Different approach to the theory of the quantum Hall effect

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It is shown that, using an improved method to solve time-dependent problems capable of handling divergences associated with quasi-infinite density of states, the theory of the integral quantum Hall effect can be worked out in the unperturbed Landau-state basis. This leads to a completely new derivation which, however, does not seem inconsistent with the widespread interpretation, mainly based on the properties of localized states in the collision-perturbed basis. Instead, due to noticeable simplifications, it offers the promise of a complete quantitative treatment.

As has been extensively described,[1] the ideas underlying the quantum Hall effect (QHE) rest on the concept of localized and extended states, respectively, ascribed to the wings and the central region of the collision-broadened Landau levels (LL). Level broadening avoids complications associated with the severe divergences in perturbation series, resulting from the high degeneracy of the unperturbed LL's. Unfortunately we are faced, on the other hand, with formidable difficulties in constructing properly localized and extended wave functions, so that, in spite of a number of attempts devoted to that question, with discussions and solutions of particular models,[1-3] there is no reliable and general quantitative theory based on the above picture.

It is evident, however, that insofar as divergence difficulties were overcome, the QHE theory could be worked out on the unperturbed basis, as well. The main purpose of this Brief Report is to demonstrate the possibility of such an alternative equivalent approach, avoiding direct construction of localized-extended states, in the framework of an improved method to solve the time-dependent Schrödinger equation in transport problems, capable of handling density-of-states divergences.[4,5] After briefly recalling the major steps of the method for clarity, we shall give an outline of its application to the QHE problem.

We start with the Laplace transform $R(\nu)$ of the Schrödinger equation of the many-body density matrix $\rho(t)$:

$$\nu R(\nu)+i\hbar^{-1}[H_0+V+F,R(\nu)]=\rho_0 , \tag{1}$$

with $\rho_0=\rho(0)$; $H_0=(\mathbf{p}+e\mathbf{A})^2/2m^*$ is the unperturbed Hamiltonian of electrons of effective mass m^*, in the Landau gauge $\mathbf{A}(0,Bx,0)$ for the magnetic field $\mathbf{B}$ in the z direction; V denotes the coupling with scatterers, and $F=eEx$ the coupling with the external electric field of strength E, applied along the x axis from $t=0$. Independent electrons are assumed. Diagonal matrix elements of V will be ignored.

Because of the presence of commutators, it is convenient to rewrite Eq. (1) in the Liouville space obtained by direct product of the Hilbert space (sustained by the eigenstates of H_0) and its dual. In that space, the Liouville kernel $\mathbb{K}_V$ is defined so that $\mathbb{K}_V X=i\hbar^{-1}[V,X]$ for any X. Quantum states are specified by double sets $cb,c_1b_1,\ldots$ of quantum numbers $c,b,c_1,b_1,\ldots$ specifying the unperturbed eigenstates of H_0 (with energy $\varepsilon_c,\varepsilon_b,\ldots$). A cb-c_1b_1 matrix element of $\mathbb{K}$ will be written as $\mathbb{K}_{c_1b}^{cb_1}$. Then, defining $\mathbb{d}=\nu\mathbb{I}+\mathbb{K}_{H_0}$, the c-b matrix element of Eq. (1) is written as

$$\mathbb{R}^c{}_b+\mathbb{d}_{cb}^{-1}(\mathbb{K}_{Vc_1b}^{cb_1}+\mathbb{K}_{Fc_1b}^{cb_1})\mathbb{R}^{c_1}{}_{b_1}=\mathbb{d}_{cb}^{-1}\rho_{0b}^c . \tag{2}$$

Note that $\mathbb{d}_{cb}=\nu+i\hbar^{-1}\varepsilon_{cb}$, with $\varepsilon_{cb}=\varepsilon_c-\varepsilon_b$. Summation over repeated indices is implicit throughout. In the next step, Eq. (2) is regarded as a linear system, and as such $\mathbb{R}$ is first written from Cramer's solution as

$$\mathbb{R}^u{}_\lambda=\mathbb{D}_{c\lambda}^{\mu b}(\mathbb{D}_{cb}^{cb})^{-1}\rho_{0b}^c/\mathbb{d}_{cb}\mathbb{D}(\mathbb{D}_{cb}^{cb})^{-1} . \tag{3}$$

$\mathbb{D}_{c\lambda}^{\mu b}$ stands for the $\mu\lambda$-cb minor and $\mathbb{D}$ for the determinant of the matrix $\mathbb{I}+\mathbb{d}^{-1}(\mathbb{K}_V+\mathbb{K}_F)$, respectively. $(\mathbb{D}_{cb}^{cb})^{-1}$ means division by the determinant of the matrix $\mathbb{S}_{cb}=\mathbb{I}+\mathbb{d}^{-1}(\mathbb{K}_V+\mathbb{K}_F\mathbb{Q}_{cb})$, which is intended to eliminate unlinked sets of transitions in the upper and lower determinants of Cramer's solution.[5,6] $\mathbb{Q}_{cb}$ denotes the projector on the subspace complementary to the cb state. Such determinant quotients can be expressed in terms of $\mathbb{S}_{cb}^{-1}$ matrix elements,[5] leading to geometrical series upon expansion. At this stage, however, as shown in Ref. 5, expression (3) of $\nu\mathbb{R}^\mu{}_\lambda$ is very close to usual perturbation series. To get a nontrivial result, we have to continue the procedure one step further, leading to

$$\nu\mathbb{R}^\mu{}_\lambda=i\hbar^{-1}eE\,\mathbb{D}_{z\lambda}^{\mu b}(\mathbb{D}_{zb}^{zb})^{-1}x_c^z\rho_{0b}^c/\mathbb{d}_{zb}\mathbb{D}(\mathbb{D}_{zb}^{zb})^{-1}+p^*(\mu,\lambda) .$$

$p^*(\mu,\lambda)$ means that the second term is obtained on interchanging λ and μ in the conjugate of the first one. On carrying out determinant divisions one obtains

$$\nu\mathbb{R}^{\mu}_{\lambda}=i\hbar^{-1}eE[\delta^{\mu}_{z}\delta^{b}_{\lambda}+\langle\mu\lambda|(\mathbb{I}+\mathbb{Q}_{zb}\mathbf{d}^{-1}\mathbb{K})^{-1}|zb\rangle]x^{z}_{c}\rho^{c}_{0b}/[\mathbf{d}_{zb}+\langle zb|\mathbb{K}(\mathbb{I}+\mathbb{Q}_{zb}\mathbf{d}^{-1}\mathbb{K})^{-1}|zb\rangle]+p^{*}(\mu,\lambda)\ . \tag{4}$$

$\mathbb{K}_V+\mathbb{K}_F$ is simply rewritten as $\mathbb{K}$, since in the linear theory $\mathbb{K}_F$ can be dropped from inside the fraction. The zero-order contribution, obtained for $\mu\lambda=zb$, is separated out, so that the index restriction $\mu\lambda\neq zb$ (as specified by $\mathbb{Q}_{zb}$) holds in $\mu\lambda$-zb matrix elements. The particular case in which x is diagonal in the unperturbed basis must be separately considered. Setting $x^{c}_{c}-x^{b}_{b}=x_{cb}$, we found similarly

$$\nu\mathbb{R}^{\mu}_{\lambda}=-(i\hbar^{-1})^{2}eE\langle\mu\lambda|(\mathbb{I}+\mathbb{Q}_{zb}\mathbf{d}^{-1}\mathbb{K})^{-1}|zb\rangle V^{z}_{c}\mathbf{d}^{-1}_{cb}x_{cb}\rho^{c}_{0b}/[\mathbf{d}_{zb}+\langle zb|\mathbb{K}(\mathbb{I}+\mathbb{Q}_{zb}\mathbf{d}^{-1}\mathbb{K})^{-1}|zb\rangle]+p^{*}(\mu,\lambda)\ . \tag{5}$$

The steady-state value of the density matrix is derived from Eqs. (4) and (5) using the elementary rule $\rho(t\rightarrow\infty)=\lim_{\nu\rightarrow+0}[\nu R(\nu)]$. A major advantage is the fractional form which will preserve convergence in the calculation of observable mean values, even if some sets of matrix elements exhibit a divergent behavior.

We now turn to the application of Eqs. (4) and (5) to the QHE. In the N,k (k denotes the usual k_y momentum component) Landau representation, the x operator is given by

$$x=l(a+a^{\dagger})/\sqrt{2}-l^{2}k\ .$$

$l=\sqrt{\hbar/eB}$ is the cyclotron radius and $a,a^{\dagger}$ are the customary annihilation and creation operators. The first term on the right-hand side refers to the relative orbital coordinate of electrons, and the second one, which relates to the center of orbit, is diagonal in the Landau representation. Hence, making use of Eqs. (4) and (5) together with the well-known current density components $J_x=i(e\hbar/m^{*}l\sqrt{2})(a-a^{\dagger})$ and $J_y=-(e\hbar/m^{*}l\sqrt{2})(a+a^{\dagger})$, the conductivity components $\sigma_{jx}=J^{\lambda}_{j\mu}\rho^{\mu}_{\lambda}/E$ ($j=x,y$) are given by

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yx}\end{bmatrix}=\begin{bmatrix}-i\\ 1\end{bmatrix}\frac{e^{2}(a\mp a^{\dagger})^{\lambda}_{\mu}}{2m^{*}\omega_{\mu\lambda}}\frac{\langle\mu\lambda|\mathbb{K}(\mathbb{I}+\mathbb{Q}_{zb}\mathbf{d}^{-1}\mathbb{K})^{-1}|zb\rangle}{\mathbf{d}_{zb}+\langle zb|\mathbb{K}(\mathbb{I}+\mathbb{Q}_{zb}\mathbf{d}^{-1}\mathbb{K})^{-1}|zb\rangle}[(a+a^{\dagger})^{z}_{c}+i\hbar^{-1}l\sqrt{2}V^{z}_{c}\mathbf{d}^{-1}_{cb}k_{cb}]\rho^{c}_{0b}+\text{c.c.} \tag{6}$$

($\mu\lambda\neq zb$). The equilibrium density matrix will be taken as

$$\rho_{0}=\exp[-\beta(H_{0}+V-\eta n)]/\mathrm{Tr}\exp[-\beta(H_{0}+V-\eta n)] \tag{7}$$

($\beta=1/k_{B}T$). η is the Fermi energy and n the number-of-fermions operator. If scattering is missing, ρ_0 becomes, in a second quantization scheme,[7]

$$\rho_{0}=\prod_{\alpha}[f_{\alpha}c^{\dagger}_{\alpha}c_{\alpha}+(1-f_{\alpha})c^{\dagger}_{\alpha}c_{\alpha}]\ , \tag{8}$$

f_α is the Fermi occupation function of the individual state $\alpha\equiv N,k$. The same form would also hold for the individual eigenstates of H_0+V, i.e., broadened LL's. These, however, are not known and we shall use, instead, expansion of the exponential operators in Eq. (7), in terms of commutators

$$\begin{aligned}&[H_{0},V]=\sum_{\alpha'\alpha}\varepsilon_{\alpha',\alpha}V^{\alpha'}_{\alpha}c^{\dagger}_{\alpha'}c_{\alpha}\ ;\\ &[H_{0},[H_{0},V]]=\sum_{\alpha'\alpha}\varepsilon^{2}_{\alpha'\alpha}V^{\alpha'}_{\alpha}c^{\dagger}_{\alpha'}c_{\alpha}\ ;\\ &\cdots\end{aligned} \tag{9}$$

By using the approximants of Baker-Campbell-Hausdorff,[8] it can be shown that interband matrix elements do not contribute the equilibrium statistics to within a very large accuracy. We shall thus restrict (9) to the case in which α and α' both belong to the same LL, and look for the limit $\varepsilon_{\alpha'\alpha}\rightarrow 0$. Then, the exponential in Eq. (7) is given to first order in $\varepsilon_{\alpha'\alpha}$ (or equivalently $[H_0,V]$) by

$$\exp[-\beta(H_{0}+V-\eta n)]=\exp(-\beta V+\tfrac{1}{2}\beta^{2}[H_{0},V]+\cdots)\times\exp[-\beta(H_{0}-\eta n)]\ . \tag{10}$$

To begin with, we consider the ideal case in which there is no scattering. Then, the zero-order term ($\mu\lambda=zb$) only remains in Eq. (4), leading to

$$\begin{bmatrix}\sigma^{(i)}_{xx}\\ \sigma^{(i)}_{yx}\end{bmatrix}=\begin{bmatrix}i\\ -1\end{bmatrix}\frac{e^{2}}{2m^{*}\omega_{zb}}(a\mp a^{\dagger})^{z}_{b}(a+a^{\dagger})^{z}_{b}\rho^{b}_{0b}+\text{c.c.} \tag{11}$$

The selection rules of $a,a^{\dagger}$ impose diagonal elements of ρ_0. When the magnetic field is varied, the Fermi level quickly jumps from a LL to the next one and remains close to the latter during filling (or emptying). Assume the $(N-1)$ LL is full and the N LL partly occupied. Due to the occupation functions arising in Eq. (8), the relevant collective b state is restricted to the $(N-1)$ and N LL's, and since $\mathbf{d}_{zb}=i\Omega$ ($\Omega=eB/m^{*}$), we find

$$\begin{aligned}&\sigma^{(i)}_{xx}=0\ ;\\ &\sigma^{(i)}_{yx}=-(e^{2}/m^{*}\Omega)\sum_{k}[N(1-f_{Nk})+(N+1)f_{Nk}]\ .\end{aligned} \tag{12}$$

We have taken $f_{N-1k}=1$, $f_{N+1k}=0$, and we shall write

$f_{Nk}=f_N$, since f_{Nk} is independent of k. Further, by summing over k, the LL's multiplicity, say eB/h, is factorized out. Hence the expected result,

$$\sigma_{yx}^{(i)}=-e^2(N+f_N)/h \ . \tag{13}$$

Coming back to the collision case, we first consider in Eq. (6) the contribution of the relative orbital motion to the Hall conductivity. Because of the LL degeneracy the upper and lower matrix elements involve strongly divergent terms, every time the $\mathbb{d}^{-1}$'s are taken between states lying inside the same LL. In a more realistic way, this amounts to considering a very high density of states in a very narrow band. Consequently, the relaxation frequency described by the $\langle zb||zb\rangle$ element in the denominator, is of much larger magnitude than $|\mathbb{d}_{zb}|=\Omega$, which will be ignored. Similarly, the $\langle\mu\lambda|zb\rangle$ elements, in the numerator of (6), contain divergent terms. The point is that all of them turn out to cancel with one another if the restriction $\mu\lambda\neq zb$ is ignored. This property is due to the selection rule of the $a,a^\dagger$, together with the particular structure of the Landau spectrum. It follows that the sum reduces to the opposite of the term $\mu\lambda=zb$, i.e.,

$$\sum_{\mu\lambda\neq zb}(a+a^\dagger)^\lambda_\mu \mathbb{d}^{-1}_{\mu\lambda}\langle\mu\lambda|\mathbb{K}(\mathbb{I}+\mathbb{Q}_{zb}\,\mathbb{d}^{-1}\mathbb{K})^{-1}|zb\rangle(a+a^\dagger)^z_c=-(a+a^\dagger)^b_z\mathbb{d}^{-1}_{zb}\langle zb|\mathbb{K}(\mathbb{I}+\mathbb{Q}_{zb}\mathbb{d}^{-1}\mathbb{K})^{-1}|zb\rangle(a+a^\dagger)^z_c \ . \tag{14}$$

As a result, the main fraction in Eq. (6) reduces to -1, which again imposes diagonal matrix elements of ρ_0. Making use of Eqs. (7) and (10), collisions are found to be eliminated from ρ_0 in the limit of zero bandwidth ($[H_0,V]\to 0$), so that the ideal value of σ_{yx} is just recovered:

$$\sigma_{yx}^{(o)}=-e^2(N+f_N)/h \ . \tag{15}$$

Consider next the center of orbit contribution in Eq. (6). Again divergent terms cancel in the numerator, apart from the index restriction, and the fraction reduces to -1, as previously. We are left with

$$\sigma_{yx}^{(oc)}=-(e^2l/\sqrt{2}m^*\omega_{zb})(a+a^\dagger)^b_z i\hbar^{-1}V^z_c\mathbb{d}^{-1}_{cb}k_{cb}\rho^c_{0b}+\text{c.c.} \tag{16}$$

As observed, owing to the presence of k_{cb}, the result is now directly dependent on *nondiagonal* matrix elements of ρ_0. To zero order of ρ_0 {i.e., $[H_0,V]=0$ in Eq. (10)}, V can be diagonalized inside the N level, together with the center of orbit coordinate $-l^2k$ and, therefore, the conductivity vanishes. In fact, the only first order term in ρ_0 yields a finite result because the denominator $\mathbb{d}_{cb}=i\hbar^{-1}\varepsilon_{cb}$ in Eq. (16) is of the same order. On the other hand, due to the $a,a^\dagger$ selection rules, the V^z_c's involve only $(N-1,k)\to(N,k')$ and $(N,k)\to(N+1,k')$ individual transitions which can be associated as follows, by using the properties of V-matrix elements between the LL's:[9]

$$\sqrt{N+1}\,V^{N+1k}_{Nk'}-\sqrt{N}\,V^{Nk}_{N-1k'}$$

$$=(l/\sqrt{2})(k-k')V^{Nk}_{Nk'}+\cdots \ ,$$

aside from terms which are ruled out in further integrations over k,k'. Combining this with the first-order limit of ρ_0 as $\varepsilon_{cb}\to 0$, one gets

$$\sigma_{yx}^{(oc)}=\frac{e^2f_N}{h}\frac{\sum_{kk'}\frac{1}{2}l^2(k'-k)^2V^{Nk}_{Nk'}\langle Nk'|\exp(-\beta V)|Nk\rangle}{\sum_k\langle Nk|V\exp(-\beta V)|Nk\rangle} \ . \tag{17}$$

This conductivity is thus given by a Boltzmann-like average of the squared momentum transfer $(k'-k)^2$ over all collision events. The latter is extremely close to the scaling value $2/l^2$. This may be understood as follows. The $V^{Nk'}_{Nk}$'s involve products of Gaussian peaks in $\exp[-l^2(k'-k)^2/4]$ and $\exp[-l^2(k'+k)^2/4]$ by polynomial functions of $(k'-k)^2$ and $(k'+k)^2$. Except for the constant term of polynomials, all others (which stem from both the oscillatory part of Landau wave functions for $N>0$ and the finite range of the collision potential) give rise to oscillating contributions which are smoothed out by randomization of scatterers. This corresponds physically to the collision breaking of cyclotron orbits and the whitening of the potential spectrum. Instead, the mean value of $l^2(k'-k)^2/2$, as calculated with the remaining Gaussian peaks, is found to be just 1, from statistical arguments. Hence, $\sigma_{yx}^{(oc)}=e^2f_N/h$ and the quantized plateaus are recovered from Eqs. (15) and (17):

$$\sigma_{yx}=\sigma_{yx}^{(o)}+\sigma_{yx}^{(oc)}=-Ne^2/h \ ,$$

the more precisely as fluctuations are expected to be very small on account of the very large number of collisions involved in the exponential expansions of Eq. (17).

The last point is concerned with the diagonal conductivity. Through inspection of the expansions arising in expression (6) of σ_{xx}, it can be shown that, given an initial b state, every z state can be associated with another one for which the related angular brackets are changed into their conjugates. This straightforwardly entails that $\sigma_{xx}=0$ *insofar as* $\mathbb{d}_{zb}$ (of magnitude Ω) *can be neglected relative to the relaxation frequency,* which is certainly true along the plateaus ($f_{N-1}=1$) but not necessarily in the intermediate region, where the N level is not completely full. Then, additional transitions occur for which the effective density of states is strongly reduced [in a ra-

tio up to the order of $\exp(-\hbar\Omega/2k_BT)$] by the exclusion principle and, accordingly, the relaxation frequency is no longer much larger than Ω, giving rise to finite contributions to σ_{xx}.

In conclusion, we believe that the interpretation of the QHE on the unperturbed Landau state basis is possible, insofar as the theory is able to deal with divergent density of states. In particular, finite and correct results seem attainable in the present method, even though some detailed problems remain. Since the formal difference rests on the use of a distinct state basis, no discrepancy is expected to arise in the physical content with existing ideas. Indeed, we notice the crucial contribution of collisions and the related LL broadening in expression (7) of the initial density matrix. Disorder effects have a decisive role in the calculation of the momentum average (17) and, consequently, in the occurrence of quantization. Localization effects can be viewed as being encompassed by an infinite relaxation rate, with the major advantage being that a detailed knowledge of localized and extended states is no longer required. We also notice that divergences are now lifted in a way similar to L'Hopital's rule leading to an analytical result [Eqs. (15) and (17)], instead of what occurs in existing methods, such as the self-consistent Born approximation,[10] which usually ends either in approximate expressions or numerically.

Finally, calculations seem much more convenient to carry through in the unperturbed basis, and the method is very likely capable of leading to a complete quantitative theory of the QHE, which is as yet unavailable.

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