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Transport properties for random walks in disordered one-dimensional media: Perturbative calculation around the effective-medium approximation

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A perturbation method for the calculation of transport properties in disordered one-dimensional media is presented. It permits considering cases of strong and weak disorder and different initial conditions in a unified way, giving new results and reproducing known ones. The effective-medium approximation (EMA) appears as the zeroth-order step of our diagrammatic scheme. The method is applied to the random-barrier and random-trap models. The range of validity of the EMA is established. Exact results beyond the EMA are obtained for the low-frequency behavior of the frequency-dependent diffusion coefficient and the modified Burnett coefficient associated with the average Green's function and with the response function of the problem.

I. INTRODUCTION

A standard model to study transport in disordered media is a random walk (RW) on a lattice in which the transition probabilities per unit time take random values.[1] The connection of this approach with related models such as continuous-time random walks[2] (CTRW) is well known. The statistical characterization of diffusion processes is usually given in terms of the long-time behavior of the moments of the displacement $\mathbf{n}(t)$ of the random walker. In particular, the asymptotic time dependence of the mean-square displacement $\langle \mathbf{n}^2 \rangle$ is considered. One generally finds $\langle \mathbf{n}^2 \rangle \sim t^{\delta}$. In the absence of disorder, $\delta=1$. The situation where $\delta<1$, indicating anomalous diffusion, can occur in the presence of disorder. An equivalent characterization of subdiffusive behavior ($\delta<1$) is given by the law with which the frequency-dependent diffusion coefficient $D(z)$ vanishes as $z\to 0$. One usually distinguishes between weak and strong disorder. For weak disorder, $\delta=1$ and the zero-frequency diffusion coefficient $D(z=0)$ is finite. Strong disorder leads to a model-dependent exponent $\delta<1$, implying $D(z=0)=0$. A different characterization of the statistical properties of the transport process can be given, for example, in terms of first-passage-time distributions.[3] In this paper we are concerned with RW models for weak and strong static[4] disorder.

A common approximation to deal with RW in statically disordered media is the effective-medium approximation[5] (EMA). Other interesting approximations are based on scaling assumptions[1(a),6] or on the CTRW and its extensions.[2] From the point of view of exact calculations in the models considered in this paper, results for weak disorder were obtained by Zwanzig[7] and extended by a diagrammatic calculation by Denteneer and Ernst.[8] For strong disorder there is the calculation of Stephen and Kariotis[9] by the replica trick and its extensions[10] and the one of Nieuwenhuizen and Ernst[11] by an integral equation approach.

These calculations leave, nevertheless, a number of important open questions. We first note the lack of a unified approach: Different methods are available for weak and strong disorder with no relation among them and with no connection with the very much used EMA. The consequence is that a standing open question was when and to which order of calculation the EMA gives a correct result for $D(z)$. The answer was known for weak disorder but not for two common models of strong disorder, namely models B and C of Ref. 1(a). For model B it was not known to which order beyond the leading one the EMA gives a correct result. For model C there was only a conjecture,[11] which turned out to be wrong, that the EMA gives the correct leading term. The situation is even much less clear for higher-order diffusion coefficients as the modified Burnett coefficient[1(b)] for which the exact leading contribution for models of strong disorder has not been reported. Another aspect is that the presence of disorder naturally leads to an effective non-Markovian diffusion process.[7] This implies a nontrivial dependence on initial conditions because of memory effects, which has not been analyzed at length.[12] Likewise, the EMA features a non-Markovian RW with transition probabilities between nearest-neighbor sites, but deeper non-Markovian effects connect the dynamics at a site with sites beyond nearest neighbors. These effects, not analyzed in previous approaches, turn out to be dominant in the calculation of high-order statistical properties such as the modified Burnett coefficient.

The above questions are addressed in this paper[13] within a calculational scheme based on a new diagrammatic perturbation method. This method separates, in a natural way, the contributions given by the EMA which appears as the zeroth-order step of our perturbation scheme. In this way we have a unified approach to models of weak and strong disorder in which the validity of the EMA and effects beyond it can be considered. The specific RW models considered are the random-barrier (RB) and the random-trap (RT) models[1] defined below. They are considered in the context of the models of disorder A, B, and C of Ref. 1(a).

The outline of the paper is as follows. In Sec. II the models to be considered are defined. We introduce an effective master equation (EME) for the non-Markovian process obtained by averaging over disorder configurations. The EME contains, in a exact way, the effects of disorder. The role of arbitrary initial conditions is discussed here. Section III sets up a diagrammatic calculation of the diffusion coefficient by a perturbative treatment of disorder. This perturbative series is reorganized in Sec. IV to introduce an effective non-Markovian medium as a zeroth-order step. This zeroth order happens to be the EMA and the perturbation around the EMA is constructed. Sections V and VI are devoted to the calculation of the diffusion coefficient $D(z)$ and modified Burnett coefficient, respectively, within the scheme of Sec. IV. Corrections beyond the EMA are calculated. In Sec. VII we calculate the diffusion and modified Burnett coefficients associated to the response function of the RB and RT models within the scheme of Sec. II. Our main conclusions are summarized in Sec. VIII. Appendix A contains some properties of the Terwiel's cumulants[14] used in our development. The evaluation of the order of some diagrams used in the main text is given in Appendix B.

II. EFFECTIVE MASTER EQUATIONS: THE RANDOM-BARRIER AND RANDOM-TRAP MODELS

Our starting point is a master equation (ME) describing RW in a disordered medium:

$$\partial_t P(t) = HP(t) + O_\xi P(t) . \tag{2.1}$$

The components $P_{\mathbf{n}}(t)$ of the vector $P(t)$ are the probabilities of finding the walker at site $\mathbf{n}$ at time t. The operator defining the master equation is splitted in a nonrandom part (H) and a random part (O_ξ). The disordered medium is described by a set of random-transition probabilities associated here with O_ξ. A configuration of the medium is defined by a particular value $\{\xi\}$ of these transition probabilities. For each configuration $\{\xi\}$, (2.1) defines a RW characterized by $P(t)$. The statistical properties of the RW in the disordered medium are described by $\langle P(t)\rangle$, the average of $P(t)$ over the distribution of possible configurations $\{\xi\}$. The average of (2.1) gives an equation for $\langle P(t)\rangle$. Such an equation is called an effective master equation. It incorporates the whole effect of the disorder on the statistical properties of the RW.

Particular examples of (2.1) are given below, but we first consider a general method to obtain an EME for a general situation (2.1). We introduce a projection operator $\mathcal{P}$ which averages over possible configurations of disorder; that is, $\mathcal{P}f \equiv \langle f \rangle$, where f is any function of the configuration $\{\xi\}$. Applying $\mathcal{P}$ and $\mathcal{Q} \equiv 1 - \mathcal{P}$ to (2.1), we find it equivalent to the following set of coupled equations:

$$\partial_t \langle P\rangle = H\langle P\rangle + \mathcal{P}O_\xi\langle P\rangle + \mathcal{P}O_\xi\mathcal{Q}P , \tag{2.2a}$$

$$\partial_t \mathcal{Q}P = H\mathcal{Q}P + \mathcal{Q}O_\xi\langle P\rangle + \mathcal{Q}O_\xi\mathcal{Q}P . \tag{2.2b}$$

The linear character of these equations allows us their formal solution. This is the way in which Klafter and Silbey[15] and, for a general initial condition, Haus and Kehr[12] established the relation between diffusion in random media and generalized ME's. We find it more convenient, for our purposes, to look for a solution of (2.2) in which O_ξ is considered as a perturbation to H. In this way we obtain a perturbative series in which the different terms can be explicitly evaluated.

In terms of $G(t|t')$, the Green's-function matrix of the nondisordered problem, (2.2b) is equivalent to the integral equation

$$\mathcal{Q}P(t) = G(t|0)\mathcal{Q}P^0 + \int_0^t dt'\, G(t|t')[\mathcal{Q}O_\xi\langle P(t')\rangle + \mathcal{Q}O_\xi\mathcal{Q}P(t')] , \tag{2.3}$$

where P^0 is the initial distribution $P^0 = P(t=0)$. Equation (2.3) can be solved iteratively as

$$\mathcal{Q}P(t) = \sum_{p=0}^{\infty} (\hat{M}\mathcal{Q}O_\xi)^p [\hat{M}\mathcal{Q}O_\xi\langle P(\cdot)\rangle + G(\cdot|0)\mathcal{Q}P^0] , \tag{2.4}$$

where we have introduced a convolution operator $\hat{M}$ acting on time-dependent vectors $f(t)$ as

$$(\hat{M}f(\cdot))_{\mathbf{n}}(t) \equiv \sum_m \int_0^t dt'\, G_{nm}(t|t') f_m(t') . \tag{2.5}$$

The index m takes values on the sites in the lattice. Substituting (2.4) into (2.2a), a closed equation for $\langle P(t)\rangle$ is finally obtained:

$$\partial_t\langle P\rangle = H\langle P\rangle + \mathcal{P}\sum_{p=0}^{\infty}(O_\xi\mathcal{Q}\hat{M})^p O_\xi\langle P\rangle + \mathcal{P}\sum_{p=0}^{\infty}(O_\xi\mathcal{Q}\hat{M})^p O_\xi\mathcal{Q}G(\cdot|O)P^0 . \tag{2.6}$$

Equation (2.6) is the desired EME. It is an integrodifferential equation describing the evolution of the average one-time probability distribution for the walker $\langle P_{\mathbf{n}}(t)\rangle$. Its convolution form indicates that, although starting from a Markovian ME, the average over disorder introduces, in general, an effective non-Markovian process. The non-Markovian character of the process is also evidentiated in the presence of a term in (2.6) involving the initial distribution P^0. We note, however, that this term involving the initial condition vanishes if $\mathbb{Q}P^0=0$, that is, if $P^0=\langle P^0\rangle$. This happens if the initial distribution is specified independently of the possible configurations of the medium, for example, if $P_n^0=\delta_{nn_0}$.

The general result (2.6) has been derived for a medium of arbitrary dimensionality and rather general forms of static disorder. In this paper we will consider two particular one-dimensional models known as random-barrier[1] and random-trap[1] models. Both models are defined starting from a general one-step Markovian ME in a chain:

$$\partial_t P_n(t)=w_{n+1}^- P_{n+1}(t)+w_{n-1}^+ P_{n-1}(t) - (w_n^+ + w_n^-)P_n(t) \ . \tag{2.7}$$

w_n^+ (w_n^-) is the transition probability per unit time to go from site n to site $n+1$ (site $n-1$). Disorder is introduced here by considering $w_n^\pm$ as random variables. The RB and the RT models differ on the symmetry requirements imposed on the transition probabilities $w_n^\pm$.

In the RB case one models a situation in which

$$w_n^- = w_{n-1}^+ \equiv w_n \ . \tag{2.8}$$

The w_n's are taken to be independent random variables identically distributed so that the medium is translationally invariant on the average. We can separate w_n in an average part $\mu \equiv \langle w_n\rangle$ and a random part ξ_n:

$$w_n=\mu+\xi_n, \quad \langle \xi_n\rangle = 0 \ . \tag{2.9}$$

The ME (2.7) can then be written as (2.1), in which

$$(HP)_n=\mu(E_n^+ + E_n^- -2)P_n \equiv \mu \mathbb{K}_n P_n \ , \tag{2.10}$$

$$(O_\xi P)_n=(1-E_n^-)\xi_n(E_n^+ -1)P_n \ . \tag{2.11}$$

We have introduced shifting operators $E_n^\pm$ acting on functions of the site n as $E_n^\pm f_n \equiv f_{n\pm 1}$.

In the same way, for the RT model, defined by

$$w_n^+ = w_n^- \equiv w_n \ , \tag{2.12}$$

where the w_n's are again independent random variables identically distributed, the master equation has the form (2.1) with H given in (2.10) and

$$(O_\xi P)_n = \mathbb{K}_n \xi_n P_n \ . \tag{2.13}$$

A complete definition of the models requires the specification of the statistical properties of ξ_n. Following the definitions of Ref. 1(a), we will refer to model A as the one for which the probability distribution of $w_n=\mu+\xi_n$ is such that the inverse moments $\beta_M \equiv \langle (w_n)^{-M}\rangle$ ($M=1,2,\ldots$) exist. This general situation corresponds to weak disorder. Strong disorder corresponds to situations in which these inverse moments diverge, and we will distinguish between models B and C. Model C is defined by a probability distribution $\rho(w_n)=(1-\alpha)(w_n)^{-\alpha}$, $0<\alpha<1$, $w_n \in (0,1)$. Model B is a limiting case of strong disorder obtained from model C with $\alpha=0$.

We can now make explicit the form of the EME (2.6) for the RB and RT models just defined. We first note that, in these cases, the Green's function is the solution of

$$\partial_t G_{nm}(t|t')=\mu \mathbb{K}_n G_{nm}(t|t'), \quad G_{nm}(t^+|t)=\delta_{nm} \ , \tag{2.14}$$

which is known to be[16]

$$G_{nm}(t|t')=e^{-2\mu(t-t')}I_{|n-m|}[2\mu(t-t')], \quad t>t' \tag{2.15}$$

$I_k(x)$ is a modified Bessel function. We first consider the explicit expression for the second and third term in the right-hand side (rhs) of (2.6) for the RB model. Their convolution form simplifies, if expressed in terms of the Laplace transforms $\tilde{G}(z)$ and $\tilde{P}(z)$ of $G(t/t')$ and $P(t)$,

$$\mathcal{P}\sum_{p=0}^{\infty}(1-E_n^-)\xi_n(E_n^+-1)\mathbb{Q}\sum_{n_1=-\infty}^{\infty}\tilde{G}_{nn_1}(z)(1-E_{n_1}^-)\xi_{n_1}(E_{n_1}^+-1)\mathbb{Q}\sum_{n_2=-\infty}^{\infty}\tilde{G}_{n_1n_2}(z)\cdots$$
$$\times\sum\cdots\sum_{n_{p-1}=-\infty}^{\infty}(1-E_{n_{p-1}}^-)\xi_{n_{p-1}}(E_{n_{p-1}}^+-1)\sum_{n_p=-\infty}^{\infty}\tilde{G}_{n_{p-1}n_p}(z)(1-E_{n_p}^-)\xi_{n_p}(E_{n_p}^+-1)\mathbb{Z}_{n_p}(z) \ . \tag{2.16}$$

We have introduced

$$\mathbb{Z}_n(z)\equiv\langle\tilde{P}_n(z)\rangle+\sum_{m=-\infty}^{\infty}\mathbb{Q}\tilde{G}_{nm}(z)P_m^0 \ . \tag{2.17}$$

Equation (2.16) can be simplified taking into account that E_n^+ and E_n^- are adjoint operators, so that expressions of the form

$$\xi_n(E_n^+-1)\sum_m A_{nm}(1-E_m^-)\xi_m F_m \tag{2.18}$$

become

$$\xi_n\sum_m[(E_n^+-1)(1-E_m^+)A_{nm}]\xi_m F_m \ . \tag{2.19}$$

By repeated application of this property, (2.16) is transformed into

$$(1-E_n^-)\sum_{p=0}^{\infty}\sum_{n_1,\ldots,n_p=-\infty}^{\infty}\mathcal{P}\xi_n \mathcal{Q} J_{nn_1}(z)\xi_{n_1}\mathcal{Q}\ldots J_{n_{p-1}n_p}(z)\xi_{n_p}(E_{n_p}^+-1)\mathbb{Z}_{n_p}(z)\ , \tag{2.20}$$

where

$$J_{nm}\equiv(E_n^+-1)(1-E_m^+)\tilde{G}_{nm}=\tilde{G}_{n+1,m}-\tilde{G}_{n+1,m+1}-\tilde{G}_{nm}+\tilde{G}_{n,m+1}=\mathbb{K}_n\tilde{G}_{nm}\ . \tag{2.21}$$

In the last equality we have used the fact that $\tilde{G}_{nm}$ depends only on the difference $n-m$.

In (2.20) we observe the appearance, in a natural way, of Terwiel's cumulants,[14] defined as

$$\langle \xi_n \xi_{n_1}\cdots\xi_{n_p}\rangle_{\mathrm{T}}\equiv\mathcal{P}\xi_n\mathcal{Q}\xi_{n_1}\cdots\mathcal{Q}\xi_{n_p}\ . \tag{2.22}$$

These quantities appear rather often when the average of a stochastic differential equation is written in integrodifferential form. Some of their useful properties, relevant for our development, are summarized in Appendix A. In terms of Terwiel's cumulants, (2.20) becomes

$$(1-E_n^-)\sum_{p=0}^{\infty}\sum_{n_1,\ldots,n_p}J_{nn_1}(z)\cdots J_{n_{p-1}n_p}(z)\{\langle\xi_n\xi_{n_1}\cdots\xi_{n_p}\rangle_{\mathrm{T}}(E_{n_p}^+-1)\langle\tilde{P}_{n_p}(z)\rangle$$
$$+\sum_m[(E_{n_p}^+-1)\tilde{G}_{n_p m}(z)]\langle\xi_n\xi_{n_1}\cdots\xi_{n_p}P_m^0\rangle_{\mathrm{T}}\}\ . \tag{2.23}$$

The statistical homogeneity of the chain implies that

$$\langle\xi_{n+k}\xi_{n_1+k}\cdots\xi_{n_p+k}\rangle_{\mathrm{T}}=\langle\xi_n\xi_{n_1}\cdots\xi_{n_p}\rangle_{\mathrm{T}}\ \forall k \tag{2.24}$$

and that J_{nm} and $\tilde{G}_{nm}$ depend only on $|n-m|$. Using these facts after letting the operator $(1-E_n^-)$ explicitly act in (2.23), (2.6) becomes

$$z\langle\tilde{P}(z)\rangle-P^0=H\langle\tilde{P}(z)\rangle+W(z)\langle\tilde{P}(z)\rangle+\Omega(z)\ , \tag{2.25}$$

where $H\langle\tilde{P}\rangle$ is the disorder-independent contribution, $W(z)\langle\tilde{P}(z)\rangle$ gives disorder effects independent of the initial condition and Ω contains the contribution associated with the initial condition

$$(H\langle\tilde{P}\rangle)_n=\mu\mathbb{K}_n\langle\tilde{P}_n\rangle\ , \tag{2.26}$$

$$(W(z)\langle\tilde{P}(z)\rangle)_n=\sum_{p=0}^{\infty}\sum_{n_1,\ldots,n_p}J_{nn_1}(z)\cdots J_{n_{p-1}n_p}(z)\langle\xi_n\xi_{n_1}\cdots\xi_{n_p}\rangle_{\mathrm{T}}\mathbb{K}_{n_p}\langle\tilde{P}_{n_p}(z)\rangle\ , \tag{2.27}$$

$$\Omega_n^{\mathrm{RB}}(z)=\tfrac{1}{2}\sum_{p=0}^{\infty}\sum_{n_1,\ldots,n_p,m}(1-E_n^-)J_{nn_1}(z)\cdots J_{n_{p-1}n_p}(z)J_{n_p m}(z)\langle\xi_n\xi_{n_1}\cdots\xi_{n_p}P_m^0\rangle_{\mathrm{T}}\ . \tag{2.28}$$

Similar manipulations to those just done for the RB model can be done for the RT. The final result for the EME has the same form as (2.25), the only difference being the explicit result for Ω:

$$\Omega_n^{\mathrm{RT}}(z)=\mathbb{K}_n\sum_{p=0}\sum_{n_1,\ldots,n_p,m}J_{nn_1}(z)\cdots J_{n_{p-1}n_p}(z)\tilde{G}_{n_p m}(z)\langle\xi_n\xi_{n_1}\cdots\xi_{n_p}P_m^0\rangle_{\mathrm{T}}\ . \tag{2.29}$$

An important consequence of these results is that whenever $\Omega^{\mathrm{RB}}=\Omega^{\mathrm{RT}}$ the RB and the RT models are exactly equivalent, as long as we are only interested in one-time properties; that is, properties which could be obtained from $\langle P_n(t)\rangle$. The equality of Ω^{RB} and Ω^{RT} often occurs because both vanish. As mentioned in connection with (2.6), this happens when $P^0=\langle P^0\rangle$. A particular interesting case of this situation is when $P_n^0=\delta_{nn_0}$, so that $\langle P(t)\rangle$ is the average Green's function of the problem. In such a case, (2.25) with $\Omega=0$ is the exact equation both for the RB and the RT models. The fact that the average Green's function is the same for the RB and the RT models has been pointed out previously by other authors.[8,9] However, a certain confusion about this point exists in the recent literature.[17] In Sec. VI we discuss a calculation of the response function. For this calculation it is necessary to use (2.25) with $\Omega\neq0$ and the differences between the RT and the RB models become apparent. In the remaining of this paper we mostly consider the average Green's function; that is, the solution of (2.25) with the initial condition $P_n^0=\delta_{nn_0}$.

Equation (2.25) for the average Green's function can be written in a compact form as

$$z\langle\tilde{P}_n(z)\rangle-\delta_{n0}=\sum_{m=-\infty}^{\infty}\hat{T}_m(z)\mathbb{K}_n(E_n^+)^m\langle\tilde{P}_n(z)\rangle\ , \tag{2.30a}$$

where the kernel $\hat{T}_m(z)$ is given by

$$\hat{T}_m(z)=(\mu+\langle \xi_m \rangle)\delta_{m0} + \sum_{p=1}^{\infty} \sum_{n_1,\ldots,n_{p-1}} \langle \xi_0\xi_{n_1}\cdots\xi_{n_{p-1}}\xi_m \rangle_{\mathrm{T}} \times J_{0n_1}(z)\cdots J_{n_{p-1}m}(z) . \tag{2.30b}$$

The term $\langle \xi_m \rangle$, which is identically zero, has been explicitly written for future comparison [see Eq. (4.2)]. Equation (2.30) is of the form proposed by Klafter and Silbey[15] but here we have an explicit expression for the memory kernel $T_m(t-t')$ in terms of Terwiel's cumulants. A main effect of averaging over disorder is the non-Markovian character of the effective process which is reflected in the presence of a memory kernel in (2.30). A second effect is the appearance in (2.30) of effective transitions of any step size m (Ref. 18), while in the original RW (2.7) only transitions between nearest neighbors appear. Equation (2.30) is the basis of our calculational scheme discussed in the next sections.

III. PERTURBATIVE TREATMENT OF DISORDER: DIAGRAMMATIC CALCULATION OF THE DIFFUSION COEFFICIENT

The EME (2.30) allows a perturbative calculation of the frequency-dependent diffusion coefficient $D(z)$ around the nondisordered system. Introducing the Fourier and Laplace transform of $P_n(t)$,

$$\langle \hat{P}_q(z) \rangle \equiv \int_0^{\infty} dt\, e^{-zt} \sum_{n=-\infty}^{\infty} e^{-iqn}\langle P_n(t) \rangle , \tag{3.1}$$

and using $P_n^0=\delta_{nn_0}$, (2.30) becomes

$$z\langle \hat{P}_q(z) \rangle - 1 = \hat{T}_q(z)K_q\langle \hat{P}_q(z) \rangle , \tag{3.2}$$

where

$$K_q=2[\cos(q)-1] \tag{3.3}$$

and

$$\hat{T}_q(z)=\sum_{m=-\infty}^{\infty} \hat{T}_m(z)e^{imq} . \tag{3.4}$$

The solution of (3.2) can be written in the form

$$\langle \hat{P}_q(z) \rangle = \frac{1}{z+q^2D_q(z)} \tag{3.5}$$

which identifies the frequency- and wave-number-dependent diffusion coefficient

$$D_q(z)=\frac{2[1-\cos(q)]}{q^2}\hat{T}_q(z) . \tag{3.6}$$

The small q behavior of $D_q(z)$ characterizes the lowest-order moments of $\langle P_n(t) \rangle$. In fact, $D_q(z)$ is expanded as

$$D_q(z)=D(z)-q^2D_2(z)+O(q^4) . \tag{3.7}$$

The frequency-dependent diffusion coefficient is defined as

$$D(z)=\lim_{q\to 0} D_q(z) \tag{3.8}$$

and the modified Burnett coefficient as

$$D_z(z)=\lim_{q\to 0}\frac{D(z)-D_q(z)}{q^2} . \tag{3.9}$$

In terms of these two quantities, the Laplace-transformed second and fourth moments of P_n can be obtained as

$$\langle n^2(z) \rangle = \int_0^{\infty} dt\, e^{-zt} \sum_{n=-\infty}^{\infty} n^2\langle P_n(t) \rangle = \frac{2}{z^2}D(z) , \tag{3.10}$$

$$\langle n^4(z) \rangle = \int_0^{\infty} dt\, e^{-zt} \sum_{n=-\infty}^{\infty} n^4\langle P_n(t) \rangle = \frac{24}{z^2}\left[D_2(z)+\frac{1}{z}[D(z)]^2 \right] . \tag{3.11}$$

We now address the calculation of $D(z)$ leaving the calculation of $D_2(z)$ for Sec. VI.

From (3.4), (3.6), (3.8), and (2.30b), the following exact expression follows for $D(z)$:

$$D(z)=\mu+\sum_{p=1}^{\infty} \sum_{n_1,\ldots,n_p=-\infty}^{\infty} \langle \xi_0\xi_{n_1}\cdots\xi_{n_p} \rangle_{\mathrm{T}} \times J_{0n_1}(z)\cdots J_{n_{p-1}n_p}(z) . \tag{3.12}$$

This gives a perturbation series around the nondisordered result $D(z)=\mu$, in which higher-order terms involve higher-order Terwiel's cumulants. In order to analyze the series it is important to note that the value of the cumulant $\langle \xi_0\xi_{n_1}\cdots\xi_{n_p} \rangle_{\mathrm{T}}$ depends only on the sequence of different and equal indexes which it contains. For example, the following cumulants have the same value:

$$\langle \xi_0\xi_1\xi_0\xi_3\xi_3\xi_1 \rangle_{\mathrm{T}} = \langle \xi_5\xi_2\xi_5\xi_1\xi_1\xi_2 \rangle_{\mathrm{T}} = \langle \xi_2\xi_4\xi_2\xi_6\xi_6\xi_4 \rangle_{\mathrm{T}} = \cdots . \tag{3.13}$$

This is so because of the statistical homogeneity of the chain and the statistical independence of random variables with different indexes. This fact indicates that it is convenient to split the sum over $(n_1,\ldots,n_p)$ in (3.12) in several parts, each part containing cumulants with the same value. For example, the $p=2$ term in (3.12) can be written as

$$\sum_{n_1,n_2} \langle \xi_0\xi_{n_1}\xi_{n_2} \rangle_{\mathrm{T}} J_{0n_1}J_{n_1n_2} = \sum_{\substack{n_1\,(\neq 0),\\ n_2\,(\neq n_1,0)}} \langle \xi_0\xi_{n_1}\xi_{n_2} \rangle_{\mathrm{T}} J_{0n_1}J_{n_1n_2} + \sum_{n_2\,(\neq 0)} \langle \xi_0\xi_0\xi_{n_2} \rangle_{\mathrm{T}} J_{00}J_{0n_2} + \sum_{n_1\,(\neq 0)} \langle \xi_0\xi_{n_1}\xi_{n_1} \rangle_{\mathrm{T}} J_{0n_1}J_{n_1n_1} + \sum_{n_1\,(\neq 0)} \langle \xi_0\xi_{n_1}\xi_0 \rangle_{\mathrm{T}} J_{0n_1}J_{n_10} + \langle \xi_0\xi_0\xi_0 \rangle_{\mathrm{T}} J_{00}J_{00} . \tag{3.14}$$

The cumulants in the rhs are not affected by the sums because of the property exemplified by (3.13). Each group of terms resulting from the splitting is conveniently represented as the average of an adequate diagram. A diagram of order p contains $p+1$ random variables and it is defined as follows: Each random variable ξ_{n_i} in the cumulant will be represented as a point on a horizontal line, each function $J_{n_i n_j}$, accompanied by a projector Q, as the segment of the line joining the points corresponding to ξ_{n_i} and ξ_{n_j}. To indicate which of the random variables ξ_{n_i} have the same index, the associated points in the diagram will be joined by a curved line. An *internal sum* over each index different from zero is also present. The number of internal sums in a diagram is called I. The order of the diagram p and I are the two important quantities to characterize a diagram, as is seen in the remaining of this paper. The first 14 diagrams with a nonvanishing average contributing to (3.12) are displayed in Fig. 1. The analytical expression for the first four is also given. Note that the properties of Terwiel's cumulants (see Appendix A) considerably reduce the number of diagrams with a nonvanishing average. For example, in (3.14) only

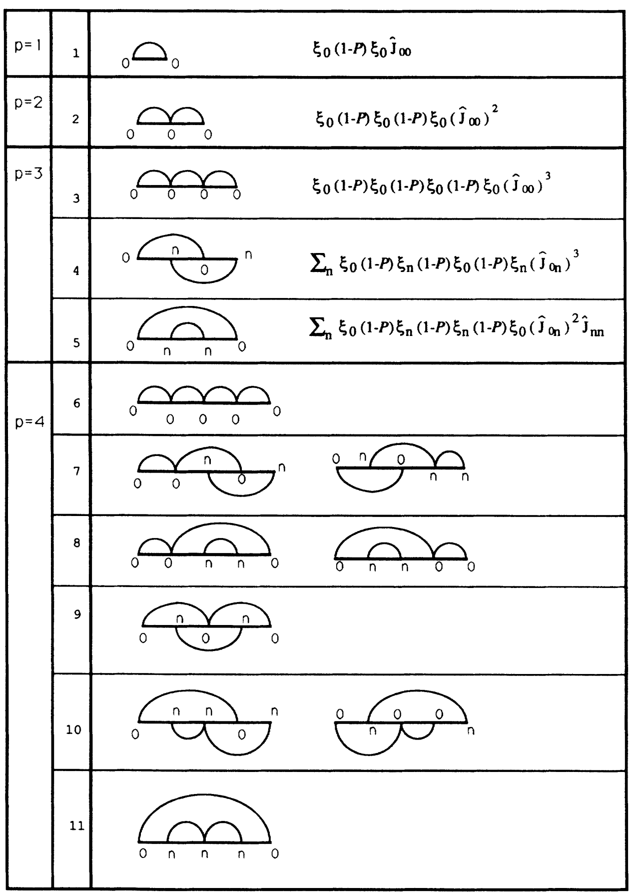

FIG. 1. The first diagrams with a nonvanishing average contributing to Eq. (3.12). The analytical expression is given for the lowest-order diagrams.

the last term is different from zero. The other ones contain cumulants consisting of two or three independent pieces, or they contain an isolated random variable which is not connected to any other point by a curved line. The average of such diagrams vanish (Appendix A) and they are not included in Fig. 1.

Up to this point we have a diagrammatic perturbation scheme[19] for $D(z)$ that we write symbolically as

$$D(z)=\mu+\left\langle \sum \text{diagrams} \right\rangle .$$

In order to know if this is a good high- or low-frequency perturbation series, we need to know the z behavior of the functions $J_{nm}(z)$. From the Laplace transform of (2.21) with (2.15) we observe

$$J_{nm}(z)=2A(z)[B(z)-1]\delta_{nm} +(1-\delta_{nm})A(z)[B(z)-1]^2[B(z)]^{|n-m|-1} , \tag{3.15}$$

where

$$A(z)=(2\mu)^{-1}\left[\frac{z}{\mu}+\frac{z^2}{4\mu^2}\right]^{-1/2} \tag{3.16}$$

and

$$B(z)=1+\frac{z}{2\mu}-\left[\frac{z}{\mu}+\frac{z^2}{4\mu^2}\right]^{1/2} . \tag{3.17}$$

The large- and small-z behavior is

$$J_{nm}(z\to\infty)\sim -2\delta_{nm}z^{-1}+(1-\delta_{nm})\mu^{-1}(\mu/z)^{|n-m|} , \tag{3.18}$$

$$J_{nm}(z\to 0)\sim -\delta_{nm}\mu^{-1}+(1-\delta_{nm})(2\mu)^{-1}(z/\mu)^{1/2} \times[1-(z/\mu)^{1/2}]^{|n-m|-1} . \tag{3.19}$$

The analysis of (3.12) at high frequencies is easier to perform. Using (3.18), we observe that the average of all the diagrams of order p for $z\to\infty$ is of order z^{-p}, so that (3.12) defines a systematic perturbation series in this case. The results of Denteneer and Ernst[8] can be reobtained order by order in z^{-1} from our perturbation theory.

The low-frequency ($z\to 0$) behavior of the series (3.12) is more difficult to analyze. To lowest order in z, each term J_{nn} contributes with a factor $-\mu^{-1}$, and each J_{nm}, $m\neq n$, with an additional factor

$$(2\mu)^{-1}(z/\mu)^{1/2}[1-(z/\mu)^{1/2}]^{|n-m|-1} .$$

The internal sums in each diagram involve this last term. Noting that

$$\sum_{\substack{n=-\infty,\\ n\neq 0}}^{\infty} (1-x)^{|n|}=\frac{2(1-x)}{x} , \tag{3.20}$$

we observe that each sum introduces a factor of order $(z/\mu)^{-1/2}$ for small z. In summary, the order in z of a diagram containing N_1 functions J_{nn}, N_2 factors J_{nm}

$(n \neq m)$, and I internal sums is

$$\mu^{-N_1-N_2}\left[\frac{z}{\mu}\right]^{(N_2-I)/2} \sim z^{(N_2-I)/2} . \tag{3.21}$$

We have explicitly written the factors μ (of order z^0) for later use because μ will be replaced by a function of z in the analysis of next section.

The result (3.21) implies that the leading contribution (of order z^0) to $D(z)$ comes from diagrams with $N_2=I$ so that $I=0$. These are the infinity of diagrams whose average contains cumulants of the form $\langle \xi_0 \cdots \xi_0 \rangle_T$, such as diagrams 1, 2, 3, and 6 of Fig. 1. Likewise, an infinity of diagrams contribute to each order in z in (3.12). As a consequence, to render our perturbation series useful for low frequencies, some rearrangement or resummation of the series becomes necessary.

A first possible resummation is that represented diagrammatically in Fig. 2. It consists in the summation of all the diagrams contributing to the leading order in z^0. These are the diagrams in (3.12) in which contiguous indexes n_i take the same value. The sum will be denoted as $\psi_n(z)$:

$$\psi_n(z) \equiv \sum_{p=0}^{\infty} (\xi_n \mathcal{Q} J_{nn})^p \xi_n . \tag{3.22}$$

We note that the quantity $\psi_n(z)$ is not a simple random function, but a random operator which has to be understood as acting on random functions. The sum in (3.22) can be explicitly carried out as follows. We apply the operator ψ_n to an arbitrary random function f,

$$\begin{aligned}\psi_n f &= \sum_{p=0}^{\infty} [J_{nn}\xi_n(1-\mathcal{P})]^p \xi_n f \\ &= \xi_n f + \sum_{p=1}^{\infty} [J_{nn}\xi_n(1-\mathcal{P})]^p \xi_n f \\ &= \xi_n f + J_{nn}\xi \psi_n f - J_{nn}\xi_n \langle \psi_n f \rangle .\end{aligned} \tag{3.23}$$

Solving for $\psi_n f$,

$$\psi_n f = M_n f - M_n J_{nn} \langle \psi_n f \rangle , \tag{3.24}$$

where

$$M_n \equiv \frac{\xi_n}{1-\xi_n J_{nn}} . \tag{3.25}$$

The term $\langle \psi_n f \rangle$ can be obtained applying $\mathcal{P}$ to (3.24) so that

$$\langle \psi_n f \rangle = \frac{1}{1+\langle M_n \rangle J_{nn}} \langle M_n f \rangle . \tag{3.26}$$

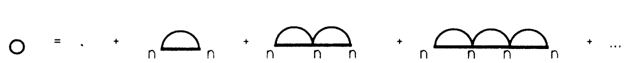

FIG. 2. Diagrammatic representation of Eq. (3.22) expressing the resummation of the diagrams contributing to order z^0 in (3.12).

Substituting into (3.24),

$$\psi_n f = M_n f - \frac{M_n J_{nn}}{1+\langle M_n \rangle J_{nn}} \langle M_n f \rangle \tag{3.27a}$$

so that the following expression is finally found:

$$\psi_n = M_n - \frac{M_n J_{nn}}{1+\langle M_n \rangle J_{nn}} \mathcal{P} M_n . \tag{3.27b}$$

After this resummation, the perturbative series (3.12) can be rewritten in terms of $\psi_n(z)$ as

$$D(z) = \mu + \sum_{p=0}^{\infty} \sum_{\substack{n_1\,(\neq 0),\\ n_2\,(\neq n_1),\\ n_p\,(\neq n_{p-1})}} \langle \psi_0(z)\psi_{n_1}(z) \cdots \psi_{n_p}(z) \rangle_{\mathrm{T}} \times J_{0n_1}(z) \cdots J_{n_{p-1}n_p}(z) . \tag{3.28}$$

The sum is now restricted to terms in which a given index n_i does not coincide with its nearest neighbors $n_{i\pm 1}$. The Terwiel's cumulant in (3.28) is

$$\langle \psi_0 \psi_{n_1} \cdots \psi_{n_p} \rangle_{\mathrm{T}} \equiv \mathcal{P}\psi_0 \mathcal{Q}\psi_{n_1} \cdots \mathcal{Q}\psi_{n_p} , \tag{3.29}$$

where each factor ψ_{n_i}, except the one at the far right, must be understood as an operator acting on its right. The series (3.28) can again be represented as a sum of averages of diagrams. The diagramatic rules are the same as before, except that each factor ψ_n, replacing the factors ξ_n in (3.12), will be represented by a circle on the horizontal line. In the new diagrammatic series there are no diagrams with a circle joined by a curved line to any of its nearest neighbors. However, there are new diagrams with a nonvanishing average: those are the irreducible diagrams containing isolated circles. In the primitive series this type of diagram vanished because $\langle \xi_n \rangle = 0$ (see Appendix A), but now $\langle \psi_n(z) \rangle \neq 0$. The lowest-order diagrams up to $p=4$ with a nonvanishing average contributing to (3.28) are displayed in Fig. 3. The order in z of each diagram is still given by (3.21) because $\langle \psi_n(z) \rangle \sim O(z^0)$. The zero-frequency diffusion coefficient $D(z=0)$ is then completely determined by the term $p=0$ in (3.28):

$$\begin{aligned} D(z=0) &= \mu + \lim_{z\to 0} \langle \psi_0(z) \rangle \\ &= \mu + \lim_{z \to 0} \frac{\langle M_0 \rangle}{1+\langle M_0 \rangle J_{00}} = \left\langle \frac{1}{w_0} \right\rangle^{-1} . \end{aligned} \tag{3.30}$$

In this way we reobtain the exact result of Ref. 7. It means that whenever the inverse moment $\beta_1 = \langle w_0^{-1} \rangle$ exists, $D(z=0)$ is finite and no anomalous diffusion appears ($\langle n^2 \rangle \sim t$).

However, the series (3.28) is not useful to calculate the low-frequency behavior beyond the limit $z=0$. Indeed, frequency-dependent corrections to (3.30) come from an infinity of terms. For example, diagrams 2 and 3 of Fig. 3, and all the diagrams obtained from them by adding circles in their interior, contribute in order $z^{1/2}$. More important is the fact that (3.29) gives rise to the appearance of the inverse moments $\beta_M = \langle w_0^{-M} \rangle$ which diverge in

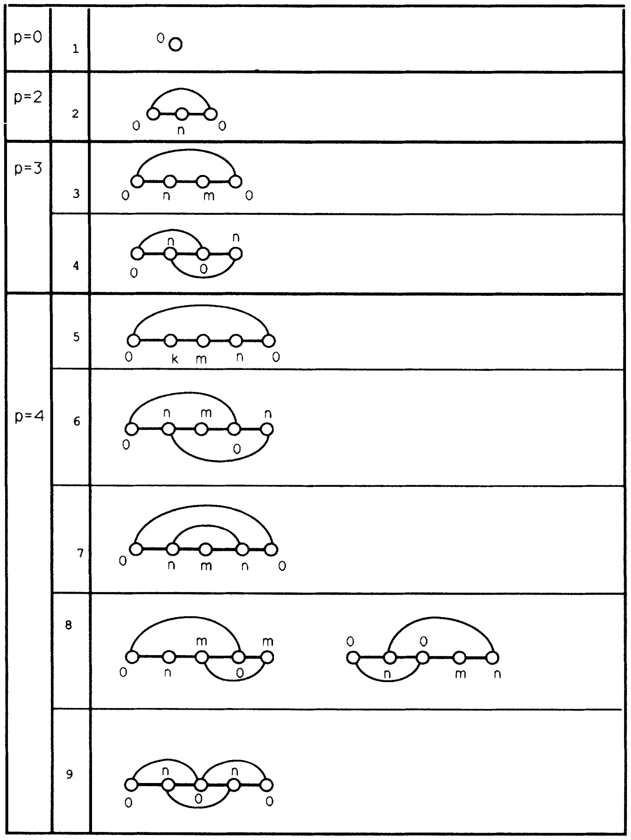

FIG. 3. The first diagrams with a nonvanishing average contributing to Eq. (3.28).

the case of strong disorder. These shortcomings identify the difficulties of a straightforward perturbation expansion around the nondisordered medium. More drastic rearrangements of the series (3.28) are necessary to have a useful perturbation theory.

IV. PERTURBATIVE EXPANSION AROUND AND EFFECTIVE NON-MARKOVIAN MEDIUM

In the preceeding sections we have considered the sources of randomness ξ_n as perturbations around the homogeneous part of the jump rates μ. This treatment has led to an efficient method for a systematic calculation of $D(z)$ at high frequencies, but it is unable to give a systematic calculation of the low-frequency behavior of $D(z)$. We already discussed that (2.30) shows that an important effect of averaging over the disorder configurations is the introduction of non-Markovian dynamics. Then, it is reasonable to expect that a perturbative representation of diffusion in a disordered medium as a perturbation around a non-Markovian dynamics would have better convergence properties that the perturbation scheme discussed in the previous section.

With this idea in mind, we rewrite the master equation for the RB model [Eqs. (2.1), (2.10), and (2.11)] by adding and subtracting a new term:

$$\partial_t P_n(t)=\int_0^t dt'\,\Gamma(t-t')\mathbb{K}_n P_n(t') + \int_0^t dt'\,(1-E_n^-)[(\mu+\xi_n)\delta(t-t')-\Gamma(t-t')] \times(E_n^+-1)P_n(t') \ . \tag{4.1}$$

We will use the explicit form of the RB model, but as shown in general before, the same results would follow for the RT model as long as the initial condition $P_n^0=\delta_{nn_0}$ is used.

The first term in the rhs of (4.1) represents a non-Markovian RW in a homogeneous medium in which only nearest-neighbor sites have influence on a given site. The second term will be interpreted as the perturbation around this medium. After averaging over disorder configurations this perturbation term will induce effective transition probabilities of any step size besides modifying the one-step transition probabilities given by $\Gamma(t-t')$. Our strategy will be to maintain $\Gamma(t-t')$ undefined and to formally construct a perturbation series for $D(z)$ along the lines of Sec. III. The idea is then to determine $\Gamma(t-t')$ *a posteriori* by imposing that the perturbation expansion around the non-Markovian dynamics defined by Γ has the best possible convergence properties in the low-frequency limit. Applying the general formula (2.6)–(4.1), we obtain an expression formally identical to (2.30), but now

$$\hat{T}_m(z)=[\hat\Gamma(z)+\langle\eta_m\rangle]\delta_{m0}+\sum_{p=1}^{\infty}\sum_{n_1,\ldots,n_{p-1}=-\infty}^{\infty}\langle\eta_0\eta_{n_1}\cdots\eta_{n_{p-1}}\eta_m\rangle_T\mathbb{J}_{0n_1}(z,\hat\Gamma)\cdots\mathbb{J}_{n_{p-1}m}(z,\hat\Gamma) \ . \tag{4.2}$$

$\hat\Gamma(z)$ is the Laplace transform of $\Gamma(t-t')$, $\eta_i\equiv\mu+\xi_i-\hat\Gamma(z)$, and $\mathbb{J}_{nm}$ is the solution of

$$z\mathbb{J}_{nm}(z,\hat\Gamma)-\delta_{nm}=\hat\Gamma\mathbb{K}_n\mathbb{J}_{nm}(z,\hat\Gamma) \ . \tag{4.3}$$

That is, $\mathbb{J}_{nm}(z,\hat\Gamma)$ is the function obtained from (3.15)–(3.17) by changing μ by $\hat\Gamma(z)$. Explicitly,

$$\mathbb{J}_{nm}(z,\hat\Gamma)=2\mathbb{A}(z)[\mathbb{B}(z)-1]\delta_{nm} +(1-\delta_{nm})\mathbb{A}(z)[\mathbb{B}(z)-1]^2[\mathbb{B}(z)]^{|n-m|-1} \ , \tag{4.4}$$

where

$$\mathbb{A}(z)=(2\hat\Gamma)^{-1}\left[\frac{z}{\hat\Gamma}+\frac{z^2}{4\hat\Gamma^2}\right]^{-1/2} \tag{4.5}$$

and

$$\mathbb{B}(z)=1+\frac{z}{2\hat\Gamma}\left[\frac{z}{\hat\Gamma}+\frac{z^2}{4\hat\Gamma^2}\right]^{1/2} \ . \tag{4.6}$$

The diffusion coefficient $D(z)$ is obtained in terms of $\hat{T}_m(z)$ from (3.4), (3.6), and (3.8):

$$D(z)=\sum_{m=-\infty}^{\infty}\hat{T}_m(z)\ .$$

This defines a perturbation series for $D(z)$ formally analogous to (3.12). As we did there, can we can sum up the infinity of terms containing cumulants of the form $\langle \eta_0 \cdots \eta_0 \rangle_T$. This introduces a random operator $\Psi_n(z)$ which is the counterpart of $\psi_n(z)$ in (3.27). We obtain

$$\Psi_n(z)=\sum_{p=0}^{\infty}(\eta_n \mathcal{Q} \mathbb{J}_{nn})^p \eta_n$$

$$=\mathbb{M}_n-\frac{\mathbb{M}_n \mathbb{J}_{nn}}{1+\langle \mathbb{M}_n \rangle \mathbb{J}_{nn}}\mathcal{P}\mathbb{M}_n\ , \tag{4.7}$$

where

$$\mathbb{M}_n=\frac{\mu+\xi_n-\hat{\Gamma}(z)}{1-[\mu+\xi_n-\hat{\Gamma}(z)]\mathbb{J}_{nn}}\ . \tag{4.8}$$

The perturbative series around $\hat{\Gamma}(z)$ for the diffusion coefficient becomes

$$D(z)=\hat{\Gamma}(z)+\sum_{p=0}^{\infty}\sum_{\substack{n_1\,(\neq 0),\\ n_2\,(\neq n_1),\\ \ldots,\\ n_p\,(\neq n_{p-1})}}\langle \Psi_0 \Psi_{n_1}\cdots \Psi_{n_p}\rangle_T \times \mathbb{J}_{0n_1}\cdots \mathbb{J}_{n_{p-1}n_p}\ . \tag{4.9}$$

The sum in the rhs of (4.9) can be represented again as a sum of averages of diagrams. The structure of such diagrams is the same as in Fig. 3, but now the circle means Ψ_n instead of ψ_n and the line means $\mathbb{J}_{nm}\mathcal{Q}$ instead of $J_{nm}\mathcal{Q}$. At this point it becomes clear how to considerably reduce the number of diagrams with a nonvanishing average contributing to (4.9): If we choose $\hat{\Gamma}(z)$ such that $\langle \Psi_n(z) \rangle = 0$, all the diagrams containing isolated circles would have a vanishing average (see Appendix A). This defining condition for $\hat{\Gamma}(z)$ can be written, through (4.7) and (4.8), as

$$\langle \mathbb{M}_n \rangle = \left\langle \frac{w_n-\hat{\Gamma}(z)}{1-[w_n-\hat{\Gamma}(z)]\mathbb{J}_{nn}(z,\hat{\Gamma})} \right\rangle = 0\ . \tag{4.10}$$

This defining condition turns out to coincide with the self-consistency condition of the EMA.[5] $\hat{\Gamma}(z)$ obtained from (4.10) is the diffusion coefficient in such an approximation. So, (4.9) is a perturbation expansion around the EMA. Equation (4.9) will permit us to establish the range of validity of the EMA and a systematic calculation of the corrections to the EMA.

The first diagrams with a nonvanishing average contributing to (4.9) with the choice of $\hat{\Gamma}(z)$ given by (4.10) are displayed in Fig. 4. To stress that circles and lines represent different functions that in Fig. 3, a solid circle has been used to represent $\Psi_n(z)$ and a double line stands for $\mathbb{J}_{nm}\mathcal{Q}$. This perturbative scheme remains well behaved for $z\to\infty$, as it was already the case of (3.12). The low-frequency behavior has to be studied for each particular model of disorder. We summarize, in Appendix B, some technical steps needed in such analysis carried out in Sec. V.

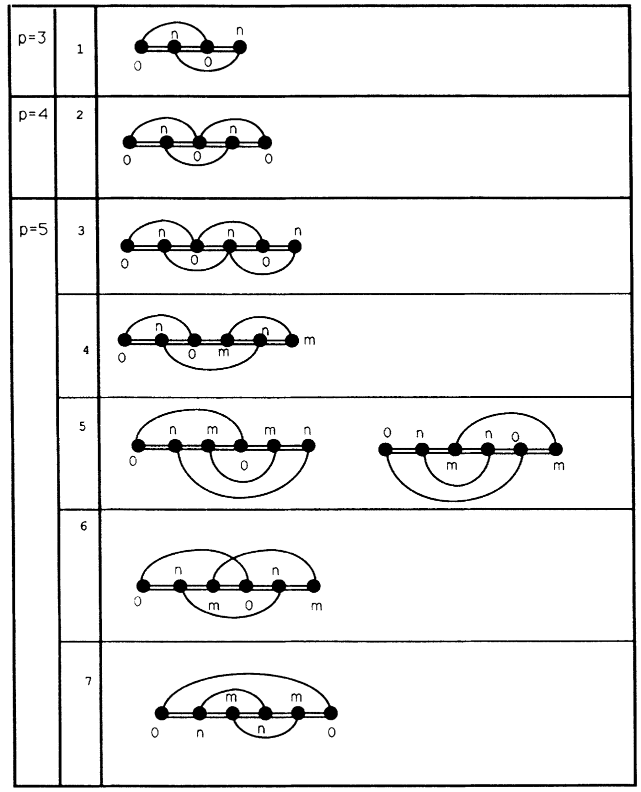

FIG. 4. The first diagrams with a nonvanishing average contributing to Eq. (4.9) with the condition (4.10).

V. DIFFUSION COEFFICIENT FOR MODELS OF WEAK AND STRONG DISORDER

In this section we discuss the explicit calculation of the low-frequency behavior of $D(z)$ for models A, B, and C defined in Sec. II.

A. Weak disorder: Model A

For weak disorder a variety of exact results are known for the low-frequency behavior of $D(z)$. We study this model here as an illustrative case and as a testing bench of our calculational scheme. We begin by considering the order of the average of the diagrams contributing to $D(z)$ in (4.9). For the case of weak disorder, it is known[1(a),(b)] that the EMA predicts $\hat{\Gamma}(z)\sim O(z^0)$, namely,

$$\hat{\Gamma}(z)=\beta_1^{-1}\left[1+\frac{\beta_2-\beta_1^2}{2\beta_1^2}(\beta_1 z)^{1/2}+\frac{3\beta_2^2-2\beta_3\beta_1-\beta_1^4}{8\beta_1^4}(\beta_1 z)+O(z^{3/2})\right],\quad \beta_{\mathrm{M}}\equiv\langle w^{-\mathrm{M}}\rangle\ . \tag{5.1}$$

This can be seen directly from (4.10) and (4.4)—(4.8). Taking into account this fact, it is shown in Appendix B that the diagrams of order p and I internal sums are of order $z^{(p-I)/2}$. Therefore, the most important diagrams of order p are those with the maximum value of I. Given that $1 \le I \le (p-1)/2$ if p is odd and $1 \le I \le p/2-1$ if p is even, we find that the most important of the diagrams of order p is of order $z^{(i[p/2]+1)/2}$, where $i[\ ]$ means the integer part.

This result shows that the perturbation series (4.9) gives a systematic way of calculating the low-frequency behavior of $D(z)$ because higher-order diagrams (larger p) contribute only to higher order in z. In particular, it shows that the EMA gives, for $D(z)$, the exact leading contribution of order z^0.

The first correction to the result of the EMA $D(z)=\hat{\Gamma}(z)$ is given by (4.9) as the average of the diagram with the lowest value of p; that is, diagram 1 of Fig. 4 with $p=3$ which is of order z:

$$\sum_{m\ (\neq 0)} \langle \Psi_0\Psi_m\Psi_0\Psi_m\rangle_{\mathrm{T}}(\mathbb{J}_{0m})^3 = \langle \Psi_0\Psi_1\Psi_0\Psi_1\rangle_{\mathrm{T}} \sum_{m\ (\neq 0)} (\mathbb{J}_{0m})^3 \ . \qquad (5.2)$$

Using (B10) and (B11), we obtain that

$$\langle \Psi_0\Psi_1\Psi_0\Psi_1\rangle_T = \langle \Psi_0\Psi_1\Psi_0\Psi_1\rangle = \langle \mathbb{M}_0^2\rangle^2 \ . \qquad (5.3)$$

From (B2) and (B7) we have, for small z,

$$\langle \mathbb{M}_0^2\rangle^2 \sim \beta_1^{-8}\left\langle \left[\frac{1}{w_0}-\beta_1\right]^2\right\rangle^2 \ . \qquad (5.4)$$

The sum in (5.2) can then be evaluated as

$$\begin{aligned}\sum_{m(\neq 0)} (\mathbb{J}_{0m})^3 &\sim 2\sum_{m=1}^{\infty}\left[\frac{1}{2\hat{\Gamma}}\left[\frac{z}{\hat{\Gamma}}\right]^{1/2}\right]^3 \times\left[1-\left[\frac{z}{\hat{\Gamma}}\right]^{1/2}\right]^{3|m|} \\ &\sim \frac{1}{4\hat{\Gamma}^3}\left[\frac{z}{\hat{\Gamma}}\right]^{3/2}\frac{[1-(z/\hat{\Gamma})^{1/2}]^3}{1-[1-(z/\hat{\Gamma})^{1/2}]^3} \\ &\sim \frac{z}{12}\beta_1^4 \ . \end{aligned} \qquad (5.5)$$

Combining (5.5) and (5.4), we obtain the following expression for the first diagram of Fig. 4 at low frequencies:

$$\frac{z}{12}\beta_1^{-4}\left\langle\left[\frac{1}{w}-\beta_1\right]^2\right\rangle^2 = \frac{z}{12}\beta_1^{-4}(\beta_2-\beta_1^2)^2 \ . \qquad (5.6)$$

This gives the first correction to the EMA calculation of $D(z)$. It agrees with the alternative exact calculation of Denteneer and Ernst.[8] It indicates that, against earlier conjectures,[20] the EMA gives an incorrect result for $D(z)$ in order z, being exact in order z^0 and $z^{1/2}$.

Higher-order corrections to the one given by (5.6) can be calculated in a similar way considering the higher-order behavior of the diagram in (5.2) and also the small-z behavior of the diagrams in Fig. 4 with higher value of p. In summary, the known results for weak disorder are well reproduced within our scheme in which the EMA contributions appear naturally separated from the rest.

B. Model B of strong disorder

We begin by solving the EMA equation (4.10) for $\hat{\Gamma}(z)$. It can be written as

$$\left\langle \frac{w}{w+R}\right\rangle = \hat{\Gamma}\left\langle\frac{1}{w+R}\right\rangle \ , \qquad (5.7)$$

where R is given in (B3). Recalling that, for model B, the probability density for a rate w is $\rho(w)=w$, $w\in(0,1)$, the averages in (5.7) lead to

$$1-R\ln(1+R)+R\ln R = \hat{\Gamma}[\ln(1+R)-\ln R] \ . \qquad (5.8)$$

This is an implicit equation for $\hat{\Gamma}$ that can be solved iteratively. Using (B7) we find

$$\begin{aligned}\hat{\Gamma}(z) = \frac{2}{|\ln z|}\Bigg[1 &- \frac{\ln|\ln z|}{|\ln z|} - \frac{\ln 2}{|\ln z|} + \frac{(\ln|\ln z|)^2}{|\ln z|^2} \\ &+ (2\ln 2-1)\frac{\ln|\ln z|}{|\ln z|^2} + (\ln 2-1)\frac{\ln 2}{|\ln z|^2} \\ &+ O\left[\frac{(\ln|\ln z|)^3}{|\ln z|^3}\right]\Bigg] \ . \end{aligned} \qquad (5.9)$$

The two first terms in (5.9) were obtained within the EMA in Ref. 5(b) and by the replica trick by Stephen and Kariotis.[9] The three first terms had been obtained by the exact calculation by integral equations of Nieuwenhuizen and Ernst.[11] These results imply that the EMA gives the exact result at least up to terms of order $|\ln z|^{-2}$. The question which we now address is to find the order in z for which the EMA result (5.9) becomes incorrect. For this purpose we need to analyze the order in z of the corrections to the EMA as given by the expansion (4.9).

In a term of (4.9) [see (B9)], the order in z of $\sum \mathbb{J}_{0n_1}\cdots \mathbb{J}_{n_{p-1}n_p}$ is given by the general result (B8), but the evaluation of the order in z of the cumulant $\langle \Psi_0\cdots\Psi_{n_p}\rangle_{\mathrm{T}}$ requires some care: The analysis in Appendix B shows [see Eq. (B13)] that a diagram of order p and I internal sums is of order

$$\left[\hat{\Gamma}\left[\frac{\hat{\Gamma}}{z}\right]^{1/2}\right]^{p+1}(z\hat{\Gamma})^{(I+1)/2} \ . \qquad (5.10)$$

This fact, combined with (B8), shows that the diagram of order p and I internal sums is of order

$$\left[\hat{\Gamma}\left[\frac{\hat{\Gamma}}{z}\right]^{1/2}\right]^{p+1}(z\hat{\Gamma})^{(I+1)/2}\left[\frac{1}{\hat{\Gamma}}\left[\frac{z}{\hat{\Gamma}}\right]^{1/2}\right]^{p}\left[\frac{\hat{\Gamma}}{z}\right]^{I/2} \sim O(\hat{\Gamma}^{2+I}),\quad \forall p \ , \qquad (5.11)$$

where $I \ge 1$. Terms with $I=0$ are absent in (4.9). The result (5.11) has two important consequences. First, the leading correction to the EMA is of order $\hat{\Gamma}^3 \sim |\ln z|^{-3}$. Therefore, all the terms explicitly written in (5.9) and obtained within the EMA are exact for $D(z)$ except the last one which needs to be modified by corrections to the

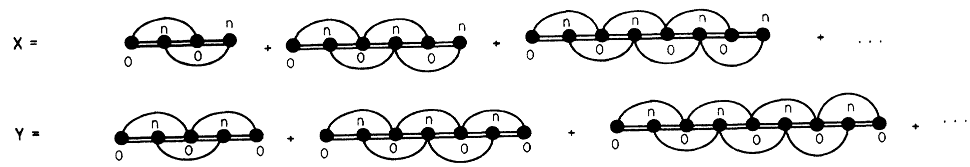

FIG. 5. Diagrammatic representation of Eqs. (5.13) and (5.14) giving the first correction to the EMA for models B and C of strong disorder.

EMA. A second fact is that the expansion around the EMA given by (4.9) is not entirely appropriate for strong disorder because diagrams of arbitrary order in p contribute to the same order in z. This is so because the order of a diagram given by (5.11) depends only on I. We note, however, that the perturbation scheme (4.9) continues to be useful since a diagrammatic resummation can be easily done. Figure 5 shows the *pair diagrams,* that is, those containing two random variables Ψ_0 and Ψ_n, so that $I=1$. According to (5.11) those are the diagrams contributing to the leading order $\hat{\Gamma}^3$. We can write, up to order $\hat{\Gamma}^3$,

$$D(z)=\hat{\Gamma}(z)+\langle X\rangle+\langle Y\rangle \tag{5.12}$$

with

$$\begin{aligned}\langle X\rangle &= \sum_{\substack{p=3,\\ p\text{ odd}}}^{\infty}\ \sum_{n\,(\neq 0)} \langle \Psi_0\Psi_n\Psi_0\Psi_n\cdots\Psi_n\rangle_{\mathrm{T}}(\mathbb{J}_{0n})^p \\ &= \sum_{\substack{p=3,\\ p\text{ odd}}}^{\infty} \langle \Psi_0\Psi_1\Psi_0\Psi_1\cdots\Psi_1\rangle_{\mathrm{T}} \sum_{n\,(\neq 0)}(\mathbb{J}_{0n})^p\end{aligned} \tag{5.13}$$

and

$$\begin{aligned}\langle Y\rangle &= \sum_{\substack{p=4,\\ p\text{ even}}}^{\infty}\ \sum_{n\,(\neq 0)} \langle \Psi_0\Psi_n\Psi_0\Psi_n\cdots\Psi_0\rangle_{T}(\mathbb{J}_{0n})^p \\ &= \sum_{\substack{p=3,\\ p\text{ even}}}^{\infty} \langle \Psi_0\Psi_1\Psi_0\Psi_1\cdots\Psi_0\rangle \sum_{n\,(\neq 0)}(\mathbb{J}_{0n})^p .\end{aligned} \tag{5.14}$$

$\langle X\rangle$ is the average of the first row of Fig. 5 and $\langle Y\rangle$ is the average of the second one. The cumulant in the terms of order p contain $p+1$ factors Ψ.

The sum $\sum(\mathbb{J}_{0n})^p$ can be computed from (4.4)–(4.6) as

$$\begin{aligned}\sum_{n\,(\neq 0)}(\mathbb{J}_{0n})^p &\sim \frac{[\mathbb{A}(z)]^p}{[\mathbb{B}(z)]^p}[\mathbb{B}(z)-1]^{2p}\sum_{n\,(\neq 0)}[\mathbb{B}(z)]^{p|n|} \\ &= \frac{2\mathbb{A}^p[(\mathbb{B}-1)]^{2p}}{1-\mathbb{B}^p} ,\end{aligned} \tag{5.15}$$

which behaves for $z\to 0$ as

$$\sum_{n\,(\neq 0)}(\mathbb{J}_{0n})^p \sim \frac{1}{2^{p-1}p}\left[\frac{1}{\hat{\Gamma}}\left[\frac{z}{\hat{\Gamma}}\right]^{1/2}\right]^p\left[\frac{\hat{\Gamma}}{z}\right]^{1/2} . \tag{5.16}$$

The small-z behavior of the cumulants in (5.13) and (5.14), containing $p+1$ factors Ψ, can be read out from (B13):

p odd:

$$\langle \Psi_0\Psi_1\cdots\Psi_1\rangle_{\mathrm{T}} \sim \left[-2\hat{\Gamma}\left[\frac{\hat{\Gamma}}{z}\right]^{1/2}\right]^{p+1}\left[\frac{(z\hat{\Gamma})^{1/2}}{2}\right]^2 \times\left[\frac{1}{(p+1)/2-1}\right]^2 , \tag{5.17}$$

p even:

$$\langle \Psi_0\Psi_1\cdots\Psi_0\rangle_{\mathrm{T}} \sim \left[-2\hat{\Gamma}\left[\frac{\hat{\Gamma}}{z}\right]^{1/2}\right]^{p+1} \times\left[\frac{(z\hat{\Gamma})^{1/2}}{2}\right]^2\frac{1}{p/2}\,\frac{1}{p/2-1} . \tag{5.18}$$

With these results we have

$$\langle X\rangle+\langle Y\rangle \sim \sum_{\substack{p=3,\\ p\text{ odd}}}^{\infty}\frac{4\hat{\Gamma}^3}{p(p-1)^2}-\sum_{\substack{p=4,\\ p\text{ even}}}^{\infty}\frac{4\hat{\Gamma}^3}{p^2(p-2)}=A\hat{\Gamma}^3 , \tag{5.19}$$

where

$$\begin{aligned}A &\equiv \sum_{k=1}^{\infty}\frac{3k+2}{2k^2(2k+1)(k+1)^2} \\ &= \frac{\pi^2}{4}-5+4\ln 2=0.239\,989\,82\ldots .\end{aligned} \tag{5.20}$$

Equation (5.19) gives the leading correction to the EMA result $D(z)=\hat{\Gamma}(z)$, so that the exact result up to order $|\ln z|^{-3}$ is given by

$$D(z)=\hat{\Gamma}(z)\{1+A[\hat{\Gamma}(z)]^2\} . \tag{5.21}$$

C. Model C of strong disorder

We recall that model C is defined by a probability distribution $\rho(w)=(1-\alpha)w^{-\alpha}$, $0<\alpha<1$, $w\in(0,1)$. The larger α is, the stronger the disorder is. The diffusion coefficient for this model was calculated within the EMA in Ref. 1(a). The leading contribution to $\hat{\Gamma}(z)$, solution of (4.10), is

$$\hat{\Gamma}(z)\sim C_0(\alpha)z^{\alpha/(2-\alpha)} \tag{5.22}$$

with

$$C_0(\alpha)=\left[\frac{\sin(\pi\alpha)}{(1-\alpha)\pi 2^{\alpha}}\right]^{2/(2-\alpha)} . \tag{5.23}$$

There are two other calculations of $D(z)$ for this model.[9,11] A direct comparison of these results with (5.22) and (5.23) has not been made and it has only been conjectured[11] that the EMA gives the correct leading result for $D(z)$. With our perturbation scheme around the EMA, we are now in a position of analyzing the validity of the result of the EMA by evaluating the corrections given by the perturbation series (4.9). The order in z of the cumulant $\langle \Psi_{n_0} \cdots \Psi_{n_p} \rangle_T$ is analyzed in Appendix B. Combining (B8) with this analysis [Eq. (B16)], we find that any diagram in (4.9) is of order $\hat{\Gamma}$, which is of order $z^{\alpha/(2-\alpha)}$, independently of the value of p and the number of internal sums I. The consequence of this fact is twofold. First, since any diagram contains a contribution modifying the lowest-order term given by the EMA, the conjecture[11] that the EMA gives the correct leading contribution turns out to be incorrect. However, it also implies that the leading exponent of z in $D(z)$ is that of $\hat{\Gamma}(z)$ and, therefore, it is correctly given by the EMA. Second is that we would need a ressummation of all the diagrams to obtain the exact low-frequency behavior for any α. Instead of searching for such a summation that would lead us to the complicated expressions of Refs. 9 and 11, we take advantage of our method being specially suitable for an expansion in α, which is an expansion in the strength of the disorder.

For small α, the coefficient $C_0(\alpha)$ of the result of the EMA [Eq. (5.23)] can be expanded as

$$C_0(\alpha)=\alpha\left[1+\frac{\alpha}{2}\ln\alpha+\alpha(1-\ln 2)+\frac{(\alpha\ln\alpha)^2}{8}\right.$$
$$+\frac{\alpha^2\ln\alpha}{2}(\tfrac{3}{2}-\ln 2)$$
$$+\frac{\alpha^2}{2}\left[3-\frac{\pi^2}{3}-3\ln 2+(\ln 2)^2\right]$$
$$\left.+O((\alpha\ln\alpha)^3)\right] . \tag{5.24}$$

The question is how the perturbation around the EMA in (4.9) modifies this coefficient for small α. The analysis in Appendix B shows that the lowest-order correction in α to the coefficient of the leading-z dependence ($z^{\alpha/(2-\alpha)}$) is of order α^{I+2}. This implies that

$$D(z)\sim C(\alpha)z^{\alpha/(2-\alpha)} , \tag{5.25}$$

and, since $I\geq 1$,

$$C(\alpha)=C_0(\alpha)+O(\alpha^3) . \tag{5.26}$$

The first correction for small α to $C_0(\alpha)$ is that of order α^3. This correction can be explicitly calculated within our formalism. Since it is associated with $I=1$, it comes from the sum of diagrams in Fig. 5. Those were the diagrams already contributing to the leading correction to the EMA in model B. It then follows that $D(z)$, to leading order in z and order α^3, is given by (5.12)–(5.14). The required expression for the cumulants involved follows from (B16):

p odd:

$$\langle \Psi_0\Psi_1 \cdots \Psi_1\rangle_T \sim [-2\hat{\Gamma}(\Gamma/z)^{1/2}]^{p+1}(z\hat{\Gamma}/4)^{1-\alpha}$$
$$\times[B(1-\alpha,(p+1)/2+\alpha-1)]^2 , \tag{5.27}$$

p even:

$$\langle \Psi_0\Psi_1 \cdots \Psi_0\rangle_T \sim [-2\hat{\Gamma}(\hat{\Gamma}/z)^{1/2}]^{p+1}(z\hat{\Gamma}/4)^{1-\alpha}$$
$$\times B(1-\alpha,p/2+\alpha)$$
$$\times B(1-\alpha,p/2+\alpha-1) , \tag{5.28}$$

where $B(x,y)$ is the β function.[21] Combining this result with (5.16), we find

$$\langle X\rangle+\langle Y\rangle \sim \hat{\Gamma}2^{2\alpha}C_0^{2-\alpha}\left[\sum_{\substack{p=3,\\ p\text{ odd}}}\frac{[B(1-\alpha,(p-1)/2+\alpha)]^2}{p}-\sum_{\substack{p=4,\\ p\text{ even}}}\frac{B(1-\alpha,p/2+\alpha)B(1-\alpha,p/2+\alpha-1)}{p}\right] . \tag{5.29}$$

Using the expansion (B18) for the β function B, (5.29) reproduces the same series already summed up in (5.20). The final result is then

$$\langle X\rangle+\langle Y\rangle \sim A\alpha^3[1+O(\alpha\ln\alpha)]z^{\alpha/(2-\alpha)} , \tag{5.30}$$

so that

$$C(\alpha)=C_0(\alpha)+A\alpha^3 \tag{5.31}$$

with A given in (5.20).

VI. THE MODIFIED BURNETT COEFFICIENT

The calculational method developed in Secs. III and IV can be used to calculate higher-order moments of $P_n(t)$ or, equivalently, successive diffusion coefficients beyond $D(z)$. This gives information about higher-order statistical properties of the transport process. In this section we discuss the calculation of the modified Burnett coefficient $D_2(z)$, defined in (3.9), for models A, B, and C. As it was the case in the calculation of $D(z)$, model A is discussed to check the consistency of our approach since exact results for the low-frequency behavior of $D_2(z)$ were already reported by Denteneer and Ernst.[8] The situation for strong disorder is much less clear. The available calculation by Nieuwenhuizen and Ernst[11] gives only the exponent of z in the leading term and makes no connection with the results from the EMA. Within our scheme we are able to establish such connection and to obtain the full correct leading term in $D_2(z)$.

We start by noting that from (3.6), (3.9), and the expansion

$$\hat{T}_q(z)=D(z)-q^2F(z)+O(q^4)\ , \tag{6.1}$$

$D_2(z)$ can be calculated as

$$D_2(z)=\tfrac{1}{12}D(z)+F(z)\ . \tag{6.2}$$

It trivially follows from (2.30a) and (3.4) that $F(z)$ vanishes when only effective transitions between nearest-neighbor sites ($m=0$) occur in the EME. This implies that, given $D(z)$, the calculation of $D_2(z)$ becomes nontrivial due to the effective non-Markovian dynamics reflected in the influence on the dynamics of a given site of other sites beyond the nearest neighbors. We also recall that the EMA neglects this contribution since the EMA corresponds to an EME of the type (2.30a) with $m=0$, so that $F(z)=0$. We will show below that, for the three models A, B, and C considered here, the contribution of $F(z)$ to $D_2(z)$ in the low-frequency limit is of the same order (model A) or more important (models B and C) than the contribution of $D(z)$. This means that the effective transition probabilities in (2.30a) with $|m|>0$ are particularly important in the calculation of statistical properties of higher order than $D(z)$.

The explicit calculation of $D_2(z)$ starts substituting (4.2) in (3.4) and applying the resummation (4.7). We then find

$$\begin{aligned}\hat{T}_q(z)&=\hat{\Gamma}(z)+\sum_{p=3}^{\infty}\ \sum_{n_1,\dots,n_p}{}' \langle\Psi_0\Psi_{n_1}\cdots\Psi_{n_p}\rangle_{\mathrm{T}}\mathbb{J}_{0n_1}\cdots\mathbb{J}_{n_{p-1}n_p}\left[1-\frac{q^2n_p^2}{2}+\frac{q^4n_p^4}{4!}+\cdots\right]\\&=D(z)-q^2\tfrac{1}{2}\sum_{p=3}^{\infty}\ \sum_{n_1,\dots,n_p}{}' n_p^2\langle\Psi_0\Psi_{n_1}\cdots\Psi_{n_p}\rangle_{\mathrm{T}}\mathbb{J}_{0n_1}\cdots\mathbb{J}_{n_{p-1}n_p}+O(q^4)\ ,\end{aligned} \tag{6.3}$$

which identities the function $F(z)$ in (6.1) as

$$F(z)=\tfrac{1}{2}\sum_{p=3}\ \sum_{n_1,\dots,n_p}{}' n_p^2\langle\Psi_0\Psi_{n_1}\cdots\Psi_{n_p}\rangle_T\times\mathbb{J}_{0n_1}\cdots\mathbb{J}_{n_{p-1}n_p}\ . \tag{6.4}$$

The prime indicates that the sums contain the same restrictions as in (3.28) or (4.9). $F(z)$ can again be represented as the average of a sum of diagrams topologically equivalent to those of Fig. 4. The difference is that now an extra factor n_p^2 accompanies the variable Ψ_{n_p} represented by the last circle in the diagram. The presence of this factor reduces the number of diagrams since some of them, such as diagrams 2 and 7 of Fig. 4, vanish because $n_p=0$ for these terms. The order in z of the different diagrams can be calculated along the lines previously discussed. The order of the sum $\sum n_p^2\mathbb{J}_{0n_1}\cdots\mathbb{J}_{n_{p-1}n_p}$ is calculated repeating the argument leading to (3.21) or (B8), but now using

$$\sum_{n\,(\neq 0)} n^2a^{|n|x}=\frac{2a^x(1+a^x)}{(1-a^x)^3} \tag{6.5}$$

instead of (3.20). We then find

$$\sum n_p^2\mathbb{J}_{0n_1}\cdots\mathbb{J}_{n_{p-1}n_p}\sim\left[\frac{1}{\hat{\Gamma}}\left[\frac{z}{\hat{\Gamma}}\right]^{1/2}\right]^p\left[\frac{\hat{\Gamma}}{z}\right]^{(I+2)/2}\ . \tag{6.6}$$

The order in z of the cumulant $\langle\Psi_0\Psi_{n_1}\cdots\Psi_{n_p}\rangle_{\mathrm{T}}$ in (6.4) is that already obtained in the calculation of $D(z)$. It is of order z^0 for model A and that given in (B13) and in (B16) for models B and C, respectively. Combining these results with (6.6), we finally find the order in z of the most important diagrams of order p:

model A:

$$(z^{(p-I)/2-1})_{I_{\max}}\sim z^{\{i[p/2]-1\}/2}\ , \tag{6.7}$$

model B:

$$\left[\frac{\hat{\Gamma}^{I+3}}{z}\right]_{I=1}\sim\frac{1}{z|\ln z|^4}\ , \tag{6.8}$$

model C:

$$(\hat{\Gamma}^2\hat{\Gamma}^{(I+1)(2-\alpha)/2}z^{-\alpha(I+1)/2-1})\sim z^{(3\alpha-2)/(z-\alpha)}\ . \tag{6.9}$$

The situation is similar to what we had found for $D(z)$. For model A, diagrams with higher p are less important, so that (6.4) gives a systematic perturbation expansion. For models B and C, the order of the diagrams is independent of p. For model B only diagrams with $I=1$ contribute to the lowest order in z. For model C the order of a diagram is also independent of I so that all diagrams contribute to $F(z)$ in the lowest order in z.

Next we explicitly calculate the leading contribution to $D_2(z)$ for models A, B, and C. For model A, the leading contribution to $D(z)$ is correctly given by the EMA as discussed in Sec. V,

$$D(z)=\beta_1^{-1}+O(z^{1/2})\ .$$

The first contribution to $F(z)$ is given according to (6.7) by the diagram with $p=3$ (diagram 1 in Fig. 4) which is also of order z^0. The consequence is that both $D(z)$ and $F(z)$ contribute to the leading order. Therefore, the EMA, for which $F(z)=0$, does not reproduce the correct result. The explicit contribution of $F(z)$ is calculated from (5.3), (5.4), and (4.4)–(4.6) as

$$\frac{1}{2}\sum_{n\,(\neq 0)}\langle\Psi_0\Psi_n\Psi_0\Psi_n\rangle_T(\mathbb{J}_{0n})^3n^2\sim\frac{1}{2}\Gamma_0^8\left\langle\left[\frac{1}{w}-\beta_1\right]^2\right\rangle^2\times\sum_{n\,(\neq 0)}n^2(\mathbb{J}_{0n})^3\sim\left\langle\left[\frac{1}{w}-\beta_1\right]^2\right\rangle^2\times\frac{1}{108}\beta_1^{-5}\ . \tag{6.10}$$

The modified Burnett coefficient for model A is then

$$D_2(z)=\left[\frac{1}{12}+\frac{1}{108}\left[1-\frac{\beta_2}{\beta_1^2}\right]^2\right]\beta_1^{-1}+O(z^{1/2})\ . \tag{6.11}$$

This result reproduces the one in Ref. 8.

For model B the leading contribution to $F(z)$ is given by the sum of diagrams with $I=1$, that is $F(z)=\langle X\rangle$, where X is defined diagramatically in Fig. 5. We recall that $\langle Y\rangle$ in Fig. 5 vanishes here because of the extra factor $n_p=0$. Using (4.4)–(4.6) and (6.5), we have

$$\langle X\rangle=\sum_{\substack{p=3,\\ p\text{ odd}}}^{\infty}\frac{1}{2}\sum_n n^2\langle\Psi_0\Psi_n\cdots\Psi_n\rangle_T(\mathbb{J}_{0n})^p\sim\sum_{\substack{n=3,\\ p\text{ odd}}}\langle\Psi_0\Psi_1\cdots\Psi_1\rangle_T\frac{2}{2^pp^3}\times\left[\frac{1}{\hat\Gamma}\left[\frac{z}{\hat\Gamma}\right]^{1/2}\right]^p(\hat\Gamma/z)^{3/2}\ , \tag{6.12}$$

and replacing (5.27), we get

$$\langle X\rangle\sim\sum_{\substack{p=3,\\ p\text{ odd}}}\frac{4}{p^3(p-1)^2}\frac{\hat\Gamma^4}{z}\sim\frac{16B}{z|\ln z|^4}\ , \tag{6.13}$$

where

$$B\equiv\sum_{k=1}^{\infty}\frac{1}{(2k+1)^3k^2}=\frac{7}{2}\zeta(3)-24+12\ln 2+\frac{7}{6}\pi^2=0.039\,503\,795\ldots\ . \tag{6.14}$$

Here, $\zeta(3)$ is a Riemann ζ function.[21] The important fact is that, for small z, this contribution is greater than the leading one of $D(z)$ coming from the EMA, so that

$$D_2(z)\sim F(z)\sim\frac{16B}{z|\ln z|^4}\ . \tag{6.15}$$

We note that this exact result disagrees with the estimation of the z dependence of the leading order of $D_2(z)$ given in Ref. 11 by an integral equation method.

For model C, all diagrams for $F(z)$ contribute to the leading order in z given by

$$F(z)=F_0z^{(3\alpha-2)/(2-\alpha)}\ . \tag{6.16}$$

We first note that, since $D(z)\sim z^{\alpha/(2-\alpha)}$ [Eq. (5.25)], the leading contribution to $D_2(z)$ is given by $F(z)$. A consequence is that, as in model B, the EMA does not give even the correct exponent of z in the leading term of $D_2(z)$. The correct dependence given by (6.16) agrees with the estimation of Nieuwenhuizen and Ernst.[11] With our perturbation method and in analogy with the calculation of $D(z)$, we are now able to calculate F_0 order by order in α. From (6.9), we see that the coefficient of the leading term in any diagram is of order α^{I+3}, so that the first contribution for small α to F_0 comes from diagrams with $I=1$; that is, again those contributing to $\langle X\rangle$ defined diagramatically in Fig. 5. From (6.6) and (B16), we obtain that, to lowest order in α, the leading z contribution is

$$\langle X\rangle\sim\sum_{\substack{p=3,\\ p\text{ odd}}}^{\infty}\frac{4\alpha^4}{p^3(p-1)^2}z^{(3\alpha-2)/(2-\alpha)}=B\alpha^4z^{(3\alpha-2)/(2-\alpha)}\ , \tag{6.17}$$

where B is defined in (6.14). In summary, $F_0=B\alpha^4$ and

$$D_2(z)=(B\alpha^4+\cdots)z^{(3\alpha-2)/(2-\alpha)}+\cdots\ . \tag{6.18}$$

VII. THE RESPONSE FUNCTION FOR THE RANDOM-BARRIER AND THE RANDOM-TRAP MODELS

The response function $\Pi_n(t)$ is defined for each configuration of the disorder as the probability for a displacement of size n, starting from a stationary situation; that is,

$$\Pi_n(t)\equiv\sum_m\mathbb{G}_{n+m,\,m}(t)P_m^{\mathrm{st}}\ , \tag{7.1}$$

where P_m^{st} is the stationary solution of (2.1) and $\mathbb{G}_{n,n'}(t)$ is the Green's function of (2.1); that is, its solution with initial condition $\mathbb{G}_{nn'}(t=0)=\delta_{nn'}$. We are interested in the average over disorder $\langle\Pi_n(t)\rangle$ of (7.1). A frequency- and wave-number-dependent diffusion coefficient $D_q^{\Pi}(z)$ can be associated to $\Pi_n(t)$ as in (3.5):

$$\langle\hat\Pi_q(z)\rangle\equiv\frac{1}{z+q^2D_q^{\Pi}(z)}\ . \tag{7.2}$$

$\hat\Pi_q(z)$ is the Fourier and Laplace transform of $\Pi_n(t)$. For the RB model, $P_m^{\mathrm{st}}=1/N$, where N is the number of sites in the chain. N must be taken as infinity at the end of the calculation. From (7.1),

$$\Pi_n(t)=(1/N)\sum_m\mathbb{G}_{n+m,m}(t)\ ,$$

and invoking the statistical homogeneity of the medium, $\langle\Pi_n(t)\rangle=\langle\mathbb{G}_{n,0}(t)\rangle$. Then, the average response function coincides with the average Green's function, the main quantity considered in this paper. This function satisfies Eq. (2.30). For the RT model, we have

$$P_n^{\mathrm{st}}=\frac{1/w_n}{\sum_m 1/w_m}\ , \tag{7.3}$$

so that, in this case, $\langle\Pi_n(t)\rangle$ does not coincide with the average Green's function.

In the remainder of this section we present a calculation of $D_q^{\text{II}}(z)$ for the RT model in the case of weak disorder as an example of the application of (2.25) with (2.29). $\Pi_n(t)$ does not satisfy a ME of the form (2.1), so that it must be calculated indirectly. Following Ref. 8, we define

$$\mathbb{P}_{nn'} \equiv N\delta_{nn'}P_n^{\text{st}} \tag{7.4}$$

and

$$\Pi_{nn'}(t) \equiv \sum_{n''} \mathbb{G}_{n,n''}(t)\mathbb{P}_{n'',n'} \ . \tag{7.5}$$

The matrix $\Pi_{nn'}(t)$ satisfies the ME (2.1) with (2.10) and (2.13):

$$\partial_t \Pi_{nn'}(t) = \mathbb{K}_n(\mu+\xi_n)\Pi_{nn'}(t) \ . \tag{7.6}$$

The initial condition for $\Pi_{nn'}(t)$ is

$$\Pi_{nn'}(t) = \mathbb{P}_{nn'} = N\delta_{nn'}P_n^{\text{st}} \ . \tag{7.7}$$

The Fourier transform of $\Pi_n(t)$ can be calculated from the Fourier transform of $\Pi_{nn'}(t)$:

$$\begin{aligned} \hat{\Pi}_q(t) &= \hat{\Pi}_{qq}(t) \ , \\ \hat{\Pi}_{qq'}(t) &= \frac{1}{N}\sum_{n,n'} e^{inq}\Pi_{nn'}(t)e^{-in'q'} \ . \end{aligned} \tag{7.8}$$

For the case of weak disorder, the central limit theorem allows us to write (7.3) as

$$P_n^{\text{st}} = \frac{D}{N}\frac{1}{w_n} \ , \tag{7.9}$$

where, for N large enough,

$$D \to \langle 1/w_n \rangle^{-1} = \beta_1^{-1} \ . \tag{7.10}$$

Then, from (7.7), $\Pi_{nn'}(t)$ is the solution of (7.6) with initial condition

$$\Pi_{nn'}(t=0) = \delta_{nn'}D/w_n \ . \tag{7.11}$$

The average over the disorder of (7.6) leads to an equation formally identical to (2.25). Its explicit form in the Laplace representation can be compactly written as

$$z\langle \tilde{\Pi}_{nn'}(z)\rangle - \delta_{nn'} = \sum_{m=-\infty}^{\infty} \hat{T}_m(z)\mathbb{K}_n(E_n^+)^m\langle \tilde{\Pi}_{nn'}(z)\rangle + \Omega_{nn'}(z) \ . \tag{7.12}$$

$\hat{T}_m(z)$ is given in (2.30b) and $\Omega_{nn'}(z)$ is given by (2.29) with $\Pi_{nn'}(t=0)$ as initial condition:

$$\Omega_{nn'}(t) = \mathbb{K}_n \sum_{p=0}^{\infty} \sum_{n_1,\dots,n_p m} J_{nn_1}(z)\cdots J_{n_{p-1}n_p}(z)\tilde{G}_{n_p m}(z) \times \langle \xi_n\xi_{n_1}\cdots\xi_{n_p}\Pi_{mn'}(t=0)\rangle_{\text{T}} . \tag{7.13}$$

This last expression can be further manipulated. By using

$$z\tilde{G}_{nn'}(z) - \delta_{nn'} = \mu\mathbb{K}_n\tilde{G}_{nn'}(z) = \mu J_{nn'}(z) \ , \tag{7.14}$$

the initial condition (7.11) and the definition of Terwiel's cumulants, we find

$$\Omega_{nn'}(z) = \frac{\mathbb{K}_n}{z}\left[(D-\mu)\delta_{nn'} - \sum_{p=0}^{\infty}\sum_{n_1,\dots,n_p} \langle \xi_n\xi_{n_1}\cdots\xi_{n_p}\xi_{n'}\rangle_{\text{T}} \times J_{nn_1}(z)\cdots J_{n_p n'}(z) \right] . \tag{7.15}$$

From (7.12) and (7.15) we finally find, for the Fourier- and Laplace-transformed response function,

$$\langle \hat{\Pi}_q(z)\rangle = \langle \hat{\Pi}_{qq}(z)\rangle = \frac{1+2(\cos q-1)[D-\hat{T}_q(z)]/z}{z+2(1-\cos q)\hat{T}_q(z)} \ . \tag{7.16}$$

$\hat{T}_q(z)$ is given in (3.4).

The exact expression for the frequency- and wavelength-dependent diffusion coefficient is extracted from (7.16) and (7.2):

$$D_q^{\text{II}}(z) = \frac{2Dz(1-\cos q)}{q^2\{z-2(1-\cos q)[D-\hat{T}_q(z)]\}} \tag{7.17}$$

and

$$D^{\text{II}}(z) = \lim_{q\to 0} D_q^{\text{II}}(z) = \langle w_n^{-1}\rangle^{-1}, \quad \forall z \ . \tag{7.18}$$

This is the known[22] exact result for the diffusion coefficient of the response function for the RT model. We have seen how it can be recovered without our formalism.

The modified Burnett coefficient can also be calculated from (7.17) and an expansion similar to (3.7):

$$D_2^{\text{II}}(z) = D\left[\frac{1}{12} + \frac{D(z)-D}{z}\right] . \tag{7.19}$$

$D(z)$ is the diffusion coefficient associated with the Green's function; that is, that given in (3.12) or (4.9) and explicitly calculated up to order z^1 combining (5.1) and (5.6). Equation (7.19) is a remarkable exact relation between the Burnett coefficient associated with the average response function and the diffusion coefficient associated with the average Green's function for the RT model and weak disorder. It permits the calculation of $D_2^{\text{II}}(z)$ up to order z^{k-1} if $D(z)$ is known up to order z^k. In particular, from (5.1) and (5.6), $D_2^{\text{II}}(z)$ is, up to order z^0,

$$D_2^{\text{II}}(z) = \beta_1^{-1}\left[\frac{\beta_2-\beta_1^2}{2\beta_1^2}(\beta_1 z)^{-1/2} + \frac{11\beta_2^2-6\beta_3\beta_1-4\beta_1^2\beta_2-\beta_1^4}{24\beta_1^4} + \tfrac{1}{12} + O(z^{1/2}) \right] . \tag{7.20}$$

Analogously, the result for $D_2^{\rm II}(z)$ up to order z given in Ref. 8 can be reobtained from the result for $D(z)$ up to order z^2, also given there.

VIII. SUMMARY AND CONCLUSIONS

In this paper we have presented a diagrammatic calculational scheme which fully incorporates the non-Markovian effective behavior caused by the averaging over disorder configurations. The method is based in an exact effective non-Markovian equation. It gives a unified approach to the calculation of transport properties in cases of weak and strong disorder. It leads to a perturbation expansion around the EMA so that it naturally identifies the domain of validity of the EMA and permits us to improve it. Our scheme has been applied to the RB and RT models of RW in the context of the models of disorder A, B and C of Ref. 1(a). Our treatment of arbitrary initial conditions in the non-Markovian effective dynamics identifies the equivalence of the RB and the RT models in the calculation of the probability distribution of the diffusing particle for a fixed (nonrandom) initial condition. The different response functions for the two models have been considered. We have also shown that the influence in the effective non-Markovian dynamics at a given site of sites beyond its nearest neighbors becomes dominant in the calculation of the modified Burnett coefficient for strong disorder. These effects are not taken into account within the EMA.

Our main specific results are for the diffusion coefficient and modified Burnett coefficient. We reproduce here their complete expressions for convenience. Model A has been used as a consistency check of our scheme. For this model of weak disorder, the diffusion coefficient of the average Green's function is, from (5.1) and (5.6),

$$D(z)=\beta_1^{-1}\left[1+\frac{\beta_2-\beta_1^2}{2\beta_1^2}(\beta_1 z)^{1/2}+\frac{11\beta_2^2-6\beta_3\beta_1-4\beta_1^2\beta_2-\beta_1^4}{24\beta_1^4}(\beta_1 z)+O(z^{3/2})\right], \tag{8.1}$$

Where the first two terms are correctly given by the EMA and the term of order z contains a contribution from the EMA and the correction given in (5.6). The modified Burnett coefficient for model A is given by (6.11). Equations (8.1) and (6.11) coincide with the results of Ref. 8. The known result for the diffusion coefficient associated with the response function of the RT model, $D^{\rm II}(z)=\beta_1^{-1}$, $\forall z$, has been recovered. We have also obtained an exact relation [Eq. (7.19)] valid for all frequencies, which permits the calculation of $D_2^{\rm II}(z)$ from the knowledge of $D(z)$.

For model B the diffusion coefficient associated with the average Green's function is given by

$$D(z)=\frac{2}{|\ln z|}\left[1-\frac{\ln|\ln z|}{|\ln z|}-\frac{\ln 2}{|\ln z|}+\frac{(\ln|\ln z|)^2}{|\ln z|^2}+(2\ln 2-1)\frac{\ln|\ln z|}{|\ln z|^2}+\frac{\ln 2(\ln 2-1)+4A}{|\ln z|^2}+O\left[\frac{(\ln|\ln z|)^3}{|\ln z|^3}\right]\right],$$
$$A=\pi^2/4-5+4\ln 2\ . \tag{8.2}$$

The three first terms reproduce the result in Ref. 11.[23] The sixth term, of order $|\ln z|^{-3}$ contains the first correction to the result of the EMA given by the coefficient A. Within the EMA, $A=0$. The modified Burnett coefficient for model B is given by (6.15).

For model C, the lowest-order contribution in z to the diffusion coefficient of the average Green's function is given, for small α, by

$$D(z)=[C_0(\alpha)+A\alpha^3+O(\alpha^4\ln\alpha)]z^{\alpha/(2-\alpha)}+\cdots\ , \tag{8.3}$$

where $C_0(\alpha)$ is given in (5.24) and A is the same coefficient as that in model B, so that the first correction to the EMA appears in the amplitude of the leading term in order α^3. The modified Burnett coefficient for model C is given by (6.18).

Finally we point out that the calculational scheme developed here can be generalized to deal with problems in several dimensions. It is also possible to use this method to treat problems of dynamic disorder. We also point out that a related scheme can be developed for the calculation of transport properties for disordered continuous media,[24] as well as passage-time properties of RW in the presence of disorder.[3]

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APPENDIX A: PROPERTIES OF TERWIEL'S CUMULANTS AND RELATED STATISTICAL QUANTITIES

Terwiel's cumulants are defined as[14]

$$\langle X_1X_2\cdots X_n\rangle_{\rm T}\equiv\mathcal{P}X_1\mathcal{Q}X_2\cdots\mathcal{Q}X_n\ , \tag{A1}$$

where $\{X_i\}$ are random quantities and $\mathcal{P}$ average over its probability distribution. By induction, the following expression for Terwiel's cumulants in terms of moments of X_i is obtained:

$$\langle X_1\cdots X_n\rangle_{\rm T}=\sum_{i=0}^{n-1}(-1)^i\sum_{1\le l_1<\cdots<l_i<n}\langle X_1\cdots X_{l_1}\rangle\langle X_{l_1+1}\cdots X_{l_2}\rangle\cdots\langle X_{l_i+1}\cdots X_n\rangle\ . \tag{A2}$$

Some examples of this formula are

$$\langle X_1\rangle_T=\langle X_1\rangle ,$$
$$\langle X_1X_2\rangle_T=\langle X_1X_2\rangle-\langle X_1\rangle\langle X_2\rangle ,$$
$$\langle X_1X_2X_3\rangle_T=\langle X_1X_2X_3\rangle-\langle X_1\rangle\langle X_2X_3\rangle-\langle X_1X_2\rangle\langle X_3\rangle+\langle X_1\rangle\langle X_2\rangle\langle X_3\rangle , \tag{A3}$$
$$\langle X_1X_2X_3X_4\rangle_T=\langle X_1X_2X_3X_4\rangle-\langle X_1\rangle\langle X_2X_3X_4\rangle-\langle X_1X_2\rangle\langle X_3X_4\rangle-\langle X_1X_2X_3\rangle\langle X_4\rangle+\langle X_1\rangle\langle X_2\rangle\langle X_3X_4\rangle$$
$$+\langle X_1\rangle\langle X_2X_3\rangle\langle X_4\rangle+\langle X_1X_2\rangle\langle X_3\rangle\langle X_4\rangle-\langle X_1\rangle\langle X_2\rangle\langle X_3\rangle\langle X_4\rangle .$$

Expressions (A2) and (A3) are valid when $\{X_i\}$ are noncommuting random operators as well as when they are simple random variables, provided that the ordinates of the factors in these formulas is preserved.

It is easy to derive from (A2) two important properties of Terwiel's cumulants.

(i) If it is possible to split the ordered set $\{X_1 \cdots X_n\}$ appearing in (A1) into two sets, $\{X_1 \cdots X_k\}$ and $\{X_{k+1} \cdots X_n\}$ without altering the ordinates and in such a way that the variables in one set are statistically independent from those in the other set, the cumulant vanishes. This can be seen from (A2) by noting that some of the moments factorize and cancel out with other terms.

(ii) If some random quantity in (A1) has a vanishing average ($\langle X_k\rangle=0$), this quantity appears only once in the cumulant and it is independent of the rest of the random variables in the cumulant, the cumulant vanishes. This is due again to the factorizing of some of the moments in (A2).

A quantity related to a Terwiel's cumulant is the moment $\langle Z_1 \cdots Z_n\rangle$, where $\{Z_i\}$ are random operators of the form

$$Z_i=V_i(1-W_i\mathcal{P}V_i) . \tag{A4}$$

V_i is a random variable and W_i is a nonrandom quantity. ψ and Ψ in (3.27b) and (4.7) [see (B1)] are random operators of the general form (A4). The algebraic steps needed to evaluate the moment

$$\langle Z_1 \cdots Z_n\rangle=\mathcal{P}V_1(1-W_1\mathcal{P}V_1)V_2(1-W_2\mathcal{P}V_2)\cdots V_n(1-W_n\langle V_n\rangle) \tag{A5}$$

are very similar to those needed to obtain (A2) from (A1). The final result is

$$\langle Z_1 \cdots Z_n\rangle=\sum_{i=0}^{n}(-1)^t\sum_{1\le l_1<\cdots<l_t\le n}\langle V_1V_2\cdots V_{l_1}\rangle W_{l_1}\langle V_{l_1}V_{l_1+1}\cdots V_{l_2}\rangle W_{l_2}\langle V_{l_2}\cdots\rangle\cdots W_{l_t}\langle V_{l_t}\cdots V_n\rangle . \tag{A6}$$

Some explicit examples are

$$\langle Z_1Z_2\rangle=\langle V_1V_2\rangle-\langle V_1\rangle W_1\langle V_1V_2\rangle-\langle V_1V_2\rangle W_2\langle V_2\rangle+\langle V_1\rangle W_1\langle V_1V_2\rangle W_z\langle V_2\rangle ,$$
$$\langle Z_1Z_2Z_3\rangle=\langle V_1V_2V_3\rangle-\langle V_1\rangle W_1\langle V_1V_2V_3\rangle-\langle V_1V_2\rangle W_2\langle V_2V_3\rangle \tag{A7}$$
$$-\langle V_1V_2V_3\rangle W_3\langle V_3\rangle+\langle V_1\rangle W_1\langle V_1V_2\rangle W_2\langle V_2V_3\rangle+\langle V_1\rangle W_1\langle V_1V_2V_3\rangle W_3\langle V_3\rangle$$
$$+\langle V_1V_2\rangle W_2\langle V_2V_3\rangle W_3\langle V_3\rangle-\langle V_1\rangle W_1\langle V_1V_2\rangle W_2\langle V_2V_3\rangle W_3\langle V_3\rangle .$$

APPENDIX B: LOW-FREQUENCY BEHAVIOR OF SOME DIAGRAMS

In this appendix we analyze the low-frequency behavior of the terms in our diagrammatic expansion. The explicit form of $\Psi_n(z)$ in (4.7) can be simplified using (4.10):

$$\Psi_n(z)=\mathbb{M}_n(1-\mathbb{J}_{nn}\mathcal{P}\mathbb{M}_n) . \tag{B1}$$

The order in z of the cumulants in (4.9) depends on the order of $\mathbb{M}_n$ and $\mathbb{J}_{nn}$. From (4.8), $\mathbb{M}_n$ can be conveniently written as

$$\mathbb{M}_n=[\hat{\Gamma}(z)+R(z)]\frac{w_n-\hat{\Gamma}(z)}{w_n+R(z)} , \tag{B2}$$

where

$$R\equiv-\hat{\Gamma}-\frac{1}{\mathbb{J}_{00}} . \tag{B3}$$

The explicit form of $\mathbb{J}_{nn}$ is obtained from (4.4)–(4.6):

$$\mathbb{J}_{nn}=\mathbb{J}_{00}=-\frac{1}{\hat{\Gamma}}\left[1-\frac{z}{(z^2+4\hat{\Gamma}z)^{1/2}}\right] . \tag{B4}$$

For models A, B, and C defined in Sec. II it is known[1] that $\hat{\Gamma}(z)$ obtained from the EMA satisfies[25]

$$\lim_{z\to 0}\frac{z}{\hat{\Gamma}(z)}=0 . \tag{B5}$$

The small z behavior of (B4) is then

$$\mathbb{J}_{nn}\sim\frac{-1}{\hat{\Gamma}}[1-\tfrac{1}{2}(z/\hat{\Gamma})^{1/2}+O((z/\Gamma)^{3/2})] , \tag{B6}$$

and that of (B3) is

$$R(z)\sim\frac{(\hat{\Gamma}z)^{1/2}}{2}+\frac{z}{4}+O\left[\frac{z^{3/2}}{\hat{\Gamma}^{1/2}}\right]. \tag{B7}$$

To estimate the order in z of a diagram it is also important to consider the sums of the form $\sum \mathbb{J}_{0n_1}\cdots\mathbb{J}_{n_{p-1}n_p}$, where there are p factors $\mathbb{J}_{n_in_j}$ and $I+1$ of the indexes are different. The sum is over the I indices different from zero. Repeating the argument leading to (3.21), and taking into account (B6) and (B7), this sum is of the order of

$$\left[\frac{1}{\hat{\Gamma}}\left[\frac{z}{\hat{\Gamma}}\right]^{1/2}\right]^{p}\left[\frac{\hat{\Gamma}}{z}\right]^{1/2}. \tag{B8}$$

Finally, due to the fact that the internal sums do not act on the cumulant [because of (3.13)], the average of a diagram is of the general form

$$\langle\Psi_0\Psi_{n_1}\cdots\Psi_{n_p}\rangle_{\mathrm{T}}\sum\mathbb{J}_{0n_1}\cdots\mathbb{J}_{n_{p-1}n_p}. \tag{B9}$$

We now study the low-frequency behavior of the cumulants in this last expression. For easier reference, we particularize the general expressions (A2) and (A6) to cumulants and moments of the random operators Ψ_n:

$$\langle\Psi_1\cdots\Psi_n\rangle_{\mathrm{T}}=\sum_{i=0}^{n-1}(-1)^i\sum_{1\le l_1<\cdots<l_i<n}\langle\Psi_1\cdots\Psi_{l_1}\rangle\langle\Psi_{l_1+1}\cdots\Psi_{l_2}\rangle\cdots\langle\Psi_{l_i+1}\cdots\Psi_n\rangle, \tag{B10}$$

$$\langle\Psi_1\cdots\Psi_n\rangle=\sum_{i=0}^{n}(-1)^i\sum_{1\le l_1<\cdots<l_i\le n}\langle\mathbb{M}_1\mathbb{M}_2\cdots\mathbb{M}_{l_1}\rangle\mathbb{J}_{l_1l_2}\langle\mathbb{M}_{l_1}\mathbb{M}_{l_1+1}\cdots\mathbb{M}_{l_2}\rangle\mathbb{J}_{l_2l_2}\langle\mathbb{M}_{l_2}\cdots\rangle\cdots\mathbb{J}_{l_il_i}\langle\mathbb{M}_{l_i}\cdots\mathbb{M}_n\rangle. \tag{B11}$$

We begin by considering the case of weak disorder. The order in z of the cumulant $\langle\Psi_0\Psi_{n_1}\cdots\Psi_{n_p}\rangle_{\mathrm{T}}$ can be estimated as follows: From (B2), (B7), and the fact that all the moments $\langle w^{-\mathbf{M}}\rangle$ are finite, we see that $\langle\mathbb{M}_j(z)^N\rangle\sim O(z^0)$, for all $N>1$ [if $N=1$ we have, from (4.10), $\langle\mathbb{M}\rangle=0$]. From (B6), also $\mathbb{J}_{nn}\sim O(z^0)$. Applying (B11) now, we find that $\langle\Psi_0\cdots\Psi_{n_p}\rangle$ is of order z^0 and the same is valid [from (B10)] for $\langle\Psi_0\cdots\Psi_{n_p}\rangle_{\mathrm{T}}$. Combining this fact with (B8), the order in z of the diagram in (B9) is $z^{(p-I)/2}$.

We next consider model B of strong disorder. To estimate the order in z of $\langle\Psi_0\cdots\Psi_{n_p}\rangle_{\mathrm{T}}$, we first calculate the small-z behavior of $\langle\mathbb{M}_j(z)^N\rangle$ for $N>1$ [if $N=1$, $\langle\mathbb{M}_j(z)\rangle=0$]:

$$\begin{aligned}\langle\mathbb{M}_j(z)^N\rangle&\sim\hat{\Gamma}^N\{1+O[(z/\hat{\Gamma})^{1/2}]\}\int_0^1 dw\frac{(w-\hat{\Gamma})^N}{(w+R)^N}\\&\sim\left[-2\hat{\Gamma}\left[\frac{\hat{\Gamma}}{z}\right]^{1/2}\right]^N\frac{(z\hat{\Gamma})^{1/2}}{2(N-1)}+\cdots.\end{aligned} \tag{B12}$$

Then, for averages of the form $\langle\mathbb{M}_{n_0}\cdots\mathbb{M}_{n_p}\rangle$ in which I of the indexes $\{n_0,\dots,n_p\}$ are different, $\{i_1,\dots,i_I\}$, being the i_j repeated m_j times ($\sum_{j=1}^{\infty}m_j=p+1$), we find

$$\begin{aligned}\langle\mathbb{M}_{n_0}\cdots\mathbb{M}_{n_p}\rangle&=\langle\mathbb{M}_{i_1}^{m_1}\rangle\langle\mathbb{M}_{i_2}^{m_2}\rangle\cdots\langle\mathbb{M}_{i_I}^{m_I}\rangle\\&\sim\left[-2\hat{\Gamma}\left[\frac{\hat{\Gamma}}{z}\right]^{1/2}\right]^{p+1}\\&\quad\times\left[\frac{(z\hat{\Gamma})^{1/2}}{2}\right]^{I+1}\frac{1}{(m_0-1)\cdots(m_I-1)}\end{aligned} \tag{B13}$$

and $\langle\mathbb{M}_{n_0}\cdots\mathbb{M}_{n_p}\rangle$ vanishes if any $m_i=1$.

In the expansion of $\langle\Psi_{n_0}\cdots\Psi_{n_p}\rangle$ in moments of $\mathbb{M}$ [Eq. (B11)], we see that the dominant term for $z\to 0$ is precisely (B13). This is so because all the other terms contain at least an additional fragmentation. This introduces at least (a) a factor $(z\hat{\Gamma})^{1/2}$, (b) some new term $\mathbb{M}_j$ is added inside some moment, introducing a factor of order $\hat{\Gamma}(\hat{\Gamma}/z)^{1/2}$ if it is repeated, or making the moment vanish if it is not, and (c) a factor $\mathbb{J}\sim O(1/\hat{\Gamma})$. Then, all the other terms in (B11) contain at least a factor of order $\hat{\Gamma}\sim O(|\ln z|^{-1})$ with respect to (B13).

Finally, in the expansion of $\langle\Psi_0\cdots\Psi_{n_p}\rangle_{\mathrm{T}}$ in moments of Ψ (B10), the term with $i+1$ moments is of order

$$\left[\hat{\Gamma}\left[\frac{\hat{\Gamma}}{z}\right]^{1/2}\right]^{p+1}(\hat{\Gamma}z)^{1/2\sum(I_k+1)}, \tag{B14}$$

where the sum runs from $k=0$ to i, and $\{I_k\}$ are the number of different subindexes of $\mathbb{M}$ inside each moment. Since

$$\sum_{k=0}^{\infty}(I_k+1)>I+1,$$

the dominant term is that containing the moment $\langle\Psi_0\cdots\Psi_{n_p}\rangle$, given for small z by (B13).

We now analyze the low-frequency behavior of $\langle\Psi_0\cdots\Psi_{n_p}\rangle_{\mathrm{T}}$ for model C of strong disorder. The small z behavior of $\langle\mathbb{M}_j(z)^N\rangle$, $N>1$, is

$$\langle \mathbb{M}_j^N \rangle = (1-\alpha)\int_0^1 w^{-\alpha}\frac{(w-\Gamma)^N}{(w+R)^N}dw$$
$$\sim R^{1-\alpha}(1-\alpha)\int_0^{1/R} dx\, x^{-\alpha}\left[1-\frac{2(\hat{\Gamma}/z)^{1/2}+O(z^0)}{1+x}\right]^N$$
$$\sim(1-\alpha)(-1)^N 2^{N+\alpha-1}(z\hat{\Gamma})^{(1-\alpha)/2}\left[\frac{\hat{\Gamma}}{z}\right]^{N/2}\hat{\Gamma}^N B(1-\alpha, N+\alpha-1) . \tag{B15}$$

$B(x,y)$ is the β function.[21] For averages of the form $\langle \mathbb{M}_{n_0}\cdots\mathbb{M}_{n_p}\rangle$ in which I of the indexes $\{n_0,\ldots,n_p\}$ are different, $\{i_1,\ldots,i_I\}$, being the I_j repeated m_j times ($\sum_{j=1}^{\infty} m_j = p+1$), we find

$$\langle \mathbb{M}_{n_0}\cdots\mathbb{M}_{n_p}\rangle = \langle \mathbb{M}_{i_1}^{m_1}\rangle\langle \mathbb{M}_{i_2}^{m_2}\rangle\cdots\langle \mathbb{M}_{i_I}^{m_I}\rangle$$
$$\sim 2^{(\alpha-1)(I+1)}(z\hat{\Gamma})^{(1-\alpha)(I+1)/2}\left[-2\hat{\Gamma}\left[\frac{\hat{\Gamma}}{z}\right]^{1/2}\right]^{p+1} B(1-\alpha, m_0+\alpha-1)\cdots B(1-\alpha, m_I+\alpha-1) . \tag{B16}$$

If some $m_i = 1$, the average vanishes. In the expansion of $\langle \Psi_0\cdots\Psi_{n_p}\rangle$ in moments of $\mathbb{M}$ [Eq. (B11)], all the dominant terms are of the order of (B16). This is so because all the other terms contain at least an additional fragmentation, introducing at least (a) a factor $(z\hat{\Gamma})^{1/2}$, (b) some new $\mathbb{M}_j$ inside some moment, introducing a factor $\hat{\Gamma}(\hat{\Gamma}/z)^{1/2}$, and (c) a factor $\mathbb{J}\sim O(\hat{\Gamma}^{-1})$. These three facts introduce a factor of order $\hat{\Gamma}(\hat{\Gamma}z)^{-\alpha/2}$ that, by (5.22), is of order z^0. So, the order in z of $\langle \Psi_0\cdots\Psi_{n_p}\rangle$ is that of (B16), although the numerical coefficients would probably be different.

In the expansion of $\langle \Psi_0\cdots\Psi_{n_p}\rangle_{\mathrm{T}}$ in moments of Ψ, the term with $i+1$ moments is of order

$$[\hat{\Gamma}(\hat{\Gamma}/z)^{1/2}]^{p+1}(z\hat{\Gamma})^{(1-\alpha)\sum(I_k+1)/2} . \tag{B17}$$

Given that $\sum_{k=0}^{i}(I_k+1) > I+1$, the order of $\langle \Psi_0\cdots\Psi_{n_p}\rangle_{\mathrm{T}}$ is again that of (B16).

The development of Sec. V C uses the dominant behavior of $\langle \Psi_0\cdots\Psi_{n_p}\rangle_{\mathrm{T}}$ for small z and α. We calculate this contribution by noting that the most important term for small z in the expansion (B10), that is, $\langle \Psi_{n_0}\cdots\Psi_{n_p}\rangle$, behaves for small α as (B16) because, by (5.24), each new fragmentation introduces a factor

$$\hat{\Gamma}(z\hat{\Gamma})^{-\alpha/2}\sim O(\alpha)z^0 .$$

Then, from (B16), (B8), and using that

$$B(1-\alpha, N+\alpha-1)\sim 1/(N-1)+O(\alpha), \quad \alpha\to 0 , \tag{B18}$$

we find that the coefficient multiplying the lowest-order z contribution $z^{\alpha/(2-\alpha)}$ is of order α^{I+2}.

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[23]The order in z of the fourth and fifth terms does not agree with the estimation given in Ref. 11.

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[25]Equation (B5) is not satisfied in the case of diffusion in a percolating structure. Our formalism can then be used to recover the known fact that the EMA does not give the exact asymptotic result in this case. This can be seen as follows: In the percolation model, $\hat{\Gamma}(z)$ is of order z [see, for example, Ref. 1(b)]. Then, $\mathbb{J}_{nn}(z) \sim O(z^{-1})$ and, according to (B1) and (B2), Ψ_n is of order z. From (4.9), we get $D(z) \sim O(z)$, according to the EMA result for the exponent of the leading z dependence. However, the coefficient of z given by the EMA is not the correct one because all the diagrams contribute.