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Generalized critical-state model for hard superconductors

Ming Xu,* Donglu Shi, and Ronald F. Fox*
Materials Science Division, Argonne National Laboratory, Argonne, Illinois 60439

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A generalized model for hard superconductors has been developed to unify all previous forms of the critical-state model. The magnetization versus applied-field curves generated from this model are shown to be physically consistent with the original Bean model. Other expressions of the critical current density can all be derived from this generalized critical-state model.

Hard superconductors have been observed to exhibit magnetic hysteresis. Various models have been developed to interpret this magnetic behavior, which is essential in understanding the nature of the mixed state of hard superconductors. For example, Bean proposed a so-called critical-state model in which the hysteresis is connected with a macroscopic parameter J_c, the critical current density.[1,2] The Bean model assumes that the critical current density J_c is a constant at a given temperature,

$$J_c(H_i,T)=J_c(T) \ , \tag{1}$$

where H_i is the local magnetic field and T is the temperature. Bean also pointed out in the model that J_c was directly determined by the microstructure of the superconductors.

Anderson[3] and Kim, Hempstead, and Strnad[4] later modified the Bean model; they suggested that J_c should vary with the local magnetic field and should have the form

$$J_c(H_i,T)=J_c(T)/(1+H_i/H_0) \ , \tag{2}$$

where H_0 is a macroscopic materials parameter with the dimension of field. The model was found to agree well with the experimental results for some conventional superconductors when the sample was assumed to be a solid cylinder.

Equation (2) indicates that the field dependence of J_c is associated with the term H_i/H_0, which may vary considerably among different systems. Watson showed that, for some systems, the condition $H_0 \gg H_i$ is satisfied and that Eq. (2) leads to a simple linear field dependence of J_c,[5] $J_c(H_i)=A-CH_i$, where A and C are constants that contain the macroscopic materials parameters.

In considering more specific pinning mechanisms, a power-law field dependence of J_c was developed by Irie and Yamafuji:[6]

$$J_c(H_i,T)=K(T)/H_i^n \ , \tag{3}$$

where K is a materials parameter and n directly reflects pinning strength.

Based on the magnetization data from cold-worked Nb-Zr wires, Fietz *et al.* found that their experimental results were excellently fitted with an empirical formula,[7]

$$J_c(H_i,T)=J_c(T)\exp(-H_i/H_0) \ . \tag{4}$$

They pointed out that Eq. (4) was obtained by several trial functions and that the critical current density calculated by using this equation agreed well with the experimental transport J_c data in Nb-25% Zr wire. They also pointed out that the Kim model was unable to fit the experimental data above 15 kOe, while Eq. (4) was valid up to 40 kOe.

In recent research of high-T_c superconductors, Eqs. (1)-(4) have been extensively used to study the critical current behavior and the motion of magnetic-flux lines,[8-12] and the validity of these models has been shown to vary from system to system. It has been shown that different hysteresis and J_c versus applied-field curves may be obtained by applying various forms of the critical-state models. A universal critical-state model for the critical current density is therefore to be established which is applicable in all situations.

In this paper we develop such an expression for the magnetization of hard superconductors. The expression is a generalization of Eqs. (1)-(4) and has the following form:

$$J_c(H_i,T)=J_c(T)/[1+H_i/H_0(T)]^{\beta} \ , \tag{5}$$

where β is a dimensionless constant. We first indicate that Eq. (5) is the general form of Eqs. (1)-(3). It is clear that $\beta=0$ and 1 give the Bean [Eq. (1)] and Kim [Eq. (2)] models, respectively. Equation (3) is obtained when the condition $H_i/H_0 \gg 1$ is satisfied in Eq. (5) and we set $K=J_c(T)H_0^{\beta}$ and $n=\beta$.

We now show that Eq. (4) can be generalized to Eq. (5) as $H_i/H_0 \ll 1$ and $\beta \gg 1$, with $H_0/\beta=\mathcal{H}_0$, where $\mathcal{H}_0$ is a finite value with dimension of field. Taking the limit of $J_c(H_i)$ with the preceding conditions, and letting H_i/H_0

$=x$, we have

$$\lim_{x\to 0} J_c(H_i,T) = J_c(T)/\lim_{x\to 0}(1+x)^{\beta}$$
$$= J_c(T)/\lim_{x\to 0}(1+x)^{(1/x)(H_i/\mathcal{H}_0)}$$
$$= J_c(T)\exp(-H_i/\mathcal{H}_0)\,. \quad (6)$$

Note that in the deriving Eq. (6), we have used the relation

$$\lim_{x\to 0}(1+x)^{1/x} = e\,.$$

It can be seen that Eq. (6) has the same form as Eq. (4) and that their physical meanings are identical.

Equation (5) has three adjustable parameters: $J_c(T)$, H_0, and β. As defined earlier, J_c is the macroscopic critical current density which is directly connected with the microstructure of the materials. Although the physical meaning of H_0 is not yet clear, it is probably an intrinsic material parameter which may be associated with the critical fields (H_{c1}, H_c, and H_{c2}) of the superconductors. It is transparent that Eq. (3) (the power-law dependence) and Eq. (4) (the exponential dependence) are the special cases of Eq. (5) with the extreme conditions of $H_0/H_i \ll 1$ and $H_0/H_i \gg 1$, respectively. The physical implication of these conditions is that Eq. (3) is applicable for materials with small H_0 values while Eq. (4) is more suitable for high-H_0 superconductors. As has been pointed out by Kim *et al.*,[4] the H_0 value is approximately the thermodynamic critical field H_c for Nb_3Sn and Nb-Zr superconductors.

We have indicated above that n $(=\beta)$ in Eq. (3) is related to the specific pinning mechanisms in the system. With the condition $H_i/H_0 \gg 1$, the case $\beta = 1$ in Eq. (5) is equivalent to $n=1$ in Eq. (3). It has been shown in deriving Eq. (6) that Eq. (4) is the special case of Eq. (5) when $\beta \gg 1$ and $H_0/H_i \gg 1$. As we have mentioned earlier, Feitz *et al.* found that only in the low-field region (3–15 kOe) did the Kim model [Eq. (2), or $\beta = 1$ in Eq. (5)] result in a reasonable fitting with the experimental magnetization data in Nb-25% Zr wire. A much more accurate fitting was obtained in a wide field region (4–40 kOe) when Eq. (4) [or $\beta \gg 1$ in Eq. (5)] was employed.[7] Further, recent studies in high-T_c superconductors indicate that Eq. (4) is more appropriate for a strongly pinned $YBa_2Cu_3O_x$ system compared with other models.[10] These previous experimental results and our generalized model suggest that, in contrast to H_0 which may be intrinsic to the superconductor, β is a microstructure-sensitive materials parameter, through which the physical meanings of and relationships between the previous models of critical current density can be understood and clarified. Thus, all hard superconductors including high-T_c ceramics may be categorized by the parameters H_0 and β with this generalized critical-state model in terms of intrinsic and extrinsic materials properties such a critical magnetic fields and microstructures.

It appears that Eqs. (2)–(4) are mathematically independent of each other. However, they can all be derived to have the form of the Bean model [Eq. (1)], if we assume $H_i/H_0 \ll 1$, $n=0$, and $H_i/H_0 \ll 1$, in Eqs. (2), (3), and (4), respectively. By taking a higher order of approximation in H_i/H_0 with the condition of $H_i/H_0 \ll 1$, we find that both Eqs. (2) and (4) result in the same linear field dependence of critical current density $J_c(H_i) = J_c(T)(1 - H_i/H_0)$.[9] Moreover, we note that Eqs. (3) and (4) are also identical when $n=1$ and $H_i/H_0 \gg 1$ is satisfied in Eqs. (3) and (4), respectively. Thus, we see close physical relationships between these models, and they are well connected by our generalized expression Eq. (5), which has a universal mathematical form.

It should be pointed out that although Eq. (1) gives a field-independent critical current density, Bean also considered an arbitrary dependence of J_c on field.[2] The differential equation for the magnetization of a slab with thickness of 2d is given by the form

$$dB/dH = H/\mathcal{H}^*(H),\ H \ll \mathcal{H}^*(H)\,, \quad (7)$$

where B is the induction $(B = 4\pi M + H)$ and $\mathcal{H}^*(H) = 4\pi J_c(H)d/c$ (c is the speed of light). With the definition of $\mathcal{H}^*(H)$, Eq. (7) can be written as

$$dB/dH = cH/4\pi J_c(H)d,\ H \ll H^*(H)\,. \quad (8)$$

The significance of Eq. (8) is that the relationship between J_c and H can be experimentally determined by the magnetization curve. Therefore, field dependence of critical current density is established without any assumption for various superconducting systems.

We now use Eq. (5) to develop an expression for magnetization, based on which the complete magnetic hysteresis can be calculated for various superconductor systems. We consider an infinite slab of thickness $2d$ with an external magnetic field parallel to the surface. There is a supercurrent density $J_c(H_i)$ associated with the local field H_i established in the slab, whose direction is perpendicular to the current. The lower critical field H_{c1} is ignored, and the applied field $H < H_{c2}$. According to Ampère's law, we have

$$-dH_i/dx = 4\pi J_c(H_i)/c\,, \quad (9)$$

where x is the distance between the slab surface to any point inside the slab. With the appropriate boundary conditions, we can derive the expressions for the local field H_i in various field regions. By definition, the magnetization M is given by

$$4\pi M = \frac{1}{d}\int_0^d H_i(x)dx - H\,, \quad (10)$$

and we obtain

$$4\pi M = -H + \frac{H_0}{h_0}\left[-\frac{(1+H/H_0)^{\beta+1}-1}{\beta+1} + \frac{(1+H/H_0)^{\beta+2}-1}{\beta+2}\right],\ \text{for } H \le H^*\,, \quad (11)$$

$$4\pi M = -H + \frac{H_0}{(\beta+2)h_0}\left\{\left[\left(1+\frac{H}{H_0}\right)^{\beta+2} - (\beta+2)h_0\right] - \left[\left(1+\frac{H}{H_0}\right)^{\beta+1} - (\beta+1)h_0\right]^{(\beta+2)/(\beta+1)}\right\},\ \text{for } H \ge H^*\,, \quad (12)$$

where H^* is the field at which the flux first completely penetrates the slab, and

$$H^* = H_0\{-1+[1+(1+\beta)h_0]^{1/(1+\beta)}\} , \tag{13}$$

where $h_0 = 4\pi J_c d(T)/cH_0$, which is a dimensionless coefficient. It should be noted that Eqs. (11) and (12) are for the initial magnetization curve as the applied field increases. For the decreasing field, we have

$$4\pi M = -H_0 - H + \frac{H_0}{(\beta+2)h_0}\left\{2\left[\frac{(1+H/H_0)^{\beta+1}+(1+H_m/H_0)^{\beta+1}}{2}\right]^{(\beta+2)/(\beta+1)} - (1+H/H_0)^{\beta+2} - \left[\left(1+\frac{H}{H_0}\right)^{\beta+1} - (\beta+1)h_0\right]^{(\beta+2)/(\beta+1)}\right\} \text{ for } H_m \ge H \ge H_m - 2H^* , \tag{14}$$

$$4\pi M = -H - \frac{H_0}{(\beta+2)h_0}\left\{\left[\left(1+\frac{H}{H_0}\right)^{\beta+2} + (\beta+2)h_0\right] - \left[\left(1+\frac{H}{H_0}\right)^{\beta+1} + (\beta+1)h_0\right]^{(\beta+2)/(\beta+1)}\right\}, \tag{15}$$

$$\text{for } 0 \le H \le H_m - 2H^* ,$$

where H_m is the maximum applied field.

It should be pointed out that Eq. (11) can also be obtained by using the original Bean differential equation [Eq. (8)] for the magnetization curve. Substituting our generalized critical-state model [Eq. (5)] into Eq. (8), we have

$$dB/dH = cH(1+H/H_0)^{\beta}/4\pi d J_c(T) . \tag{16}$$

Solving this differential equation with the same approach introduced earlier, we can obtain Eq. (11), which is one of the principal results of this study.

In Fig. 1 we plot the initial magnetization versus field (M vs H) curves for different β values and fixed H_0 (=1 T). We set $J_c = 10^5$ A/cm^2 and $d = 0.5$ mm. It should be noted that the magnetization curves are plotted only for the increasing field part ($H < H^*$ and $H > H^*$). As shown in Fig. 1, the magnitude of the magnetization increases dramatically with decreasing β. At 8 T, the magnetization for $\beta = 0.5$ is 12 times higher than that for $\beta = 1.5$. This indicates that the value of β reflects the flux-pinning strength of the superconductors, because the pinning energy is proportional to the area under the M vs T curve. The peaks of the magnetization curves correspond to the full penetration field H^*. It is consistent with the general pinning mechanisms that stronger pinning (smaller β values) requires higher field (H^*) for the flux lines to fully penetrate the superconductor at a given temperature.

Figure 2 shows the M vs H curves for different H_0 and fixed β (=1.5). As can be seen in Fig. 2, the M vs H curve is also considerably changed with H_0. This variation of the magnetization curve can be explained by Eq. (13). As shown in this equation, for a given β, H^* is weakly dependent on the term $[1+(1+\beta)h_0]^{1/1+\beta}$ and is determined mostly by the prefactor H_0. Therefore, the peak of the M vs H curve shifts to higher field (H^*) as H_0 increases. It can be seen in Fig. 2 that the peak field (close to H^*) of the M vs H curve is located at 1, 1.5, and 4 T for H_0 values of 0.5, 1, and 5 T, respectively, which is consistent with Eq. (13).

Thus, Eqs. (11)–(14) can give magnetization curves with various β and H_0 values. For a hard superconductor, if H_0 is given, Eqs. (11)–(14) can be used to fit the experimental M vs H data at a given temperature. The fitted curves will result in a finite β value that is directly related

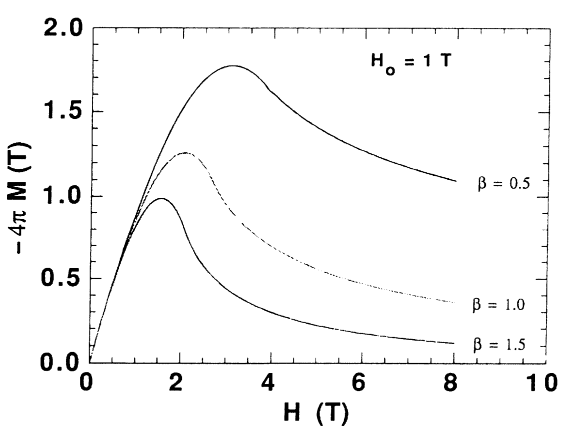

FIG. 1. Calculated magnetization vs field based on Eqs. (11) and (12) for various β indicated and fixed H_0 (=1 T).

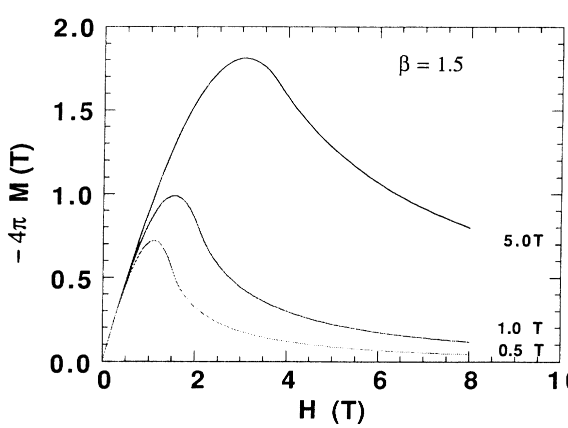

FIG. 2. Calculated magnetization vs field based on Eqs. (11) and (12) for various H_0 indicated and fixed β (=1.5).

to the microstructure and the pinning strength of the material.

In conclusion, we have developed a generalized critical-state model for hard superconductors. This model unifies the major forms of the critical-state model developed previously. We have mathematically shown that all other forms of the critical-state model can be derived from this generalized form [Eq. (5)]. We have also shown that the fitting parameters such as β and H_0 are physically consistent with the Bean critical-state model.

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*Also at School of Physics, Georgia Institute of Technology, Atlanta, GA 30332.

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