egg" diagram with one line proportional to $R_{1j}$ and the other line proportional to $n_1 n_j$. The last term corresponds to an "egg" diagram with both lines proportional to $n_1 n_j$. The nearest-neighbor index $j$ can occupy $z$ equivalent sites, including site 2. In general, $M_1^{(2)}=\aleph_1+\aleph_2$ is negative below the MF transition temperature $T_{c,\rm MF}$ and vanishes above $T_{c,\rm MF}$.

Because $f=1$ at the MF transition temperature, $M_1$ diverges to $-\infty$ at $T_{c,\rm MF}$. As discussed in Ref. 23, this divergence signals a shift in the transition temperature away from the MF result. The dimensionless transition temperature $T_c^*=T_c/zJ$ can be expanded in powers of $1/z$ as

$$T_c^*=T_0(\alpha)+\frac{1}{z}T_1(\alpha) , \quad (37)$$

where both the MF transition temperature $T_0=T_{c,\rm MF}/zJ$ and the $1/z$ coefficient $T_1$ depend on $z$ only through the dimensionless parameter $\alpha$. Inserting this expansion into Eq. (29) for $M$ and setting $M=0$ furnishes the result

$$T_1(\alpha)=-\lim_{T^*\to T_0}\frac{M_1(\alpha,T^*)}{dM_0/dT^*} , \quad (38)$$

which is negative for all $\alpha$ and diverges to $-\infty$ as $\alpha\to 2$. Since $T_1<0$, fluctuations lower the transition temperature from the MF value.

Just as the divergence in $M_1$ signals a shift in $T_c^*$, the divergence in $T_1$ likewise signals a shift[23] in the critical parameter $\alpha_c$, below which superconductivity is impossible. Writing

$$\alpha_c=\alpha_0+\frac{1}{z}\alpha_1 , \quad (39)$$

the first-order correction to the MF value $\alpha_0=2$ is given by

$$\alpha_1=-\lim_{\alpha\to\alpha_0}\frac{T_1(\alpha)}{dT_0/d\alpha} . \quad (40)$$

Because $\alpha_1>0$, fluctuations increase $\alpha_c$ from its MF value.

The decrease in $T_c^*$ and the increase in $\alpha_c$ from their MF values implies that the phase space available for superconductivity is diminished by the coupling of phase and charge fluctuations on neighboring grains. In the next section, I show that $T_c^*$ is increased and $\alpha_c$ is decreased as $\lambda$ decreases, so that quasiparticle dissipation reclaims some of the phase space lost by fluctuations.

## III. TRANSITION TEMPERATURE AND CRITICAL PARAMETER $\alpha$

This section uses the formalism of Sec. II to calculate the fluctuation corrections to the transition temperatures $T_c$ and the critical parameter $\alpha_c$. Both the diagonal and nearest-neighbor capacitances are retained in the Lagrangian of Eq. (7). As discussed earlier, the off-diagonal capacitance is generated by the virtual tunneling of quasiparticles between neighboring grains. The main conclusion of this section is that quasiparticle dissipation enhances the phase coherence of the array by increasing $T_c$ and decreasing $\alpha_c$.

Since the MF Hamiltonian is unchanged by quasiparticle dissipation (except for the modified definition of $U$), the MF results of Ref. 24 can be adopted in this paper. To evaluate $T_1$ and $\alpha_1$, the MF expressions for $dM_0/dT^*$ and $dT_0/d\alpha$ are required. By a simple extension of the work in Ref. 24, I find that

$$\lim_{T^*\to T_0}\frac{dM_0}{dT^*}=-\frac{M_0\beta^{*2}}{1-f}\left[\frac{1}{4}-\frac{1}{2\alpha}+\frac{2}{\alpha}A_2\right] . \quad (41)$$

In the limit $T^*\to T_0$, the function $A_2(\tau)$ is independent of $\tau$:

$$\lim_{T^*\to T_0}A_2(\tau)\equiv A_2=\frac{1}{Z_{00}}\sum_m m^2 e^{-2m^2/\alpha T_0} , \quad (42)$$

$$Z_{00}=\sum_m e^{-2m^2/\alpha T_0} , \quad (43)$$

where all sums run from $-\infty$ to $\infty$. As $T^*$ approaches $T_0$, the scaling function $f$ approaches 1, $M_0$ approaches 0, and $dM_0/dT^*$ diverges to $-\infty$. The other required relation is provided by Eq. (34) of Ref. 24:

$$\lim_{\alpha\to\alpha_0}\frac{dT_0}{d\alpha}=\frac{3}{16}T_0^2 e^{1/T_0} , \quad (44)$$

which diverges to $\infty$ as $\alpha\to 2$ and $T_0\to 0$.

Finally, the contribution $M_1^{(2)}$ of the "egg" diagram

must be calculated near $T_0$. The linearized relations for the components $\aleph_1$ and $\aleph_2$ are furnished in Eqs. (A1) and (A2). The integrands of those expressions involve combinations of the functions $F_i$ and $A_i$, which must be evaluated to linear order in $M_0$. These linearized combinations are provided in Appendix B. To avoid excruciating detail, I do not reproduce the final results for the integrals over imaginary time.

Combining the MF results of Eqs. (41) and (44) with the integrals for $\aleph_1$ and $\aleph_2$, I obtain the fluctuation correction $T_1$ plotted in Fig. 2 for $\lambda=0$, 1, and $\infty$. Since $T_1$ is negative for all values of $\lambda$ and $\alpha$, the true transition temperature is always lower than its MF value. The plot of $T_1(\alpha)$ for $\lambda=\infty$, corresponding to $\Delta C=0$, was previously presented in Ref. 23. But unlike the numerical results of Ref. 23, the curves in Fig. 2 are obtained analytically by linearizing $M_1$ for small $M_0$. Because $-T_1$ decreases as $\lambda$ decreases from $\infty$ to 0, quantum fluctuations are suppressed by the nearest-neighbor capacitance. For $\lambda=0$ or 1, $-T_1$ has a minimum as a function of $\alpha$. This minimum becomes more pronounced and moves towards smaller values of $\alpha$ as $\lambda$ decreases.

The dependence of $T_1$ on $\lambda$ is simple to explain. To first order in $1/z$, the effective Hamiltonian of the array contains only diagonal and nearest-neighbor Coulomb interactions, $B_{11}>0$ and $B_{12}=B_{11}/z(1+\lambda)$. When a Cooper pair tunnels between neighboring, charge-neutral grains 1 and 2, it creates a "dipole" of $n_1=-1$ and $n_2=+1$. Because the off-diagonal matrix element $B_{12}$ is positive, the opposite charges of the dipole attract. So the charging energy cost for the tunneling of a Cooper pair is diminished by the off-diagonal Coulomb interaction. Since the long-range order of the array is forged by the tunneling of Cooper pairs, the phase coherence of the lattice is enhanced as $B_{12}$ increases or as $\lambda$ decreases. In this manner, the virtual tunneling of quasiparticles promotes the long-range order and enhances the transition temperature of the array. The "dipole" mechanism was previously discussed in Ref. 26, which used MF theory to study the off-diagonal Coulomb interactions in the regime $\alpha T^* \ll 1$. In agreement with the more sophisticated calculation presented here, Ref. 26 found that the off-diagonal interactions enhance the phase coherence of the array.

However, in disagreement with Ref. 26, I now find that the transition temperature is never reentrant. Despite the minimum in $-T_1$, the transition temperature $T_c^*=T_0+T_1/z$ is a single-valued function of $\alpha$. So for a given $\alpha$, only a single transition from the normal state to the superconducting state is possible. By contrast, Ref. 26 found that $T_c^*$ is a double-valued function of $\alpha$ when $\lambda<\lambda_c$ and $\alpha<\alpha_0=2$.

Two inconsistent procedures were responsible for the reentrance in Ref. 26. While the capacitance matrix $\underline{A}$ was numerically inverted to obtain the exact Coulomb matrix $\underline{B}$ to all orders in $1/z$, the transition temperature was calculated with MF theory. Because the fluctuation correction to $T_c^*$ is of order $1/z$, this approach is flawed. The present work consistently calculates both the Coulomb matrix and the transition temperature. To order $1/z$, the transition temperature is a single-valued, nonreentrant function of $\alpha$. Although no proof is forthcoming, I conjecture that the array remains nonreentrant to any order in $1/z$.

A surprising feature of Fig. 2 is the minimum in $-T_1$ as a function of $\alpha$. But as Fig. 3 demonstrates, $T_c^*$ does not necessarily have a maximum as a function of $\alpha$. The solid curve in Fig. 3 is the MF transition temperature $T_c^*=T_0$ with $z=\infty$. The two dashed curves are calculated with $z=4$, so that $T_c^*=T_0+T_1/4$. Both curves lie far below the MF transition temperature and approach the limit of $\frac{3}{8}$ as $\alpha\to\infty$. Generally, $T_0\to\frac{1}{2}$ and $T_1\to-\frac{1}{2}$ as $\alpha\to\infty$, independent of the off-diagonal capacitance.

The fluctuation correction $\alpha_1$ can now be derived. Employing the results of Appendix B, I find that

$$\lim_{\alpha\to\alpha_0}\lim_{T^*\to T_0}\aleph_1=-\tfrac{7}{10}M_0\ , \tag{45}$$

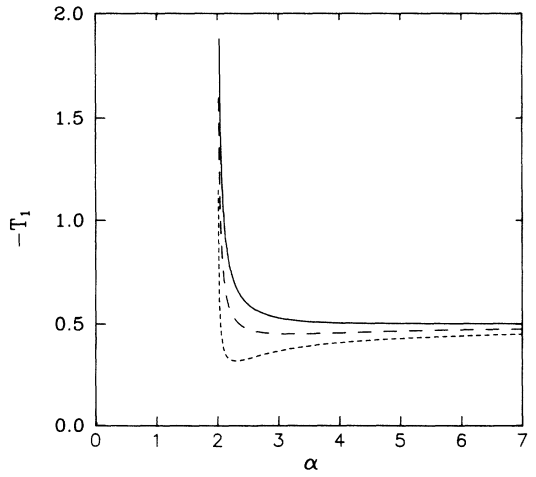


FIG. 2. The fluctuation correction $-T_1$ to the transition temperature vs $\alpha$ for $\lambda=\infty$ (solid), 1 (long dash), and 0 (short dash).

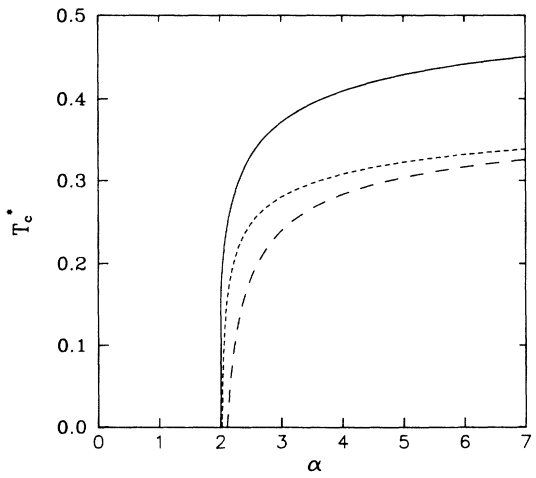


FIG. 3. The total transition temperature $T_c^*$ vs $\alpha$ for $z=\infty$ (solid) and $z=4$ with $\lambda=\infty$ (long dash) or 0 (short dash).

$$\lim_{\alpha\to\alpha_0}\lim_{T^*\to T_0}\aleph_2=\frac{1}{2(1+\lambda)}M_0 \ . \tag{46}$$

From Eq. (38) for $T_1$ and Eq. (40) for $\alpha_1$, it follows that

$$\alpha_1=\frac{7}{5}-\frac{1}{(1+\lambda)} \ , \tag{47}$$

which is one of the central results of this work. Since $\alpha_1>0$ for all $\lambda$, the coupling of charge and phase fluctuations always increases $\alpha_c$ from its MF value. Because $\alpha_1$ decreases as $\lambda$ decreases, the off-diagonal capacitance again enhances the phase coherence of the array. When $C_0=0$, $\alpha_1=\frac{2}{5}$ is substantially reduced from its value of $\frac{7}{5}$ when $\Delta C=0$.

These results imply that the superconducting phase space in a plot of $T_c^*$ versus $\alpha$ is curtailed by the coupling of charge and phase fluctuations on neighboring grains. The off-diagonal capacitive interactions recover some but not all of this lost phase space. This behavior is seen in the numerical results of Fig. 3. The numerical value of $\alpha_c$ is the point at which $T_0+T_1/z$ vanishes in this figure. As pictured, $\alpha_c$ increases and $T_c^*$ decreases as $z$ decreases from $\infty$ to 4 and as $\lambda$ increases from 0 to $\infty$. Comparison of the numerical and analytic results for $\alpha_1$ reveals that the analytic result is quite a bit larger. The reason for this discrepancy is that unlike the numerical result, the analytic result uses an expansion of $T_0(\alpha)$ to first order in $1/z$. While the analytic result is exact to first order in $1/z$, the numerical result of Fig. 3 contains an infinite number of higher-order terms.

Most of this discussion would have to be revised if $\alpha$ was replaced by the parameter

$$\gamma=\frac{zJ}{e^2}(z\Delta C+C_0) \ . \tag{48}$$

To first order in $1/z$, $\alpha$ and $\gamma$ are related by

$$\gamma=\alpha\left[1+\frac{1}{z(1+\lambda)^2}\right] \ . \tag{49}$$

If the critical value of $\gamma$ is expanded as

$$\gamma_c=\gamma_0+\frac{1}{z}\gamma_1 \ , \tag{50}$$

then $\gamma_0=\alpha_0=2$. Using Eq. (47) for $\alpha_1$, I find that

$$\gamma_1=\frac{7}{5}-\frac{1}{1+\lambda}+\frac{2}{(1+\lambda)^2} \ . \tag{51}$$

Like $\alpha_1$, $\gamma_1$ is also positive so that fluctuations increase the critical value of $\gamma$. But unlike $\alpha_1$, $\gamma_1$ is a nonmonotonic function of $\lambda$ with a minimum of $\frac{51}{40}$ at $\lambda=3$. When $C_0\gg z\Delta C$, $\alpha_1=\gamma_1=\frac{7}{5}$, in agreement with Ref. 24. But if $C_0\ll z\Delta C$, $\gamma_1=\frac{12}{5}$ and

$$\left[zJ\frac{z\Delta C}{e^2}\right]_c=2+\frac{12}{5}\frac{1}{z} \ , \tag{52}$$

in agreement with Ferrell and Mirhashem.[15]

The difference between $\alpha$ and $\gamma$ can be appreciated by writing those parameters as

$$\alpha=zJ\frac{C_0+z\Delta C}{e^2}\frac{1}{B_{11}} \ , \tag{53}$$

$$\gamma=zJ\frac{C_0+z\Delta C}{e^2}A_{11} \ , \tag{54}$$

where $\underline{B}$ is the Coulomb matrix and $\underline{A}=\underline{B}^{-1}$ is the capacitance matrix. In the limit $\lambda=\infty$, $\underline{A}=\underline{B}=\underline{I}$ and $\gamma=\alpha$. But for any finite $\lambda$, $B_{11}>1/A_{11}$ and $\gamma>\alpha$, in accord with Eq. (49).

Many of the conclusions of this paper would have to be modified if $\gamma$ replaced $\alpha$ as the critical parameter. Because $\gamma_c$ is a nonmonotonic function of $\lambda$, the transition temperature is also a nonmonotonic function of $\lambda$ at fixed $\gamma$. Nevertheless, for a given $\lambda$, $T_c^*$ is a single-valued function of either $\alpha$ or $\gamma$. So the phase transition is nonreentrant in either case.

The $1/z$ expansion was constructed in terms of $\alpha$ rather than $\gamma$ in order to eliminate the $1/z$ corrections from MF theory. The Hamiltonian $H$ was separated into three parts so that all the diagonal terms in the charging energy

$$\frac{2e^2}{C_0+z\Delta C}\sum_{i,j}B_{ij}n_in_j \tag{55}$$

are contained in $H_{\rm eff}$. The constant $U$ was defined so that $M_0$ is a function only of the dimensionless parameters $T^*$ and $\alpha=zJ/U$. As a result, $\langle H_2\rangle_{\rm MF}=0$ and the fluctuation corrections to MF theory are generated by an expansion in powers of $H_2$.

If $\gamma$ was used in place of $\alpha$, the MF theory would contain explicit corrections of order $1/z$. Consequently, theoretical predictions are much simpler in terms of $\alpha$ rather than $\gamma$: Only in the former case are the transition temperature and the critical parameter monotonic functions of $\lambda$. The phase coherence of the array is enhanced as $\lambda$ decreases only for fixed $B_{11}$ and $\alpha$, not for fixed $A_{11}$ and $\gamma$.

As mentioned previously, the results of this paper at zero temperature agree with Ferrell and Mirhashem,[15] whose method is quite different from the one employed here. Their method exploits analogies with Bethe's cluster expansion of the Ising model.[29] Unlike the $1/z$ expansion in this paper, the method of Ferrell and Mirhashem was developed for zero temperature. So their approach cannot be used to calculate the transition temperature or the short-range order parameter, discussed in the next section.

## IV. SHORT-RANGE ORDER PARAMETER

In this section, I calculate the short-range order parameter $S=\langle\cos(\phi_1-\phi_2)\rangle$, where grains 1 and 2 are nearest neighbors. The short-range order parameter measures the phase coherence between neighboring grains, which is created by the tunneling of Cooper pairs below $T_{c0}$. In the normal state of the array above $T_c$ but below $T_{c0}$, long-range order is absent but short-range order is still present. Therefore, unlike the true order parameter $M$, the short-range order parameter $S$ is nonzero

above $T_c$. The calculation of $S$ in this section generalizes the calculation in Ref. 24, where $\Delta C$ was neglected. The main conclusion of this section is that quasiparticle dissipation enhances the short-range order of the lattice below $T_c$, but does not affect the short-range order above $T_c$ to first order in $1/z$.

Like the true order parameter, the short-range order parameter can be expanded in powers of $1/z$ as

$$S = S_0(\alpha, T^*) + \frac{1}{z} S_1(\alpha, T^*) , \tag{56}$$

where the MF value of $S$ is

$$S_0(\alpha, T^*) = M_0(\alpha, T^*)^2 . \tag{57}$$

As discussed in Ref. 24, the $1/z$ correction to $S$ is

$$S_1 = S_1^{(1)} + 2M_0 M_1 . \tag{58}$$

So to first order in $1/z$,

$$\langle \cos(\phi_1 - \phi_2) \rangle - \langle \cos\phi_1 \rangle^2 = \frac{1}{z} S_1^{(1)} . \tag{59}$$

The right-hand side of Eq. (59) is produced by the coupling of fluctuations on neighboring grains 1 and 2. While $\langle \cos\phi_1 \rangle^2$ vanishes in the normal state, $S_1^{(1)}$ is nonzero at all temperatures below $T_{c0}$. Therefore, the short-range phase coherence above $T_c$ is caused by the product of fluctuations.

In Fig. 4, $S_1^{(1)}$ is represented by a single line which joins grains 1 and 2. This line is proportional to the product of fluctuations on grains 1 and 2. Separating the charge and phase fluctuations, I write $S_1^{(1)}$ as

$$S_1^{(1)}(\alpha, T^*) = \Sigma_1(\alpha, T^*) + \frac{1}{1+\lambda} \Sigma_2(\alpha, T^*) , \tag{60}$$

which is also represented in Fig. 4. In the first diagram, corresponding to $\Sigma_1$, the line is proportional to $R_{12}$. In the second, corresponding to $\Sigma_2/(1+\lambda)$, the line is proportional to $n_1 n_2$. Both $\Sigma_1$ and $\Sigma_2$ are independent of $\lambda$. In the limit $\lambda = \infty$, $S_1^{(1)} = \Sigma_1$ is given by

$$\Sigma_1(\alpha, T^*) = zJ \int_0^\beta d\tau \{ [F_1(\tau) - M_0^2]^2 + F_2(\tau)^2 \} , \tag{61}$$

where the functions $F_1$ and $F_2$ are defined in Eqs. (A7) and (A8). The charge fluctuation correction $\Sigma_2$ can be written

$$\Sigma_2(\alpha, T^*) = -\frac{4zJ}{\alpha} \int_0^\beta d\tau \, A_2(\tau)^2 , \tag{62}$$

where $A_2$ is defined in Eq. (A4).

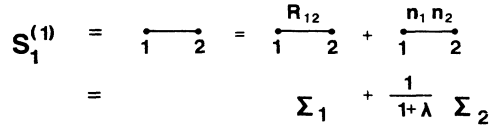


FIG. 4. The diagrams which contribute to $S_1^{(1)}$. The two contributions $\Sigma_1$ and $\Sigma_2/(1+\lambda)$ are shown.

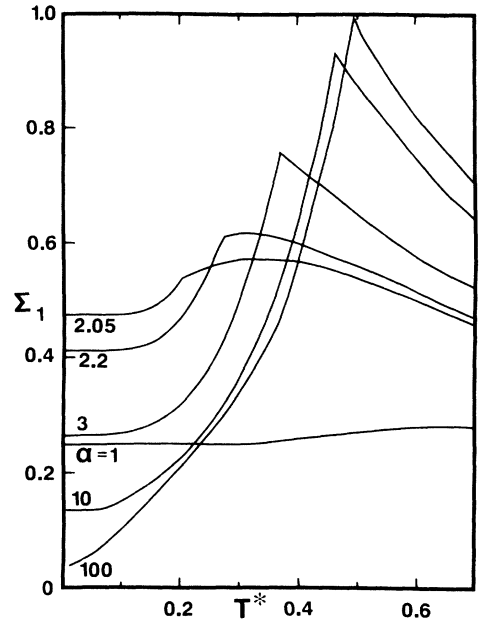


FIG. 5. The fluctuation contribution $\Sigma_1$ vs $T^*$ for different values of $\alpha$.

In Fig. 5, I plot the phase fluctuation term $\Sigma_1$ versus $T^*$ for various values of $\alpha$. Since $\Sigma_1 > 0$, the coupling of phase fluctuations enhances the short-range phase coherence in both the superconducting and normal states. The cusp in $\Sigma_1$ occurs at the MF transition temperature. As discussed in Ref. 24, $\Sigma_1$ has a peak in the normal state[27] at a temperature $\bar{T}^*$ proportional to $1/\alpha$. This peak appears in Fig. 5 for $\alpha = 2.05$ and 2.2. When $\bar{T}^* < T_0$, the peak is absent.

The charge fluctuation term $\Sigma_2$ is plotted in Fig. 6. Like $\Sigma_1$, $\Sigma_2$ is also positive so the virtual tunneling of

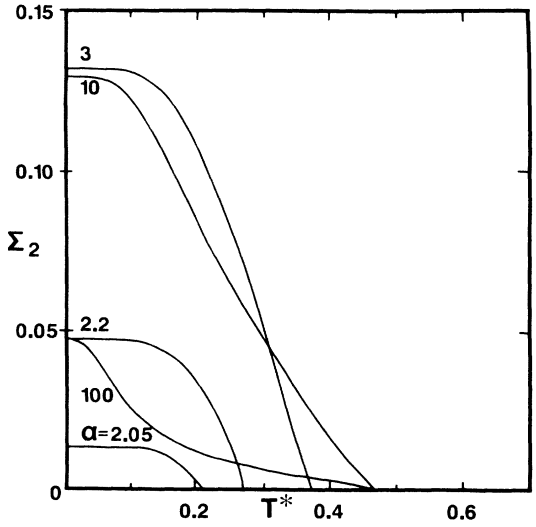


FIG. 6. The fluctuation contribution $\Sigma_2$ vs $T^*$ for different values of $\alpha$.

quasiparticles enhances the short-range phase coherence of the array. For a given $\alpha$, $\Sigma_2$ is a monotonically decreasing function of $T^*$ which vanishes above $T_0$. So to order $1/z$, the short-range order of the normal state is unchanged by the coupling of charge fluctuations on neighboring grains. In the superconducting state, the dissipative correction $\Sigma_2$ is a nonmonotonic function of $\alpha$ which vanishes when $\alpha=\infty$ or $\alpha<2$. At zero temperature, $\Sigma_2$ has a maximum of about 0.13 when $\alpha\approx 5$.

## V. APPLICATIONS AND DISCUSSION

This paper has demonstrated that both the long-range and short-range phase coherence of a granular superconductor are enhanced by the virtual tunneling of quasiparticles between grains. In Sec. III, I found that the transition temperature is increased and the critical parameter $\alpha_c$ is decreased by quasiparticle dissipation. Hence, the $\alpha-T^*$ phase space available for superconductivity is enlarged by the off-diagonal capacitance in the Lagrangian. Similarly, the short-range order parameter calculated in Sec. IV increases as $\lambda$ decreases.

If $\gamma$ replaced $\alpha$ as the critical parameter of the theory, these conclusions would have to be modified. At a fixed $\gamma$, the transition temperature is no longer a monotonically decreasing function of $\lambda$. The critical value $\gamma_c$ is actually *larger* for $\lambda=0$ than for $\lambda=\infty$. Since $T_c^*$ is a decreasing function of $\lambda$ when $\gamma\gg 1$, the curves of $T_c^*$ versus $\gamma$ must cross for different $\lambda$. Obviously, the results of this work are more straightforward in terms of $\alpha$, which is the natural choice for the critical parameter of the theory. Because the MF theory is parametrized in terms of $\alpha$ rather than $\gamma$, the $1/z$ expansion about the MF solution is also much simpler in terms of $\alpha$.

Regardless of which parameter is used, however, the physical predictions of the theory are the same. The results for either $\alpha_c$ or $\gamma_c$ can be rewritten as

$$\frac{zJ}{e^2}(z\Delta C+C_0)_c=2+\frac{1}{z}\Gamma(\lambda_c)\ , \tag{63}$$

$$\Gamma(\lambda)=\frac{7}{5}-\frac{1}{1+\lambda}+\frac{2}{(1+\lambda)^2}\ , \tag{64}$$

where

$$\lambda_c=\left[\frac{C_0}{z\Delta C}\right]_c \tag{65}$$

is the ratio of the diagonal and nearest-neighbor capacitances at their critical values. As mentioned earlier, the nearest-neighbor capacitance $\Delta C$ may contain two contributions: a geometric contribution from the electrostatic potential and a virtual contribution from quasiparticle tunneling. When both contributions are present, $\Delta C$ equals the sum of the virtual and electrostatic capacitances. If in addition, the diagonal capacitance $C_0$ equals zero, then $\lambda_c=\infty$ and $\Gamma=\frac{7}{5}$. This situation was discussed by Mirhashem and Ferrell in Ref. 21. In the remainder of this paper, I neglect the electrostatic contribution to the nearest-neighbor capacitance but I retain the diagonal capacitance $C_0$. It is straightforward to extend the following discussion to the general case with electrostatic contributions to both $C_0$ and $\Delta C$.

Using Eqs. (1) and (4) for $\Delta C$ and $J$ in terms of the normal-state conductance $\sigma_N$ and the zero-temperature gap $\Delta(0)$, Eq. (63) can be rewritten in terms of the critical resistance parameter

$$r=\frac{\sigma_N^{-1}}{R_Q}\ , \tag{66}$$

where $R_Q=h/(2e)^2=6.5\ \mathrm{k\Omega}$ is the quantum resistance. If $r_c$ is the critical value of $r$ which separates the normal and superconducting states, then

$$\frac{3}{32}\frac{z^2}{r_c^2}+\frac{x}{2}\frac{z}{r_c}=2+\frac{1}{z}\Gamma(\lambda_c)\ , \tag{67}$$

where $x$ is defined by

$$x=\frac{C_0\Delta(0)}{e^2}\ . \tag{68}$$

This new dimensionless parameter depends only on the properties of a single grain. Since $C_0$ is proportional to the grain diameter, $x$ can be considered the dimensionless grain diameter of the theory. In terms of $x$ and $r_c/z$, $\lambda_c$ is written

$$\lambda_c=\frac{16}{3}\frac{r_c}{z}x\ . \tag{69}$$

For a fixed value of $x$, Eqs. (66) and (67) provide the critical value of the junction conductance $\sigma_N$. When $r>r_c$, $\alpha<\alpha_c$ and the granular array is normal. When $r<r_c$, $\alpha>\alpha_c$ and the granular array is superconducting.

Like every other critical parameter, $r_c/z$ can be expanded in powers of $1/z$ as

$$\frac{r_c}{z}=r_0+\frac{1}{z}r_1\ , \tag{70}$$

where both $r_0$ and $r_1$ depend on the dimensionless grain diameter $x$. Using this expansion in Eq. (67) for $r_c/z$, I find that

$$r_0=\tfrac{1}{8}x+\tfrac{1}{8}\sqrt{x^2+3}\ , \tag{71}$$

$$r_1=-\frac{2\Gamma(\lambda_0)r_0^2}{\sqrt{x^2+3}}\ , \tag{72}$$

where $\lambda_0=16r_0x/3$. Because $r_1<0$, the coupling of charge and phase fluctuations suppresses $r_c/z$ from its MF value. The resistive components $r_0$ and $-r_1$ are plotted versus $x$ in Fig. 7. The MF threshold $r_0(x)$ is a monotonically increasing function of $x$ which approaches $x/4$ as $x\to\infty$. The fluctuation correction $r_1(x)$ has a shallow maximum at $x\sim 1$ and approaches $-7x/40$ as $x\to\infty$. For all possible values of $z$, $r_c/z$ diverges in the limit of infinite self-capacitance. Therefore, the normal-state resistivity of the array becomes irrelevant when the grains become very large.

If the grains are very small, however, $x=0$ and

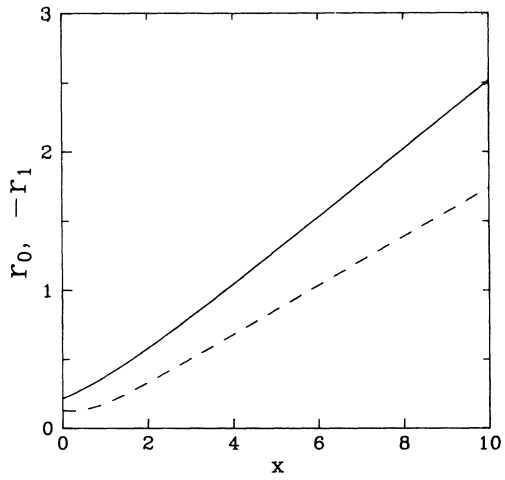


FIG. 7. The resistive components $r_0$ (solid) and $-r_1$ (dashed) vs $x$.

$$\frac{r_c}{z}=\frac{\sqrt{3}}{8}\left[1-\frac{3}{5}\frac{1}{z}\right], \tag{73}$$

in agreement with Ferrell and Mirhashem.[15] The scale of the grain size is determined by the parameters $x$ and $\lambda_0 \sim x$. So it is safe to neglect the self-capacitance when $x \ll 1$ or, equivalently, when $C_0 \ll e^2/\Delta(0)$. Since $x < 10^{-2}$ in the experiments of Refs. 18 and 19, the transition temperature in those experiments is controlled by the normal-state conductance of a Josephson junction.

Unfortunately, Eq. (73) for $r_c/z$ disagrees with the experimental observations in an essential way. In a wide variety of granular and amorphous films, the measured value[1,2] of $r_c$ is very close to 1. These experiments suggest that the resistive threshold does not depend on $z$, in disagreement with the theoretical prediction. For a square array with $z=4$, the MF result $r_c=0.866$ is not far from the observed threshold. But fluctuations lower this threshold further from the observed value. Although some measurements[30,31] of $r_c$ in granular films do favor a lower value in closer agreement with the analytic prediction, the disagreement with the experimental work of Haviland *et al.*[2] is discouraging.

Other theories of quasiparticles dissipation run into similar difficulties. The spin-wave approximation[12] replaces the periodic Josephson potential by a quadratic potential. Using this approach, the resistive threshold again depends on the coordination number of the lattice. Even more troubling, the threshold $r_c$ vanishes in the limit $x \rightarrow 0$.

Many workers[13,14,20] have studied a different Hamiltonian in which shunt resistors are placed across each junction in the array. The Ohmic dissipation in the shunt resistors is quite different than the quasiparticles dissipation in the junctions between the grains. Because the shunt resistance does not depend on temperature, the energy spectrum of the normal electrons does not contain an energy gap. Therefore, *real* quasiparticle excitations are allowed even at zero temperature and the normal-state resistivity of the array approaches a finite value at $T=0$. The quasiparticle model, on the other hand, assumes that only virtual tunneling is possible at $T=0$ because of the gap $\Delta(0)$ in the quasiparticle energy. So the normal-state resistivity diverges as $T \rightarrow 0$.

The predictions of the Ohmic model for $r_c$ also depend on the coordination number $z$. Within the Caldeira-Leggett formalism,[32] the dissipation of normal electrons in the shunt resistors is modeled by coupling the phase differences across each junction to a collection of harmonic oscillators. Various techniques have been used to study the Caldeira-Leggett Hamiltonian for a Josephson array. A perturbative expansion[13,20] of the resistivity in powers of $J$ yields a threshold of $r_c=z$ while the spin-wave approximation[14] yields a threshold $r_c=z/2$, both for a hypercubic lattice in any dimension. Obviously, these predictions also disagree with the experimental measurements.

Several ideas have been suggested to explain this disagreement. One possibility[1] is that the theoretical predictions for a lattice cannot be directly applied to random, granular systems. The effective coordination number in a disordered material may be determined by the weakest bond in the percolative backbone. According to this suggestion,[1] $z$ should be replaced by 2 in a disordered material. But even with $z=2$, the $1/z$ expansion still disagrees with the observed threshold.

Recently, Pang[33] and Fisher *et al.*[34] have separately obtained resistive thresholds $r_c$ that are independent of the coordination number $z$. In agreement with most measurements in granular or amorphous films, Pang[33] finds that $r_c=1$. Taking a different route, Fisher *et al.*[34] find that $r_c=8/\pi$ in close agreement with new measurements on ordered, fabricated Josephson arrays.[3] Both theories hinge on the special properties of two-dimensional systems. For a three-dimensional array, nonuniversal values of $r_c$ are expected.[34]

Of course, the $1/z$ expansion is not really valid in two dimensions, anyway. It is well known that the low-energy excitations of the Hamiltonian $H$ are spin waves.[4] The MF Hamiltonian replaces those Goldstone modes by a discrete set of eigenvalues. Since spin waves destroy the long-range order in two dimensions, an expansion about MF theory cannot be taken seriously for two-dimensional arrays. In two dimensions, the MF values of the order parameter and transition temperature must be exactly canceled by the fluctuation corrections, summed to all orders in $1/z$. So the phase coherence of the array is completely destroyed and the spin-wave spectrum totally recovered only after all the fluctuation corrections have been calculated. Therefore, the results of this formalism for $r_c$ should probably be restricted to three dimensions.

Beside the predictions for $r_0$ and $r_1$, the other important result of this paper is a quantitative theory for the transition temperature as a function of the intergrain capacitance $\Delta C$, the self-capacitance $C_0$, the tunneling probability $J$, and the coordination number $z$. This theory incorporates the coupling of charge and phase fluctuations on neighboring grains, which was neglected by MF theory. Like the predictions for $r_c$, the predictions for $T_c$ are also strictly valid only in three dimensions. Unlike the MF analysis of Ref. 26, the present work finds

that the superconducting transition is never reentrant, at least to order $1/z$.

This result disagrees with Simanek[5] and Kampf and Schon,[35] who find that a reentrant phase transition is possible, even without dissipation. Unlike this paper, Refs. 5 and 35 use nonperiodic eigenfunctions to evaluate the order parameter. By treating the phase as an extended variable over the range $[-\infty,\infty]$ instead of the range $[-\pi,\pi]$, Refs. 5 and 35 violate the quantization condition on the Cooper-pair numbers $n_i$, which was carefully maintained in the present work. The reentrance obtained by Simanek and by Kampf and Schnon is a consequence of treating the phase as an extended variable and violating the Cooper-pair quantization.

Under special conditions, Zwerger *et al.*[36] have shown that the phase dynamics[37] of the Hamiltonian is *unchanged* when nonperiodic eigenfunctions are allowed. In particular, when the dissipation is Ohmic, a phase-periodic description and the nonperiodic description of Caldeira and Leggett[32] yield the same voltage response for the array.[13] If quasiparticle dissipation is used instead, then the two descriptions are inequivalent and the phase cannot be treated as an extended coordinate. But regardless of the source of dissipation, it is *always* valid to calculate the order parameter of a periodic Hamiltonian using periodic eigenfunctions. With such a phase-periodic description, the transition temperature of a granular array is nonreentrant to order $1/z$.

To conclude, this paper has used the $1/z$ expansion to study the effects of an off-diagonal grain capacitance, which is induced by the virtual tunneling of quasiparticles at zero temperature. Due to the coupling of phase and charge fluctuations on neighboring grains, quasiparticle dissipation reduces the critical value of $\alpha$ and enhances the transition temperature. The predictions of this theory should be valid in a three-dimensional granular superconductor or Josephson array. It is hoped that future experiments in three-dimensional systems will substantiate the predictions of this work.

## ACKNOWLEDGMENTS

I would like to acknowledge useful conversations with Dr. B. Mirhashem.

## APPENDIX A

In this appendix, I collect the results for the linearized, fluctuation corrections $\aleph_1$ and $\aleph_2$ and for the functions $A_i$ and $F_i$. It is straightforward to show that in the limit of small $M_0$, $\aleph_1$ and $\aleph_2$ are given by

$$\lim_{T^*\to T_0}\aleph_1=(zJ)^2\int_0^\beta d\tau_1\int_0^{\tau_1}d\tau_2 F_1(\tau_1-\tau_2)\{F_3(\tau_1,\tau_2)+F_4(\tau_1,\tau_2)-2M_0[F_1(\tau_1)-F_1(\tau_1-\tau_2)]\}\ , \tag{A1}$$

$$\lim_{T^*\to T_0}\aleph_2=-\frac{4(zJ)^2}{\alpha(1+\lambda)}\int_0^\beta d\tau_1\int_0^{\tau_1}d\tau_2[A_1(\tau_1-\tau_2)A_4(\beta-\tau_1)-A_1(\tau_2+\beta-\tau_1)A_4(\tau_2)]$$
$$+\frac{16(zJ)^2}{\alpha^2(1+\lambda)^2}\left[A_2\int_0^\beta d\tau_1\int_0^{\tau_1}d\tau_2 A_3(\beta-\tau_2,\tau_1-\tau_2)-\tfrac{1}{2}\beta^2A_2^2M_0\right]\ . \tag{A2}$$

Notice that both $\aleph_1$ and $\aleph_2$ are functions only of $\alpha$ and $T^*$.

The functions $A_1$ and $F_i$ are defined below:

$$A_1(\tau)=\langle\sin\hat{\phi}_1(\tau)\hat{n}_1(0)\rangle_{\rm MF}\ , \tag{A3}$$

$$A_2(\tau)=\langle\hat{n}_1(\tau)\hat{n}_1(0)\rangle_{\rm MF}\ , \tag{A4}$$

$$A_3(\tau_1,\tau_2)=\langle\cos\phi_1(\tau_1)\hat{n}_1(\tau_2)\hat{n}_1(0)\rangle_{\rm MF}\ , \tag{A5}$$

$$A_4(\tau_1,\tau_2)=\langle\cos\phi_1(\tau_1)\sin\hat{\phi}_1(\tau_2)\hat{n}_1(0)\rangle_{\rm MF}\ , \tag{A6}$$

$$F_1(\tau)=\langle\cos\hat{\phi}_1(\tau)\cos\hat{\phi}_1(0)\rangle_{\rm MF}\ , \tag{A7}$$

$$F_2(\tau)=\langle\sin\hat{\phi}_1(\tau)\sin\hat{\phi}_1(0)\rangle_{\rm MF}\ , \tag{A8}$$

$$F_3(\tau_1,\tau_2)=\langle\cos\hat{\phi}_1(\tau_1)\cos\hat{\phi}_1(\tau_2)\cos\hat{\phi}_1(0)\rangle_{\rm MF}\ , \tag{A9}$$

$$F_4(\tau_1,\tau_2)=\langle\sin\hat{\phi}_1(\tau_1)\sin\hat{\phi}_1(\tau_2)\cos\hat{\phi}_1(0)\rangle_{\rm MF}\ . \tag{A10}$$

Since $A_4\sim M_0$, the function $A_2$ in Eq. (A2) can be replaced by its normal-state value given by Eq. (42).

## APPENDIX B

In this appendix, I collect the various combinations of the functions $A_i$ and $F_i$ which are required to evaluate $\aleph_1$ and $\aleph_2$, given by Eqs. (A1) and (A2). To lowest order in $M_0$,

$$\lim_{T^*\to T_0}F_1(\tau)=\frac{1}{2Z_{00}}\sum_m e^{-2U[m^2\beta+(2m+1)\tau]}\ , \tag{B1}$$

$$\lim_{T^*\to T_0}[F_3(\tau_1,\tau_2)+F_4(\tau_1,\tau_2)]=\frac{\alpha M_0}{2Z_{00}}\sum_m\frac{1}{4m^2-1}\Big(-\tfrac{1}{2}e^{-2U[m^2\beta-(2m+1)\tau_1]}-\tfrac{1}{2}e^{-2U[(m+1)^2\beta+(2m+1)\tau_2]}$$
$$+\tfrac{1}{2}e^{-2U[(m-1)^2\beta-(2m+1)\tau_2+4m\tau_1]}+\tfrac{1}{2}e^{-2U[m^2\beta+(2m+1)\tau_1-4m\tau_2]}$$
$$-e^{-2U[m^2\beta+(2m+1)(\tau_1-\tau_2)]}\Big), \tag{B2}$$

$$\lim_{T^*\to T_0}A_1(\tau)=\frac{i\alpha M_0}{4Z_{00}}\sum_m e^{-2U[m^2\beta+(2m+1)\tau]}\frac{1}{2m+1}, \tag{B3}$$

$$\lim_{T^*\to T_0}A_3(\tau_1,\tau_2)=-\frac{\alpha M_0}{2Z_{00}}\sum_m e^{-2U\beta m^2}\frac{m^2}{4m^2-1}+\frac{\alpha M_0}{4Z_{00}}\sum_m e^{-2U[m^2\beta+(2m+1)\tau_1]}\left[\frac{m+1}{2m+1}+\frac{m}{2m+1}e^{2U\tau_2(2m+1)}\right], \tag{B4}$$

$$\lim_{T^*\to T_0}A_4(\tau_1,\tau_2)=\frac{i}{2Z_{00}}\sum_m me^{-2U[m^2\beta-(2m-1)(\tau_1-\tau_2)]}, \tag{B5}$$

where $Z_{00}$ is defined by Eq. (43) and all sums run from $-\infty$ to $+\infty$.


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