Spin-wave singularities: Free energy and equation of state in $O(n)$ spin models near T_c

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We give a detailed derivation of the thermodynamics of $O(n)$ spin models, correct to $O(\epsilon=4-d)$, using a generalization of the renormalization-group trajectory integral and noncritical matching technique first introduced by Rudnick and Nelson. The results are valid throughout the critical region, both with and without external magnetic field. We especially emphasize the coexistence-curve behavior for spins with a continuous symmetry ($n \geq 2$), deriving detailed expressions for the renormalized spin-wave stiffness (superfluid density) and longitudinal susceptibility. We generalize the parametric representation for the equation of state to include spin-wave singularities, yielding corrections to the so-called "linear model." Our expression for the free energy differs in some details from that given previously by Nicoll and Chang, necessitating minor corrections to previous applications to the weakly interacting Bose gas at a constant density.

I. INTRODUCTION

The low-temperature, ordered phase of ferromagnets with a continuous symmetry (spin dimensionality $n > 1$) exhibits coexistence curve singularities in various thermodynamic functions. For example, the longitudinal susceptibility $\chi_L = \partial M/\partial h$, where M is the magnetization and h the external magnetic field, diverges as $h \rightarrow 0$:

$$\chi_L \sim h^{-\epsilon/2}, \quad h \rightarrow 0, \quad T < T_c \ , \tag{1.1}$$

where $\epsilon = 4-d$, and d, which we henceforth take to be in the range $2 < d < 4$, is the spatial dimensionality. This divergence is a direct consequence of the slow, power-law decay of correlations in the ordered phase. In particular, the longitudinal pair-correlation function decays as (see, e.g., Ref. 9)

$$G_L(\mathbf{r}) \equiv \langle \mathbf{s}(\mathbf{r})\cdot\hat{\mathbf{M}}\, \mathbf{s}(\mathbf{0})\cdot\hat{\mathbf{M}} \rangle \sim |\mathbf{r}|^{-2(d-2)} \ , \quad h=0, \quad |\mathbf{r}| \rightarrow \infty \ , \tag{1.2}$$

where $\mathbf{s}(\mathbf{r})$ is the spin at site $\mathbf{r}$. The fact that $\chi_L(h=0)=\infty$ for $d<4$ follows directly from the spatial integral of (1.2).

The coexistence curve divergence [Eq. (1.1)] should be contrasted with that occurring *at* $T = T_c$:

$$\chi_L \sim h^{1/\delta - 1} \ , \tag{1.3}$$

where $\delta > 1$ is the critical exponent which describes the vanishing of $M \sim h^{1/\delta}$ at T_c. Accompanying (1.3) is the analogous power-law decay of critical correlations

$$G_L(\mathbf{r}) \sim 1/|\mathbf{r}|^{d-2+\eta} \ , \quad h=0, \quad T=T_c, \quad |\mathbf{r}| \rightarrow \infty \ , \tag{1.4}$$

which defines the critical exponent η.

In this paper we reexamine the connection between coexistence-curve singularities, such as (1.1) and (1.2), and their critical-point counterparts, such as (1.3) and (1.4). Using straightforward renormalization-group recursion-relation techniques, we rederive various thermodynamic functions, valid throughout the critical regime, both above and below T_c, with and without an applied field. These functions properly exhibit each set of singularities in the appropriate limit. We also exhibit the full free-energy function, which, to our knowledge, has not been fully analyzed previously.

We will use the original trajectory integral and noncritical matching technique of Rudnick and Nelson,[1] circumventing—by means of simple spin-wave theory—the difficulties these authors encountered near the coexistence curve. Some of the results presented here have been derived to various levels of completeness by other authors. In our eyes, however, their derivations, which often involve very sophisticated field-theoretic techniques, seem enormously complicated.[2,3] The most complete discussion has been given by Nicoll and Chang.[3] They used a more involved version of the trajectory integral technique; many of their intermediate expressions bear a strong (presumably noncoincidental) resemblance to our own, but we have not attempted a detailed comparison.[4] We find precisely their result for the equation of state, but our result for the free energy (which we believe to be correct) differs in some details.

We view our work as a final demonstration of the simplicity and utility of the Rudnick and Nelson[1] technique. Our main claim to originality is in supplying the one ingredient missing from the original discussion, namely, the present understanding of spin waves in the ordered phase of vector ferromagnets.

The outline of the rest of this paper is as follows: In Sec. II we recapitulate the model, its recursion relations, and their solutions. In Sec. III we derive the equation of state by combining the results of spin-wave theory with those of Sec. II. Asymptotic scaling equations are derived and the corresponding *parametric* forms are exhibited for general n—these contain spin-wave singularities in the angular variable θ. In Sec. IV we calculate the free energy and demonstrate consistency by deriving from it the correct equation of state. Finally, in Sec. V we give a short rederivation of the helicity modulus (or superfluid density), as well as explore various quantities derived from the free energy such as the entropy (or density, depending on how the variable r is interpreted) and specific heat. Appendix A contains some details of the recursion relation solutions; Appendix B gives some insight into the linear spin-wave approximation; in Appendix C some spin-wave integrals are evaluated.

II. MODEL AND RECURSION RELATIONS

We work with the Landau-Ginzburg-Wilson continuous-spin Hamiltonian

$$H_{\rm LGW}=\int d^dx\left[\frac{R_0^2}{2}|\nabla \mathbf{s}|^2+\frac{1}{2}r|\mathbf{s}|^2+u|\mathbf{s}|^4-\mathbf{h}\cdot\mathbf{s}\right], \tag{2.1}$$

where $-\infty<s^\alpha<\infty$, and $1\le\alpha\le n$, where n is the spin dimensionality. An underlying lattice, with lattice spacing a or, equivalently, a momentum space cutoff $k_\Lambda\sim\pi/a$, must be assumed in order for the partition function

$$Z_{\rm LGW}=\int DSe^{-H_{\rm LGW}}, \tag{2.2}$$

defined as a functional integral over all spin configurations, to be well defined.

The model (2.1) undergoes a ferromagnetic phase transition as the temperaturelike variable r decreases through a critical value $r_c(u)$ in zero external field $h=0$. In mean-field theory, defined here as the limit in which the coefficient R_0^2 of $|\nabla\mathbf{s}|^2$ tends to infinity (so that fluctuations are effectively suppressed), one has $r_c(u)\equiv 0$. The *spontaneous magnetization*

$$\mathbf{M}\equiv\frac{1}{V}\int d^dx\,\mathbf{s}(\mathbf{x})=\langle\mathbf{s}(\mathbf{x})\rangle \tag{2.3}$$

becomes nonzero below r_c, increasing with a characteristic exponent β:

$$M\equiv|\mathbf{M}|\sim|t|^\beta, \tag{2.4}$$

for small $t=r-r_c<0$. In mean-field theory we have

$$M_{\rm MF}=(-r/4u)^{1/2}\Longrightarrow\beta_{\rm MF}=\tfrac{1}{2}. \tag{2.5}$$

When $M>0$ (i.e., when $r<r_c$ or $h>0$), it is convenient to expand the fluctuations in the spin variable around the uniform magnetization $\mathbf{M}$. If we define

$$\sigma(x)=[\mathbf{s}(\mathbf{x})-\mathbf{M}]\cdot\hat{\mathbf{M}},\quad \hat{\mathbf{M}}\equiv\mathbf{M}/M,$$
$$\mathbf{s}^\perp(\mathbf{x})=\mathbf{s}(\mathbf{x})-\sigma(\mathbf{x})\hat{\mathbf{M}}, \tag{2.6}$$

then (2.1) can be rewritten in the form

$$H_{\rm LGW}=H_0+H_1+H_2+H_3+H_4, \tag{2.7}$$

where

$$H_0=(\tfrac{1}{2}rM^2+uM^4-hM)V,$$
$$H_1=-\int d^dx\,\tilde{h}\sigma,$$
$$H_2=\tfrac{1}{2}\int d^dx(R_0^2|\nabla\mathbf{s}^\perp|^2+R_0^2|\nabla\sigma|^2+r_T|\mathbf{s}^\perp|^2+r_L\sigma^2), \tag{2.8}$$
$$H_3=\int d^dx(w_1\sigma|\mathbf{s}^\perp|^2+w_2\sigma^3),$$
$$H_4=\int d^dx(u_1\sigma^4+2u_2|\mathbf{s}^\perp|^2\sigma^2+u_3|\mathbf{s}^\perp|^4),$$

in which we have defined

$$\tilde{h}=h-rM-4uM^3,$$
$$r_L=r+12uM^2,\quad r_T=r+4uM^2, \tag{2.9}$$
$$w_1=w_2=4uM,\quad u_1=u_2=u_3=u.$$

The first term H_0 should be recognized as the Landau mean-field free energy, which yields (2.5) when $h=0$.

Rudnick and Nelson[1] have derived differential recursion relations to one-loop order for the Hamiltonian (2.7), with $R_0=1$ and $k_\Lambda=1$. We reproduce them here in more detail and with minor misprints corrected:

$$\frac{d\tilde{h}}{dl}=\left[3-\frac{1}{2}\epsilon\right]\tilde{h}-\frac{w_1K_4(n-1)}{1+r_T}-\frac{3w_2K_4}{1+r_L}+O(uw,w^3), \tag{2.10}$$

$$\frac{dr_L}{dl}=2r_L+\frac{12K_4u_1}{1+r_L}+\frac{4(n-1)u_2K_4}{1+r_T}-\frac{18K_4w_2^2}{(1+r_L)^2}-\frac{2(n-1)K_4w_1^2}{(1+r_T)^2}+O(u^2,uw^2,w^4), \tag{2.11}$$

$$\frac{dr_T}{dl}=2r_T+\frac{4(n+1)u_3K_4}{1+r_T}+\frac{4K_4u_2}{1+r_L}-\frac{4K_4w_1^2}{(1+r_L)(1+r_T)}+O(u^2,uw^2,w^4), \tag{2.12}$$

$$\frac{dw_1}{dl}=\left[1+\frac{1}{2}\epsilon\right]w_1-\frac{4(n+1)w_1u_3K_4}{(1+r_T)^2}-\frac{16w_1u_2}{(1+r_L)(1+r_T)}-\frac{12w_2u_2K_4}{(1+r_L)^2}+\frac{12w_1^2w_2K_4}{(1+r_L)^2(1+r_T)}$$
$$+\frac{4w_1^3K_4}{(1+r_T)^2(1+r_L)}+O(wu^2,w^3u,w^5), \tag{2.13}$$

$$\frac{dw_2}{dl}=\left[1+\frac{1}{2}\epsilon\right]w_2-\frac{4(n-1)w_1u_2K_4}{(1+r_T)^2}-\frac{36w_2u_1K_4}{(1+r_L)^2}+\frac{36K_4w_2^3}{(1+r_L)^3}+\frac{\frac{4}{3}K_4w_1^3(n-1)}{(1+r_T)^3}+O(wu^2,w^3u,w^5)\ , \tag{2.14}$$

$$\frac{du_1}{dl}=\epsilon u_1-\frac{36u_1^2K_4}{(1+r_L)^2}-\frac{4(n-1)u_2^2K_4}{(1+r_T)^2}+\frac{216K_4u_1w_2^2}{(1+r_L)^3}+\frac{8K_4u_2w_1^2(n-1)}{(1+r_T)^3}-\frac{54w_2^4K_4}{(1+r_L)^4}-\frac{\frac{2}{3}(n-1)w_1^4K_4}{(1+r_T)^4}$$
$$+O(u^3,u^2w^2,uw^4,w^6)\ , \tag{2.15}$$

$$\frac{du_2}{dl}=\epsilon u_2-\frac{16K_4u_2^2}{(1+r_L)(1+r_T)}-\frac{12u_1u_2K_4}{(1+r_L)^2}-\frac{4(n+1)u_2u_3K_4}{(1+r_T)^2}+\frac{4K_4u_3w_1^2(n+1)}{(1+r_T)^3}+\frac{36K_4u_2w_2^2}{(1+r_L)^3}$$
$$+\frac{12u_1w_1^2K_4}{(1+r_L)^2(1+r_T)}+\frac{4K_4u_2w_1^2}{(1+r_T)^2(1+r_L)}+\frac{48K_4u_2w_1w_2}{(1+r_L)^2(1+r_T)}+\frac{16K_4u_2w_1^2}{(1+r_T)(1+r_L)^2}-\frac{4w_1^4K_4}{(1+r_T)^3(1+r_L)}$$
$$-\frac{36K_4w_1^2w_2^2}{(1+r_T)(1+r_L)^3}-\frac{12w_1^3w_2k_4}{(1+r_L)^2(1+r_T)^2}+O(u^3,u^2w^2,uw^4,w^6)\ , \tag{2.16}$$

$$\frac{du_3}{dl}=\epsilon u_3-\frac{4(n+7)u_3^2K_4}{(1+r_T)^2}-\frac{4u_2^2K_4}{(1+r_L)^2}+\frac{24K_4u_3w_1^2}{(1+r_T)^2(1+r_L)}+\frac{8K_4u_2w_1^2}{(1+r_L)^2(1+r_T)}-\frac{4w_1^4K_4}{(1+r_L)^2(1+r_T)^2}$$
$$+O(u^3,u^2w^2,uw^4,w^6)\ . \tag{2.17}$$

All terms in these recursion relations are evaluated at rescaling parameter l, with $l=0$ corresponding to the unrenormalized parameters. Rudnick and Nelson did not distinguish between u_1, u_2, and u_3, and ignored a number of the terms on the right-hand sides (RHS's) of (2.13)–(2.17). In addition, the recursion relations for R_0, which we do not display, turn out to contain terms of $O(w^2)$ which could violate the assumption that $R_0\equiv 1$ to $O(\epsilon)$ when $M>0$. It transpires that these differences do not affect the solutions to the recursion relations to $O(\epsilon)$. The reason for this is quite simple: Initially, at $l=0$, we have $M\le O(1)$ and $r,u\le O(\epsilon)$, implying that $w\le O(u)\le O(\epsilon)$. Hence, initially, terms of order w^3 in (2.13) and (2.14) and terms of order w^2u and w^4 in (2.15)–(2.17) are much smaller—by relative factors of u or u^2—than the other terms in (2.13)–(2.17). However, the recursion relations will be integrated out to a value $l=l^*$, defined in such a way that $r_L(l^*)=O(1)$. It will be shown below that at $l=l^*$ we will still have $u\le O(\epsilon)$, but that $M=O(1/\sqrt{u})\ge O(1/\sqrt{\epsilon})$, and hence $w=O(\sqrt{u})\le O(\sqrt{\epsilon})$. Thus, near $l=l^*$, one will have $u\sim w^2$. The terms with higher powers of w will therefore be of the *same* order as the others and thus nominally should be kept. In fact, when $l\le l^*$ it is readily apparent that only the *first* terms on the right-hand sides of (2.11), (2.14), and (2.15) need be kept; the rest are smaller by relative order u. Initially, it would seem that the first term *always* dominates. However, since we are dealing with *exponential* growth, this turns out not to be the case. To complete the argument one must show that the neglected terms give small contributions on a *logarithmic* scale. It will turn out that in the regime in which $w=O(u)$, the next-to-leading terms are important, while the rest can indeed be neglected: In the regime $w\gg u$, as stated above, only the leading term is important.

To see this, the interval $[0,l^*]$ is divided into two parts: One, $[0,l_\theta]$, in which $r_L\le O(\epsilon^\theta)$, and the other, $[l_\theta,l^*]$, in which $r_L\ge O(\epsilon^\theta)$, where $0<\theta<1$ is an arbitrary exponent (we require only that $\epsilon^\theta\gg\epsilon$). Let us estimate l^*-l_θ: Assuming that in the range $l\ge l_\theta$ only the leading terms need to be kept, we find,

$$\left.\begin{aligned}&r_L(l)\sim r_L(l_\theta)e^{2(l-l_\theta)}\\&w_i(l)\sim w_i(l_\theta)e^{[1-(1/2)\epsilon](l-l_\theta)},\quad i=1,2\end{aligned}\right\}l\ge l_\theta\ , \tag{2.18}$$

which implies [using $w_i(l^*)=O(\sqrt{u(l^*)})$]

$$e^{(l^*-l_\theta)}=O(\epsilon^{-\theta/2})\ ,$$
$$w_i(l_\theta)=O(\sqrt{u(l^*)}\epsilon^\theta),\quad i=1,2\ . \tag{2.19}$$

Substituting these estimates back into the higher-order terms in the recursion relations, we find, self-consistently, that the relative corrections to (2.18) are $O(u(l^*)/\epsilon^\theta,u(l^*)/\epsilon^{1-\theta})$ for r_L, and $O(u(l^*),u(l^*)\epsilon^\theta)$ for w_i. Since $u(l^*)\le O(\epsilon)$, these corrections are indeed much smaller than unity as long as $\epsilon\ll 1$. Finally, we note that

$$u(l)-u(l_\theta)=O(\epsilon u(l^*),u(l^*)^2)(l-l_\theta)\ , \tag{2.20}$$

so that

$$u(l^*)=u(l_\theta)[1+O(\epsilon\ln(\epsilon),u(l^*)\ln(\epsilon))]\ , \tag{2.21}$$

implying that $u(l)$ is essentially constant in the interval $[l_\theta,l^*]$.

The above considerations imply that to leading nontrivial order in ϵ, and for $r_L\le O(1)$, we need only work with the following reduced recursion relations:

$$\frac{d\tilde{h}}{dl}=\left[3-\frac{1}{2}\epsilon\right]\tilde{h}-\frac{wK_4(n-1)}{1+r_T}-\frac{3wK_4}{1+r_L}\ , \tag{2.22}$$

$$\frac{dr_L}{dl}=2r_L+\frac{12K_4u}{1+r_L}+\frac{4(n-1)K_4u}{1+r_T}-\frac{18K_4w^2}{(1+r_L)^2}-\frac{2(n-1)K_4w^2}{(1+r_T)^2}\ , \tag{2.23}$$

$$\frac{dr_T}{dl}=2r_T+\frac{4(n+1)uK_4}{1+r_T}+\frac{4K_4u}{1+r_L}-\frac{4K_4w^2}{(1+r_L)(1+r_T)}\ , \tag{2.24}$$

$$\frac{dw}{dl}=\left[1+\frac{1}{2}\epsilon\right]w-4(n+8)wuK_4\ , \tag{2.25}$$

$$\frac{du}{dl}=\epsilon u-4(n+8)u^2K_4\ , \tag{2.26}$$

where, to this order, we have $R_0\equiv 1$, $w_1=w_2\equiv w$, and $u_1=u_2=u_3\equiv u$. A further simplification is achieved when we note that the solution to (2.25), recalling (2.9), is $w(l)=4M(0)e^{[1-(1/2)\epsilon]l}u(l)$. Since the renormalization-group transformation used here is quasilinear, the renormalization of the magnetization is given precisely by the spin-rescaling factor $\exp\frac{1}{2}\int_0^l[\eta(l')+d-2]dl'$. To the present order we have $\eta=0$, so that

$$M(l)=M(0)e^{(1/2)(d-2)l}=M(0)e^{[1-(1/2)\epsilon]l}\ , \tag{2.27}$$

which then yields

$$w(l)=4u(l)M(l)\ . \tag{2.28}$$

The solution to (2.26) is straightforward and yields

$$u(l)=u(0)e^{\epsilon l}/Q(l)\ ,$$

where

$$Q(l)\equiv 1-\tilde{u}+\tilde{u}e^{\epsilon l}\ ,$$

and

$$\tilde{u}\equiv u(0)/u^*=4(n+8)K_4u(0)/\epsilon\ , \tag{2.29}$$

where $u^*=\epsilon/4(n+8)K_4$ is the nontrivial fixed-point solution for u. The solutions for $\tilde{h}$, r_L, and r_T are more complicated. We will illustrate the general methodology in Appendix A, but quote only the answers here. (Rudnick and Nelson[1] have done most of their explicit calculations only when $M=h=0$. We feel it worthwhile to outline the $M>0$ calculation in more detail, as some subtleties do appear.) One finds, to lowest nontrivial order in ϵ,

$$r_L(l)=T_L(l)-2(n+2)K_4u(l)+6K_4u(l)T_L(l)\ln[1+T_L(l)]+2(n-1)K_4u(l)T_T(l)\ln[1+T_T(l)]$$
$$+144K_4u(l)^2M(l)^2\left[\ln[1+T_L(l)]+\frac{T_L(l)}{1+T_L(l)}\right]+16(n-1)K_4u(l)^2M(l)^2\left[\ln[1+T_T(l)]+\frac{T_T(l)}{1+T_T(l)}\right]\ , \tag{2.30}$$

$$r_T(l)=T_T(l)-2(n+2)K_4u(l)+6K_4u(l)T_L(l)\ln[1+T_L(l)]+2(n-1)K_4u(l)T_T(l)\ln[1+T_T(l)]\ , \tag{2.31}$$

$$\tilde{h}(l)=h(l)-t(l)M(l)-4u(l)M(l)^3+2(n+2)K_4u(l)M(l)$$
$$-2(n-1)K_4u(l)M(l)T_T(l)\ln[1+T_T(l)]-6K_4u(l)M(l)T_L(l)\ln[1+T_L(l)]\ , \tag{2.32}$$

where

$$T_L(l)=t(l)+12u(l)M(l)^2\ ,$$
$$T_T(l)=t(l)+4u(l)M(l)^2\ ,$$
$$t(l)=t(0)e^{2l}/Q(l)^{(n+2)/(n+8)}\ , \tag{2.33}$$
$$t(0)\equiv r(0)+2(n+2)K_4u(0)+O(\epsilon u,u^2)\ ,$$
$$h(l)\equiv h(0)e^{[3+(1/2)\epsilon]l}\ .$$

The variable $t(0)$ is precisely $r-r_c(u)$.

III. SPIN WAVES AND THE EQUATION OF STATE

From Eq. (2.27) the magnetization is given by

$$M(0)=M(l^*)e^{-[1-(1/2)\epsilon]l^*}\ . \tag{3.1}$$

The calculation of the renormalized magnetization $M(l^*)$ will be done within the linear spin-wave approximation, which, as we shall demonstrate, is valid precisely in the limit $u(l^*)\ll 1$, $r_L(l^*)=O(1)$.

The linear spin-wave approximation is simply the quadratic fluctuation correction to the Landau mean-field solution. The only inputs are therefore the two orthogonal curvatures of the Landau free-energy surface at the mean-field minimum. Note that when the external field vanishes the transverse curvature must vanish due to the continuous global spherical symmetry of the spins. This requirement will provide a consistency check on the renormalization-group calculation. From (2.8) (with $R_0=1$, $u_1=u_2=u_3\equiv u$, and $w_1=w_2\equiv w$), it is easy to see that the minimum occurs for $\mathbf{s}^\perp=0$ and $\sigma=\overline{M}(l)-M(l)$, where $\overline{M}$ satisfies

$$(\tilde{h}+r_L M-8uM^3)=(r_L-12uM^2)\overline{M}+4u\overline{M}^3 . \tag{3.2}$$

It should be emphasized that $\overline{M}(l)\neq M(l)$, although when $l=l^*$ they differ only by terms of order w; but even then their *external field* dependence (which comes from wave numbers $k<e^{-l^*}$) is very different. This difference is crucial to the derivation of the correct equation of state (see below). When expanded around this minimum the Hamiltonian takes the form

$$\begin{aligned}H(l)=\int d^dx &\Big\{\tfrac{1}{2}|\nabla \mathbf{s}^\perp|^2+\tfrac{1}{2}|\nabla\tilde{\sigma}|^2+\tfrac{1}{2}[r_T-4u(M^2-\overline{M}^2)]|\mathbf{s}^\perp|^2\\ &+\tfrac{1}{2}[r_L-12u(M^2-\overline{M}^2)]\tilde{\sigma}^2+4u\overline{M}(\tilde{\sigma}^3+\tilde{\sigma}|\mathbf{s}^\perp|^2)+u(\tilde{\sigma}^2+|\mathbf{s}^\perp|^2)^2\Big\}\\ &+e^{dl}[\tfrac{1}{2}r(0)M(0)^2+u(0)M(0)^4-h(0)M(0)]V\\ &+[\tfrac{1}{2}r_L(\overline{M}-M)^2+4uM(\overline{M}-M)^3+u(\overline{M}-M)^4-\tilde{h}(\overline{M}-M)]V ,\end{aligned} \tag{3.3}$$

where $\tilde{\sigma}=\sigma+(M-\overline{M})$. The square of the transverse and longitudinal curvatures are, respectively, $\kappa_T(l)^2\equiv r_T-4u(M^2-\overline{M}^2)$, and $\kappa_L(l)^2\equiv r_L-12u(M^2-\overline{M}^2)$. Using (2.30)–(2.33) and (3.2), we find

$$\begin{aligned}\kappa_T^2&=\frac{h(l)}{M(l)}+\left[\frac{1}{\overline{M}(l)}-\frac{1}{M(l)}\right]\\ &\qquad\times[\tilde{h}(l)+r_L(l)M(l)-8u(l)M(l)^3]\\ &=\frac{h(l)}{\overline{M}(l)}+\left[\frac{M(l)}{\overline{M}(l)}-1\right][r_L(l)-r_T(l)-8u(l)M(l)^2] .\end{aligned} \tag{3.4}$$

The second term in both cases $O(u(l)^2)$ and hence is beyond the resolution of the present calculation. Therefore, to the order we are working we may take

$$\kappa_T^2(l)=h(l)/M(l) , \tag{3.5}$$

which indeed vanishes in the ordered phase when $h(0)=0$. It should be noted that, because of the spherical symmetry, the *transverse* susceptibility χ_T is always given *exactly* by $M(l)/h(l)$, although (3.5), which is essentially the mean-field inverse susceptibility, is only approximate since fluctuations with $k<e^{-l}$ have not yet been accounted for. The longitudinal curvature is given by

$$\kappa_L^2=r_L+3(\kappa_L^2-r_T) , \tag{3.6}$$

which is always nonzero and yields essentially the mean-field inverse longitudinal susceptibility χ_L^{-1}. The true inverse longitudinal susceptibility *vanishes* at $h=0$ due to spin-wave effects (see below), though not as rapidly as χ_T^{-1}. Again, this is an effect strictly of wave numbers with $k<e^{-l}$.

We now proceed to calculate the equation of state. Since the field $\sigma(x)$ was defined to have zero mean, $\langle\sigma(x)\rangle=0$, we will have

$$\langle\tilde{\sigma}(\mathbf{x})\rangle=M(l)-\overline{M}(l) . \tag{3.7}$$

But the left-hand side, to lowest order in the fluctuations around the mean-field minimum, is just

$$\langle\tilde{\sigma}(\mathbf{x})\rangle=-\left\langle\tilde{\sigma}(\mathbf{x})\int d^dy\, 4u\overline{M}[\tilde{\sigma}(\mathbf{y})^3+\tilde{\sigma}(\mathbf{y})|\mathbf{s}^\perp(\mathbf{y})|^2]\right\rangle_0 ,$$

where $\langle\ \rangle_0$ indicates an average with respect to the quadratic Hamiltonian

$$H_0=\tfrac{1}{2}\int d^dx(|\nabla\mathbf{s}^\perp|^2+|\nabla\tilde{\sigma}|^2+\kappa_T^2|\mathbf{s}^\perp|^2+\kappa_L^2\sigma^2) . \tag{3.8}$$

In Appendix B we give more insight into this form of perturbation theory. Combining (3.7) and (3.8), we find

$$\begin{aligned}M(l)-\overline{M}(l)&=-4u(l)\overline{M}(l)\int d^dy\langle\tilde{\sigma}(\mathbf{x})\tilde{\sigma}(\mathbf{y})\rangle_0[3\langle\tilde{\sigma}(\mathbf{y})^2\rangle_0+(n-1)\langle|\mathbf{s}^\perp(y)|^2\rangle_0]\\ &=\frac{-4u(l)\overline{M}(l)}{\kappa_L^2(l)}\int_q\left[\frac{3}{q^2+\kappa_L^2(l)}+\frac{n-1}{q^2+\kappa_T^2(l)}\right] ,\end{aligned} \tag{3.9}$$

where $\int_q\equiv\int d^dq/(2\pi)^d$. The integrals in (3.9) are straightforward and are evaluated in Appendix C. The result is

$$M-\overline{M}=-\frac{\overline{M}}{\kappa_L^2}\left\{6K_4u[1-\kappa_L^2\ln(1+\kappa_L^2)+\kappa_L^2\ln(\kappa_L^2)]\right.$$
$$\left.+2(n-1)K_4u\left[1-\kappa_T^2\ln(1+\kappa_T^2)-\frac{2}{\epsilon}\kappa_T^2\left[\frac{\epsilon\pi/2}{\sin(\epsilon\pi/2)}(\kappa_T^2)^{-\epsilon/2}-1\right]\right]\right\} . \quad (3.10)$$

Now it is straightforward to show that to first order in $(M-\overline{M})/M$, (3.2) can be written in the form

$$\tilde{h}\approx r_L(\overline{M}-M)\approx\kappa_L^2(\overline{M}-M) . \quad (3.11)$$

To the same order, M and $\overline{M}$ are interchangeable on the RHS of (3.10); one must, however, be careful to use (3.5) for κ_T^2 in the last, singular term of (3.10). Using (2.32) for $\tilde{h}$, (3.10) becomes, after a large number of cancellations,

$$\frac{h}{M}=T_T+6K_4uT_L\ln(T_L)+2(n-1)K_4u\left[\frac{-2}{\epsilon}\right]\frac{h}{M}\left[\left[\frac{h}{M}\right]^{-\epsilon/2}-1\right] , \quad (3.12)$$

where all quantities are evaluated at $l=l^*$. Note that we have, correctly to the order we are working, used (T_L,T_T) and (r_L,r_T) interchangeably in the $O(u)$ terms on the right-hand side. We may substitute the solutions from Sec. II into (3.12) to get the complete equation of state:

$$\frac{h_0}{M_0}=\frac{t_0Q^{6/(n+8)}+4u_0M_0^2}{Q}-\frac{(n-1)}{(n+8)}\tilde{u}\frac{h_0}{M_0}\frac{1}{Q}\left[\left[\frac{M_0}{h_0}\right]^{\epsilon/2}-e^{\epsilon l}\right]+\frac{6K_4u_0e^{(\epsilon-2)l^*}T_L\ln(T_L)}{Q} , \quad (3.13)$$

where the subscript zero denotes the *unrenormalized* initial values of the parameters, and Q and $\tilde{u}$ were defined in (2.29) and (2.33). In order to compare with results of Ref. 3, we define

$$P=1+\tilde{u}\frac{n-1}{n+8}\left[\left[\frac{M_0}{h_0}\right]^{\epsilon/2}-1\right]+\frac{9}{n+8}(Q-1) , \quad (3.14)$$

and define the matching parameter l^* by $T_L(l^*)=1$ [*not* $T_T(l^*)=O(1)$, as assumed in Ref. 1], i.e.,

$$1/S=(t_0Q^{6/(n+8)}+12u_0M_0^2)/Q , \quad (3.15)$$

where $S=e^{2l^*}$, which implies that

$$Q=1+\tilde{u}(S^{\epsilon/2}-1) . \quad (3.16)$$

The equation of state then reads

$$\frac{h_0}{M_0}=\frac{t_0Q^{6(n+8)}+4u_0M_0^2}{P} . \quad (3.17)$$

Equations (3.14)–(3.17) constitute the full equation of state, valid throughout the critical region. These expressions agree precisely with those derived by Nicoll and Chang[3] (see their Sec. III, with the correspondences $\Gamma_1=h_0$, $Y=P^{-1}$, $Y_2=Q^{-1}$, $\overline{\Gamma}_2=S^{-1}$, $\lambda_4=\epsilon$, $\lambda_2=2$, and $u_4=12K_4u$). See also Ref. 5, Sec. VI, for some applications of these equations.

As an application of (3.17), let us derive the low-field magnetization for $t_0<0$ ($T<T_c$) and identify the spin-wave singularity in the susceptibility. If M_0 remains positive when $h_0\to0$, we see from (3.14) that P is dominated by

$$P\approx\tilde{u}\frac{n-1}{n+8}\left[\frac{M_0}{h_0}\right]^{\epsilon/2},\quad h_0\to0,\quad t_0<0 , \quad (3.18)$$

so that the equation state reads

$$\tilde{u}\frac{n-1}{n+8}\left[\frac{h_0}{M_0}\right]^{1-\epsilon/2}\approx t_0Q^{6/(n+8)}+4u_0M_0^2 . \quad (3.19)$$

The zero-field magnetization is therefore given by

$$M_0(h_0=0)=(-t_0/4u_0)^{1/2}Q^{3/(n+8)},\quad t_0<0 , \quad (3.20)$$

while (3.15) and (3.16) yield

$$Q=1-\tilde{u}+\tilde{u}(-2t_0)^{-\epsilon/2}Q^{(\epsilon/2)(n+2)/(n+8)} ,\quad h_0=0,\quad t_0<0 . \quad (3.21)$$

Writing, for small h_0, $M_0=M_0(h_0=0)+\delta M_0$, and denoting the solution to (3.21) by $Q_-(\tilde{u},t_0)$, we find

$$\delta M_0=\tilde{u}\frac{n-1}{n+8}\frac{1}{8u_0M_0(0)}\left[\frac{h_0}{M_0(0)}\right]^{1-\epsilon/2}\left\{\left[1-\frac{\epsilon}{2}\frac{n+11}{n+8}\left[1-\frac{1-\tilde{u}}{Q_-}\right]\right]\Big/\left[1-\frac{\epsilon}{2}\frac{n+2}{n+8}\left[1-\frac{1-\tilde{u}}{Q_-}\right]\right]\right\} . \quad (3.22)$$

So that the small-field susceptibility $\chi=\partial\delta M_0/\partial h_0$ diverges as

$$\chi\approx\tilde{u}\frac{n-1}{n+8}(1-\epsilon/2)\frac{C(Q_-)}{8u_0M_0(0)^2}\left[\frac{h_0}{M_0(0)}\right]^{-\epsilon/2}, \quad h_0\to 0,\ t_0<0 \tag{3.23}$$

where $C(Q_-)$ is the expression in braces in (3.22). This expression gives the precise $O(\epsilon)$ amplitude of the spin-wave singularity right up to T_c. In Ref. 5 it is shown how to cast such expressions into scaling form in order to best illustrate the various crossovers involved. A simple example will suffice here. For simplicity we take $\tilde{u}=1$ (i.e., $u=u^*$; see Ref. 5 for the method treating the general case—which involves the introduction of a second scaling variable $\sim u_0/t_0^{\epsilon/2}$). Define the scaling combinations

$$\tau\equiv\frac{t_0}{(4u_0M_0^2)^{1/2\beta}},\quad \beta=\frac{\frac{1}{2}(1-\frac{1}{2}\epsilon)}{[1-\frac{1}{2}\epsilon(n+2)/(n+8)]},$$
$$\mathcal{Q}\equiv(4u_0M_0^2)^{\omega_Q}Q,\quad \omega_Q=\frac{\epsilon}{2-\epsilon}=\frac{4-d}{d-2}. \tag{3.24}$$

One then finds that $\mathcal{Q}$ satisfies

$$\mathcal{Q}^{(\epsilon-2)/\epsilon}=3+\tau\mathcal{Q}^{6/(n+8)},\quad \tilde{u}=1. \tag{3.25}$$

This represents the Griffiths scaling form for $\mathcal{Q}$, valid for all $\tau\geq\tau_{\rm coex}$, negative and positive, where $\tau_{\rm coex}=-2^{6\epsilon/(n+8)(2-\epsilon)}$ represents the coexistence curve. Proceeding now to the equation of state, we define the scaling variable for h_0:

$$\zeta^2\equiv 4u_0h_0^2/(4u_0M_0^2)^\delta,\quad \delta=(6-\epsilon)/(2-\epsilon). \tag{3.26}$$

The relation between ζ and τ then becomes

$$\zeta=(1+\tau\mathcal{Q}^{6/(n+8)})\Big/\left[\frac{9}{n+8}\mathcal{Q}+\frac{n-1}{n+8}\zeta^{-\epsilon/2}\right], \tag{3.27}$$

which represents the equation of state in Griffiths scaled form. This should be solved to yield $\zeta=Z(\tau)$, or

$$h_0=DM_0^\delta Z(ct_0/M_0),\quad u_0=u^*, \tag{3.28}$$

where $D=(4u^*)^{-2/(6-\epsilon)}$ and $c=(4u^*)^{-1/2\beta}$. When $t_0=\tau=0$, this yields $h_0=DZ_0M_0^\delta$, the usual definition of the exponent δ. The constant $Z_0=Z(0)$ satisfies

$$Z_0\left[\frac{9}{n+8}3^{\epsilon/(2-\epsilon)}+\frac{n+1}{n+8}Z_0^{-\epsilon/2}\right]=1.$$

In the opposite limit $\tau\to\tau_{\rm coex}$, a careful analysis of (3.27) yields precisely (3.22) and (3.23) with $\tilde{u}=1$. Thus (3.27) and (3.28) are a succinct way of representing the crossover between the spin-wave "fixed point" at $\tau=\tau_{\rm coex}$ and the critical fixed point at $t_0=h_0=0$ [the value of τ is not specified in this limit, depending along which particular path in the (t_0,h_0) plane one approaches the critical point].

Equations (3.24)–(3.27) may also be cast in so-called parametric form.[6] Here one uses a polar-coordinate-type representation in the (t_0,h_0) plane to simplify expressions. One writes

$$h_0=R^{\beta\delta}h(\theta),\quad t_0=Rt(\theta),\quad M_0=R^\beta m(\theta), \tag{3.29}$$

where $R\geq 0$ is the radial variable and $-1\leq\theta\leq 1$ is the angular variable. Since the system is symmetric under $h_0\to -h_0$, a convenient normalization is obtained by taking the positive t_0 axis as $\theta=0$, the coexistence curve as $\theta=\pm 1$ (where $h_0\to\pm 0$, respectively), and the positive and negative h_0 axes as $\theta=\pm\theta_0$, respectively, where $0<\theta_0<1$ is chosen for convenience. The utility of this representation is apparent when one realizes that in the Ising case $n=1$, Eqs. (3.14)–(3.17) (with $\tilde{u}=1$) correspond *exactly* to the choices

$$h(\theta)=\theta(1-\theta^2)/\sqrt{8u^*},\quad t(\theta)=(1-\tfrac{3}{2}\theta^2),$$
$$m(\theta)=\theta/\sqrt{8u^*}, \tag{3.30}$$

with $\theta_0^2=\frac{2}{3}$ and the exponents β,δ displayed in (3.24) and (3.26) and evaluated at $n=1$. One also finds the correspondence $Q=R^{\beta(3-\delta)}$. This is the so-called *linear model.*

For $n>1$, life cannot be so simple. A spin-wave singularity, roughly of the form $(1-\theta^2)^{(1/2)\epsilon}$, must appear in either $h(\theta)$ or $m(\theta)$ when $\theta\to\pm 1$. In fact, this is apparent in Ref. 7 where this term appears as an unexponentiated logarithm in their Eq. (25). The authors, however, were unable to interpret this term unambiguously. To see how (3.30) must be modified when $n>1$, it is simplest to leave the functions $t(\theta)$ and $m(\theta)$ untouched [in fact, one may always choose $m(\theta)$ linear[6]]. This yields (3.15) and (3.16) (when $\tilde{u}=1$) for general n, with the same relation between Q and R. We now modify $h(\theta)$ to obtain the equation of state (3.14) and (3.17) for general n. If we write

$$h(\theta)=\theta(1-\theta^2)/(8u^*)^{1/2}\tilde{h}(\theta), \tag{3.31}$$

we find that consistency yields the equation

$$\tilde{h}(\theta)=\nu_n(1-\theta^2)^{-\epsilon/2}\tilde{h}(\theta)^{\epsilon/2}+(1-\nu_n), \tag{3.32}$$

where $\nu_n=(n-1)/(n+8)$ vanishes when $n=1$. For $\theta^2\to 1$ one finds

$$\tilde{h}(\theta)\approx\nu_n^{1/(1-\epsilon/2)}(1-\theta^2)^{-\epsilon/2/(1-\epsilon/2)}+O(1), \tag{3.33}$$

and hence

$$h(\theta)\approx 2^{-1/2}\nu_n^{-1/(1-\epsilon/2)}(1-\theta^2)^{1/(1-\epsilon/2)},\quad \theta^2\to 1. \tag{3.34}$$

It is apparent that the linear model misses all of the essential physics of the coexistence-curve behavior when $n>1$. If the spin-wave singularities are expanded naively in powers of ϵ, as in Ref. 7, one encounters a term proportional to $\epsilon(1-\theta^2)\ln(1-\theta^2)$, which drives h_0 unphysically negative as $\theta\to 1$, yielding, for example, a negative-going susceptibility.[4]

IV. FREE ENERGY

In this section we compute the free energy. From Ref. 1 we have, to one-loop order,

$$F(r_0,u_0,h_0)=\int_0^l e^{-dl'}G_0(l')dl' + e^{-dl}F(r(l),u(l),h(l)) \ , \quad (4.1)$$

where

$$G_0(l)=\frac{1}{2}K_d\left[\ln[1+r_L(l)]+(n-1)\ln[1+r_T(l)]-\frac{2}{d}n\right] , \quad (4.2)$$

and we have dropped a trivial constant proportional to $\ln(2\pi)$ from the total free energy. The trajectory integral is a tedious, but straightforward application of the techniques used in Appendix A. Most of the answer has already been given in Ref. 1. We begin by writing $\ln(1+r)=[\ln(1+r)-r+\frac{1}{2}r^2]+r-\frac{1}{2}r^2$, the idea being, once again, to isolate the small r from the large r behavior. The first (bracketed) term contains the large $r=O(1)$ dependence, the second contains the small r dependence, and the last is slowly varying and must be integrated exactly. One finds, then,

$$\int_0^l e^{-dl}G_0(l')dl'=I_1(l)+I_2(l)-I_2(l=0) \ , \quad (4.3)$$

where

$$I_1(l)=-\tfrac{1}{4}K_d\int_0^l e^{-dl'}[r_L(l')^2+(n-1)r_T(l')^2]dl' \ , \quad (4.4)$$

$$I_2(l)=\frac{K_d}{2d}(n-1)e^{-dl}\left[[r_T(l)^2-1]\ln[1+r_T(l)]-\frac{1}{2}r_T(l)^2-\frac{2}{d-2}r_T(l)+\frac{2}{d}\right] + \frac{K_d}{2d}e^{-dl}\left[[r_L(l)^2-1]\ln[1+r_L(l)]-\frac{1}{2}r_L(l)^2-\frac{2}{d-2}r_L(l)+\frac{2}{d}\right] . \quad (4.5)$$

The expression (4.5) for $I_2(l)$ differs in some details from that given in Ref. 1—the major difference being the $-\frac{1}{2}r^2$ terms, which are needed for later cancellation. All other differences disappear when one takes, correct to order ϵ, $d=4$ in the various coefficients on the right-hand side of (4.5). The remaining integral $I_1(l)$ is evaluated by using $r_L\approx T_L$ and $r_T\approx T_T$, substituting (2.27) and (2.33), then performing the integral exactly. (This is possible since the entire integrand is then slowly varying, a function only of $e^{\epsilon l}$.) The result is

$$I_1(l)=\frac{t_0^2}{16u_0}\frac{n}{n-4}[Q(l)^{(4-n)/(n+8)}-1]+\frac{1}{2}t_0M_0^2[Q(l)^{-(n+2)/(n+8)}-1]+u_0M_0^4[Q(l)^{-1}-1] \ . \quad (4.6)$$

To complete the calculation, we require $F(l)$. This is given by the first fluctuation correction using the Hamiltonian (3.3). We find $e^{-dl}F(l)=F_0+F_1(l)+F_2(l)$, where

$$F_0=\tfrac{1}{2}r_0M_0^2+u_0M_0^4-h_0M_0 \ , \quad (4.7)$$

$$F_1(l)=e^{-dl}\{\tfrac{1}{2}r_L(l)[\overline{M}(l)-M(l)]^2+4u(l)M(l)[\overline{M}(l)-M(l)]^3+u(l)[\overline{M}(l)-M(l)]^4-\tilde{h}(l)[\overline{M}(l)-M(l)]\} \ , \quad (4.8)$$

$$F_2(l)=\frac{1}{2}e^{-dl}\int_q\left[\ln\left[\frac{1}{q^2+\kappa_L^2}\right]+(n-1)\ln\left[\frac{1}{q^2+\kappa_T^2}\right]\right] . \quad (4.9)$$

Note that by (3.2), $\overline{M}(l)$ minimizes $F_1(l)$, i.e., $\partial F_1(l)/\partial\overline{M}(l)=0$. The integral $F_2(l)$ can be evaluated to give $F_2(l)=\overline{F}_2(l)+F_{\rm SW}(l)$, where (see Appendix C)

$$\overline{F}_2(l)=-e^{-dl}\frac{(n-1)K_d}{2d}\left[(\kappa_T^4-1)\ln(1+\kappa_T^2)-\frac{1}{2}\kappa_T^4-\frac{2}{d-2}\kappa_T^2+\frac{2}{d}\right] - e^{-dl}\frac{K_d}{2d}\left[(\kappa_L^4-1)\ln(1+\kappa_L^2)-\frac{1}{2}\kappa_L^4-\frac{2}{d-2}\kappa_L^2+\frac{2}{d}\right] , \quad (4.10)$$

$$F_{\rm SW}(l)=(n-1)\frac{K_d}{2d}\left[\frac{-2}{\epsilon}\right]\kappa_T^4\left[\kappa_T^{-\epsilon}\frac{4}{d}\frac{\pi\epsilon/2}{\sin(\pi\epsilon/2)}-1\right]e^{-dl}+\frac{K_d}{2d}e^{-dl}\kappa_L^4\left[\ln(\kappa_L^2)-\frac{1}{2}\right] . \quad (4.11)$$

We have displayed the exact $\kappa_T^{4-\epsilon}$ singularity in (4.11) for completeness. This is the spin-wave contribution to F. Note the similarity between $I_2(l)$ and $\overline{F}_2(l)$. We take advantage of this similarity by expanding $I_2+\overline{F}_2$ in the small differences $\kappa_L^2-r_L=12u(\overline{M}^2-M^2)$ and $\kappa_T^2-r_T=4u(\overline{M}^2-M^2)$:

$$I_2+\overline{F}_2\approx-\frac{1}{2}e^{-dl}(\overline{M}^2-M^2)[-2(n+2)K_4u+6K_4uT_L\ln(1+T_L)+2(n-1)K_4uT_T\ln(1+T_T)] \approx -M(\overline{M}-M)e^{-dl}(r_T-T_T) \ , \quad (4.12)$$

where we have used (r_L, T_L, κ_L^2) and (r_T, T_T, κ_T^2) interchangeably inside the terms of order u, and set $d=4$ in the various coefficients on the right-hand side. The last line follows from (2.31) and (2.33). We now combine (4.12) and (4.8), and use (3.11) to evaluate $\overline{M}-M$ (correct to the order we are working):

$$I_2+\bar{F}_2+F_1 \approx e^{-dl}\frac{1}{T_L}\left[\frac{1}{2}\tilde{h}^2-\tilde{h}^2-\tilde{h}M(r_T-T_T)\right]$$
$$=-e^{-dl}(M^2/2T_L)[(h/M-T_T)^2-(r_T-T_T)^2] , \tag{4.13}$$

where we have used $\tilde{h}=h-r_TM$ [see (2.31) and (2.32)]. The expressions for F_0, $I_1(l)$, and $I_2(l=0)$ combine to yield

$$F_0-I_2(l=0)+I_1(l)=A_{\text{reg}}+\left[\frac{t_0^2}{16u_0}\right]\frac{n}{n-4}(Q^{(4-n)/(n+8)}-1)-h_0M_0+\frac{\frac{1}{2}t_0M_0^2Q^{6/(n+8)}+u_0M_0^4}{Q} , \tag{4.14}$$

where

$$A_{\text{reg}}=\left[\frac{nK_d}{2d}\right]\left[(1-r_0^2)\ln(1+r_0)+\frac{1}{2}r_0^2+\frac{2}{d-2}r_0-\frac{2}{d}\right] \tag{4.15}$$

is the regular part of the free energy.

It is tempting to replace κ_T^2 by h/M in (4.11), according to (3.4) and (3.5). Unfortunately, this leads to incorrect results: We will ultimately be interested in deriving the correct equation of state from our free energy, and although $\kappa_T^2 \approx h/M$, the equality breaks down under differentiation. The correct replacement is actually $h/\overline{M}$ (see below) or, preferably, to preserve the exact value $r_T-4u(M^2-\overline{M}^2)$ until after differentiation. One may then safely interchange M and $\overline{M}$. Let us define $\overline{M}_0 \equiv e^{-(1-\epsilon/2)l}\overline{M}$, and [cf. (3.14)]

$$\tilde{P}=1+\frac{9}{n+8}(Q-1)+\frac{n-1}{n+8}\tilde{u}\left[\frac{4}{d}\left[\frac{h_0}{\overline{M}_0}\right]^{-\epsilon/2}-1\right] . \tag{4.16}$$

The total free energy then reads

$$A \equiv F+h_0M_0=A_{\text{reg}}+\frac{t_0^2}{16u_0}\frac{4}{n-4}\left[Q^{(4-n)/(n+8)}-\frac{n}{4}\right]$$
$$+\frac{1}{16u_0}\left[\frac{h_0}{\overline{M}_0}\right]^2(Q-\tilde{P})+\frac{R^2}{16u_0Q}-\left[\frac{SM_0^2}{2T_L}\right]\left[\left[\frac{h_0}{M_0}-\frac{R}{Q}\right]^2-\left[\frac{r_T}{S}-\frac{R}{Q}\right]^2\right]+\left[\frac{K_4}{8}\right]e^{-dl}\kappa_L^4\left[\ln(\kappa_L^2)-\frac{1}{2}\right] , \tag{4.17}$$

where $S=e^{2l}$ [see (3.16)], and we have defined, for convenience,

$$R=t_0Q^{6/(n+8)}+4u_0M_0^2 . \tag{4.18}$$

Equation (4.17) deserves some comment: Strictly speaking, F should depend only on h_0, while A should depend only on M_0. Thus one should actually write

$$F=F(r_0,u_0,h_0,M_0(h_0)), \quad A=A(r_0,u_0,h_0(M_0),M_0) . \tag{4.19}$$

The function $M_0(h_0)$, or, equivalently, $h_0(M_0)$, is determined by minimization:

$$\left.\frac{\partial F}{\partial M_0}\right|_{M_0(h_0)}=0 \quad \text{or} \quad \left.\frac{\partial A}{\partial h_0}\right|_{h_0(M_0)}=0 , \tag{4.20}$$

where the partials denote derivatives with respect to the fourth and third arguments of F and of A, respectively, in (4.19). Alternatively, the usual thermodynamic relations imply

$$\frac{dF}{dh_0} \equiv \frac{\partial F}{\partial h_0}+\frac{\partial F}{\partial M_0}\frac{dM_0}{dh_0}=-M_0(h_0) ,$$
$$\frac{dA}{dM_0} \equiv \frac{\partial A}{\partial M_0}+\frac{\partial A}{\partial h_0}\frac{dh_0}{dM_0}=h_0(M_0) . \tag{4.21}$$

However, by (4.20), the second terms vanish, yielding simply

$$\frac{\partial F}{\partial h_0}=-M_0(h_0) \quad \text{or} \quad \frac{\partial A}{\partial M_0}=h_0(M_0) . \tag{4.22}$$

Note that since $A=F+h_0M_0$, the second half of (4.20) is equivalent to the first half of (4.22), and vice versa. This still leaves two entirely distinct ways of calculating the equation of state. The main test of the free energy (4.17) is that it should yield the same result [Eqs. (3.14)–(3.17)] by either route. We begin by verifying the h_0 derivative. This is quite simple since the only explicit h_0 dependence is in κ_T^2 and κ_L^2 through $\overline{M}$. From (3.2) we find (for fixed l)

$$\frac{\partial\overline{M}}{\partial h}=\frac{1}{\kappa_L^2}\ , \tag{4.23}$$

and hence

$$\frac{\partial\kappa_T^2}{\partial h}=\frac{8u\overline{M}}{\kappa_L^2},\quad \frac{\partial\kappa_L^2}{\partial h}=\frac{24u\overline{M}}{\kappa_L^2}\ . \tag{4.24}$$

It is now straightforward to derive

$$0=\frac{\partial A}{\partial h_0}=\frac{SM_0}{Q\kappa_L^2}\frac{h_0}{M_0}(Q-P)-\left[\frac{SM_0}{T_L}\right]\left[\frac{h_0}{M_0}-\frac{R}{Q}\right]+\left[\frac{K_d}{4}\right]\kappa_L^2\ln(\kappa_L^2)\left[\frac{24u\overline{M}_0}{\kappa_L^2}\right]\ . \tag{4.25}$$

Setting $T_L=1\approx\kappa_L^2$ [which yields (3.15)], the last term vanishes, and (4.25) can be manipulated into the form

$$h_0/M_0=R/P\ , \tag{4.26}$$

which, using (4.18), corresponds precisely to (3.17), the correct equation of state. The derivative with respect to M is somewhat more tedious. Let us define

$$\begin{aligned}V=&\,432K_4u^2[\ln(1+T_L)+T_L/(1+T_L)]+48(n-1)K_4u^2[\ln(1+T_T)+T_T/(1+T_T)]\\&+3456u^3M^2K_4[1/(1+T_L)+1/(1+T_L)^2]+128(n-1)K_4u^3M^2[1/(1+T_T)+1/(1+T_T)^2]\\W=&\,144K_4u^2[\ln(1+T_L)+T_L/(1+T_L)]+16(n-1)K_4u^2[\ln(1+T_T)+T_T/(1+T_T)]\ .\end{aligned} \tag{4.27}$$

Note that $\partial/\partial M(WM^3)=VM^2$, and that both V and W are $O(u^2)$. It is then straightforward to show that

$$\begin{aligned}&\frac{\partial\overline{M}}{\partial M}=\frac{M(M-\overline{M})V}{\kappa_L^2}\ ,\\&\frac{\partial r_L}{\partial M}=(24u+V)M\ ,\\&\frac{\partial\kappa_L^2}{\partial M}=VM\left[1+\frac{24u\overline{M}(M-\overline{M})}{\kappa_L^2}\right]\ ,\\&\frac{\partial r_T}{\partial M}=(8u+W)M\ ,\\&\frac{\partial\kappa_T^2}{\partial M}=M\left[W+\frac{V8u\overline{M}(M-\overline{M})}{\kappa_L^2}\right]\ .\end{aligned} \tag{4.28}$$

Thus $\partial\overline{M}/\partial M=O(u^2)$; i.e., $\overline{M}$ essentially does not depend on M. This can be seen more directly from (3.2): To the extent that r_L-12uM^2 and r_T-4uM^2 are approximately M independent, the coefficients in (3.2) are M independent. Similarly, κ_L^2 and κ_T^2 are weakly M dependent, while r_L and r_T are dominated by the M dependence of T_L and T_T. To lowest order, then, we have

$$\frac{\partial A}{\partial M_0}\approx\frac{R}{8u_0Q}8u_0M_0+\frac{S}{T_L}\left[\frac{R}{Q}+\frac{8u_0M_0^2}{Q}\right]\left[h_0-\frac{RM_0}{Q}\right]\ . \tag{4.29}$$

Using $T_L=S(R+8u_0M_0^2)/Q$, essentially everything cancels, and we are left with

$$\frac{\partial A}{\partial M_0}\approx h_0\ , \tag{4.30}$$

which is the correct answer. It is apparent then that the equation of state [Eq. (4.26)] can emerge from $\partial A/\partial M_0$ only in higher order. However, keeping the higher-order terms in (4.28) is not sufficient—many other terms of the same order will arise from higher-order Feynman graphs and from better approximate solutions to the recursion relations. This problem seems to be a general feature of loop expansions: Different paths leading to the same physical quantity may require different orders in perturbation theory to achieve equivalent results.

Having demonstrated satisfactory consistency of our free energy, we compare it to that derived by Nicoll and Chang.[3] Their result can be written in the form (see Ref. 5, Sec. VI)

$$A_{\rm NC}-A_{\rm reg}=\frac{t_0^2}{16u_0}\frac{4}{n-4}\times\left[Q^{(4-n)/(n+8)}-\frac{n}{4}\right]+\frac{R^2}{16u_0P}\ , \tag{4.31}$$

with (3.14), (3.15), (3.16), and (4.18) defining P, Q, R, and S, respectively. They state that the equation of state should be derived by differentiating only the explicit M_0 dependence in (4.31), i.e., that which appears in R. This indeed yields (4.26). However, one should demonstrate that derivatives, with respect to the implicit M_0 dependence, do not contribute further terms. The simplest way to compare (4.31) to (4.17) is to use the equation of state to substitute for h_0/M_0 in (4.17) and use M_0 and $\overline{M}_0$ interchangeably. This yields

$$A = A_{\text{reg}} + \frac{t_0^2}{16u_0}\frac{4}{n-4}\left[Q^{(4-n)/(n+8)} - \frac{n}{4}\right] - \frac{K_d}{16}S^{-[2-(1/2)\epsilon]} + \frac{R^2}{16u_0}\left[\frac{Q}{P^2} - \frac{\tilde{P}}{P^2} + \frac{1}{Q} - \frac{1}{(1+R/8u_0M_0^2)} \times\left[\frac{Q}{P^2} - \frac{2}{P} + \frac{1}{Q}\right]\right], \tag{4.32}$$

where we have dropped the term proportional to $(r_T/S - R/Q)^2$ since it is apparently of higher order in u and in any case approximately independent of both M_0 and h_0. The last term in (4.32) can be rewritten:

$$\frac{R^2}{16u_0\tilde{P}}\left[1 + \frac{\tilde{P}R/P^2}{R+8u_0M_0^2}\frac{1}{Q}(P-Q)^2 - \left[\frac{P-\tilde{P}}{P}\right]^2\right]. \tag{4.33}$$

To the extent that $\tilde{P} \simeq P$ [compare (3.14) and (4.16)] and $R \ll 1$, (4.33) reproduces the last term in the Nicoll-Chang result [Eq. (4.31)]. It is easy to check that $(P-\tilde{P})/P$ is always $O(\epsilon)$, while $P-Q$ is roughly $O(\epsilon)$, unless $(h_0/M_0)^{\epsilon/2}$ is small—however, in this case the prefactor $R/P = h_0/M_0$ is small, and so the whole term is always small. A better approximation is to take $\tilde{P}$ in place of P in (4.31). This is so because, for complete consistency, the equation of state should be derived not only from the M_0 dependence in R, but also from that in P. One finds [recall that $h_0/\overline{M}_0$ in (4.16) has been replaced by R/P]

$$\frac{\partial \tilde{P}}{\partial M_0} = \frac{4}{d}\frac{\partial P}{\partial M_0} = \frac{-16u_0M_0}{R}\frac{(\tilde{P}-P)P}{P-\frac{1}{2}d(\tilde{P}-P)}, \tag{4.34}$$

so that neglecting the M_0 dependence of P in (4.31) entails errors of relative order $(\tilde{P}-P)/P = O(\epsilon)$ in the equation of state. Alternatively, if one takes $R^2/\tilde{P}$ in (4.31), the extra factor of $4/d$ cancels the error term linear in $(\tilde{P}-P)/P$, leaving errors only of $O([(\tilde{P}-P)/P]^2)$.

Finally, recall the $\kappa_L^4[\ln(\kappa_L^2) - \frac{1}{2}]$ term. This term is constructed so as to vanish when differentiated at fixed l and then evaluated at $\kappa_L^2 = 1$. However, if $\kappa_L^2 = 1$ is imposed *before* differentiation, this term, which then takes the value $\frac{1}{16}K_dS^{-[2-(1/2)\epsilon]}$, serves to maintain the identity $\partial A/\partial l^* = 0$—canceling contributions from the now M_0- and h_0-dependent functions Q and S appearing elsewhere in the free energy. This term is also crucial for correct evaluation of other derivatives, such as the entropy (or density, depending on how thermodynamic variable r is identified) $-(\partial A/\partial t_0)_{M_0}$. Therefore, the lack of this term in the Nicoll-Chang free energy represents a definite discrepancy with our own expression. For the reasons given above, we believe our expression to be the correct one.

In summary, then, the correct form for the free energy, closest in spirit to that of Nicoll and Chang, reads

$$\tilde{A}_{\text{NC}} - A_{\text{reg}} = \frac{t_0^2}{16u_0}\frac{4}{n-4}\left[Q^{(4-n)/(n+8)} - \frac{n}{4}\right] + \frac{R^2}{16u_0\tilde{P}} - \frac{1}{16}K_dS^{-[2-(1/2)\epsilon]}, \tag{4.35}$$

with the equation of state to be derived by differentiation with respect to the explicit M_0 dependence in R [according to (4.18)] and in $\tilde{P}$ [according to (4.34)]. By construction [explicitly verified for the free energy (4.17)], all other implicit M_0 dependence—embodied in the choice of the matching scale $l = l^*$ [i.e., Eq. (3.15)]—will cancel out under differentiation. In Sec. V we explore the effects of the extra $S^{-[2-(1/2)\epsilon]}$ term on quantities derived from the free energy. In particular, we reexamine the derivation of the helicity modulus Υ at constant density in Ref. 5. We also give an enormously simplified rederivation of the helicity modulus which agrees precisely with the expression calculated by Rudnick and Jasnow.[8] Our approach relies on the identification of Υ with the small **k** behavior of the Green's function,[9] rather than the method of comparing free energies for periodic and antiperiodic boundary conditions.[8]

V. HELICITY MODULUS, DENSITY, AND SPECIFIC HEAT

A. Helicity modulus

In the ordered phase (in zero external field), the Green's function

$$G(\mathbf{k}) = \langle |\mathbf{s}_\mathbf{k}|^2\rangle = \langle |\mathbf{s}_\mathbf{k}\cdot\hat{\mathbf{M}}|^2\rangle + \langle |\mathbf{s}_\mathbf{k}^\perp|^2\rangle \tag{5.1}$$

has the small-**k** behavior

$$G(\mathbf{k}) = |\mathbf{M}_0|^2\delta(\mathbf{k}) + b_T/|\mathbf{k}|^2 + b_L/|\mathbf{k}|^\epsilon + O(1), \tag{5.2}$$

where b_L is related to the amplitude of the divergence of the longitudinal susceptibility [Eq. (1.1)], and b_T is the amplitude of the transverse spin-wave singularity. One has the exact correspondence[9]

$$b_T = (n-1)|\mathbf{M}_0|^2k_BT/\Upsilon, \tag{5.3}$$

where Υ is the *helicity modulus*, which is related to the superfluid density via

$$\rho_s = (m/\hbar)^2\Upsilon, \tag{5.4}$$

m being the mass of a ^{4}He atom. One may also define Υ in terms of an integral over a current-current correlation function[8,9] (which involves an average over a four-spin operator, rather than a two-spin operator). The latter is more closely related to the definition of Υ in terms of the free-energy increment due to "twist" boundary conditions.[8,10] We concentrate on the former definition, which may be restated as

$$\frac{\Upsilon}{k_B T|\mathbf{M}_0|^2}=\lim_{k\to 0}\frac{1}{k^2 G_\perp(\mathbf{k})} , \tag{5.5}$$

where $G_\perp(\mathbf{k})=[1/(n-1)]\langle|\mathbf{s}_k^\perp|^2\rangle$ is the transverse part of the Green's function. Since the renormalization-group transformation used to generate the recursion relations in Sec. II is quasilinear, the small-$\mathbf{k}$ part of the Green's function transforms exactly as

$$\frac{dG_\perp}{dl}=-[2-\eta(l)]G_\perp . \tag{5.6}$$

To $O(\epsilon)$ one has $\eta(l)\equiv 0$, which yields

$$G_\perp(\mathbf{k},l=0)=e^{2l^*}G_\perp(\mathbf{k}e^{l^*},l^*) , \tag{5.7}$$

and hence

$$\begin{aligned}\Upsilon/k_B T M_0^2&=\lim_{k\to 0}1/k^2 G_\perp(\mathbf{k},l^*)\\&=\Upsilon(l^*)/k_B T M(l^*)^2 ,\end{aligned} \tag{5.8}$$

so that

$$\Upsilon=e^{-(d-2)l^*}\Upsilon(l^*) . \tag{5.9}$$

Thus we need only calculate $\Upsilon(l^*)$, which to $O(\epsilon)$ involves only the lowest-order spin-wave corrections to Υ. From the Hamiltonian (3.3) we find, to $O(u(l^*))$,

$$\begin{aligned}G_\perp(\mathbf{k})^{-1}=k^2+\kappa_T^2&+4(n+1)u(l^*)I_T+4u(l^*)I_L\\&-\tfrac{1}{2}w(l^*)^2[4(n-1)I_T+6I_L]/\kappa_L^2\\&-4w(l^*)^2 I_{LT}(\mathbf{k})+O(u(l^*)^2) ,\end{aligned} \tag{5.10}$$

where

$$\begin{aligned}&I_L=\int_q\frac{1}{q^2+\kappa_L^2},\quad I_T=\int_q\frac{1}{q^2+\kappa_T^2} ,\\&I_{LT}(\mathbf{k})=\int_q\frac{1}{(\mathbf{k}+\mathbf{q})^2+\kappa_L^2}\,\frac{1}{q^2+\kappa_T^2} .\end{aligned} \tag{5.11}$$

On the coexistence curve the right-hand side of (5.10) must vanish at $k=0$. This determines κ_T^2. Setting $k=0$, we find

$$\begin{aligned}(\kappa_T^2)_{\text{coex}}=&[-4(n+1)u(l^*)+2(n-1)w(l^*)^2]I_T\\&+[-4u(l^*)+3w(l^*)^2/\kappa_L^2]I_L\\&+4w(l^*)^2 I_{LT}(k=0) .\end{aligned} \tag{5.12}$$

Hence the only contribution to $\Upsilon(l^*)$ comes from the k dependence of $I_{LT}(\mathbf{k})$:

$$\Upsilon(l^*)/k_B T=1-\lim_{k\to 0}4w(l^*)^2\frac{1}{k^2}[I_{LT}(\mathbf{k})-I_{LT}(0)] . \tag{5.13}$$

At this point one encounters problems with the sharp cutoff we have been using: The domain of integration for $I_{LT}(\mathbf{k})$ in (5.11) is defined as the region of $|\mathbf{q}|<1$ such that $|\mathbf{k}+\mathbf{q}|<1$ as well, i.e., the intersection between two hyperspheres whose centers are separated by $\mathbf{k}$. This yields a contribution to $I_{LT}(\mathbf{k})\propto|\mathbf{k}|$, and hence the limit in (5.13) yields a divergent result. This problem is solved in the original physical model by imposing proper periodicity at the boundaries of the Brillouin zone (i.e., Umklapp processes in the Fourier space representation of the us^4 interaction). The spherical Brillouin zone we use here complicates matters since it cannot be repeated periodically via translation by reciprocal lattice vectors. We instead solve the problem by fiat: Since $I_{LT}(\mathbf{k})-I_{LT}(0)$ is well defined if the cutoff is allowed to diverge to infinity, we define $\Upsilon(l^*)$ from the leading k^2 dependence of this cutoffless expression. The result we will then derive agrees with that of Rudnick and Jasnow[8] (who encountered precisely this problem and solved it in this same way) and with a field-theoretic derivation of universal amplitude ratios involving Υ (see below).[11]

To the requisite order, (5.13) may be evaluated with $\kappa_T^2=0$ on the right-hand side. We find

$$\lim_{k\to 0}\frac{1}{k^2}[I_{LT}(\mathbf{k})-I_{LT}(0)]=\int_q\frac{1}{q^2}\,\frac{(\epsilon/d)q^2-\kappa_L^2}{(q^2+\kappa_L^2)^2} . \tag{5.14}$$

To $O(\epsilon)$ we may also take $d=4$ in this integral so that

$$\Upsilon(l^*)/k_B T=M(l^*)^2+\tfrac{1}{4}K_4+O(u(l^*)) , \tag{5.15}$$

where we have used the matching conditions $\kappa_L(l^*)\approx 1$ and $8u(l^*)M(l^*)\approx 1$ in the second term on the right-hand side. The final result is then

$$\Upsilon/k_B T=M_0^2+e^{-(d-2)l^*}\tfrac{1}{4}K_4+O(\epsilon^2) , \tag{5.16}$$

which corresponds precisely to the result of Rudnick and Jasnow.[8]

A very similar calculation for the helicity modulus was carried out in Appendix B of Ref. 11 in the context of verifying two-scale-factor universality. There it was shown that the ratio $\xi_\Upsilon(-t_0)/\xi(t_0)$ tends to a universal constant as $t_0\to 0^+$. Here ξ is the usual correlation length defined by the exponential decay of the spin-spin correlation function above T_c, while $\xi_\Upsilon=(\Upsilon/k_B T)^{-1/(d-2)}$ is the natural hydrodynamic length which diverges as T_c is approached from below. Universality of this ratio is a consequence of hyperscaling and is therefore valid for $2<d<4$.

B. Density

We define the density ρ_0 via

$$\rho_0=\left(\frac{\partial A}{\partial r_0}\right)_{M_0}=\left(\frac{\partial F}{\partial r_0}\right)_{h_0} . \tag{5.17}$$

We call this a density since in the problem of superfluidity in a dilute Bose gas, $r_0\propto-\mu$ is related to the chemical potential, and ρ_0 is related via a multiplicative temperature-dependent factor to the boson density ρ.[5] We carry out the above derivative on the free-energy expression (4.17) [or (4.35)] at fixed l, then set $l=l^*$. For simplicity, we will take $h_0=0$. At fixed l, Q is r_0 independent. For $t_0<0$, we also have $R=0$ (coexistence curve). As mentioned earlier, the last term in (4.17) is designed to vanish under differentiation. The only t_0 (hence r_0) dependence that contributes to ρ_0 in the end comes from the first two terms in (4.17). Hence

$$\rho_0=\rho_{0,\mathrm{reg}}+\frac{t_0}{8u_0}\frac{4}{n-4}\left[Q_-^{(4-n)/(n+8)}-\frac{n}{4}\right], \quad h_0=0,\ t_0<0, \tag{5.18}$$

where Q_- satisfies (3.21), and

$$\rho_{0,\mathrm{reg}}=(nK_4/4)[1-r_0\ln(1+r_0)]. \tag{5.19}$$

This should be compared to Eq. (6.35) in Ref. 5 which is far more complicated. The extra complexity is a direct result of the missing $S^{-[2-(1/2)\epsilon]}$ term in (4.31) which would otherwise serve to *cancel* the extra terms. The numerical difference is, however, probably very small. Correcting the subsequent equations in Ref. 5 is very simple. In particular, the coefficient of $\dot{Q}^{(4-n)/(n+8)}$ in Eq. (6.48) and of $Z(\Xi)^{1/5}$ in Eq. (6.65) should simply be set to unity.

For $t_0>0$ one needs an expression for the limit $\kappa_T^2\approx h/M$ when $h\to 0$, i.e., $\chi(l)^{-1}$, the inverse susceptibility. From the equation of state, or by direct diagrammatic evaluation, one finds

$$\chi^{-1}(l=0)=e^{-2l}\chi(l)^{-1}=t_0Q^{-(n+2)/(n+8)}\{1-2(n+2)K_4u(l)\ln[t(l)]\},\quad h_0=0,\quad t_0>0. \tag{5.20}$$

With the matching condition $t(l^*)=1$, one determines $Q\equiv Q_+$ via

$$Q_+=1-\tilde{u}+\tilde{u}t_0^{-\epsilon/2}Q_+^{(\epsilon/2)(n+2)/(n+8)} \tag{5.21}$$

[cf. (3.21)], and only the first term in (5.20) survives. From these equations one finds that $P=Q_+$ when $h_0=0$ and $t_0>0$. Hence only the fourth term in (4.17) contributes any further t_0 dependence, and one finds

$$\rho_0=\rho_{0,\mathrm{reg}}+\frac{t_0}{8u_0}\frac{n}{n-4}(Q_+^{(4-n)/(n+8)}-1),\quad h_0=0,\quad t_0>0. \tag{5.22}$$

The compressibility (more commonly interpreted as the specific heat) is then given by

$$\begin{aligned}\kappa_0=-\frac{\partial\rho_0}{\partial t_0}&=\kappa_{0,\mathrm{reg}}+\frac{1}{2(4-n)u_0}\\&\times\left\{Q_-^{(4-n)/(n+8)}\left[\left[1-\frac{\epsilon}{2}\frac{6}{n+8}[1-(1-\tilde{u})Q_-^{-1}]\right]\Bigg/\left[1-\frac{\epsilon}{2}\frac{n+2}{n+8}[1-(1-\tilde{u})Q_-^{-1}]\right]\right]-\frac{n}{4}\right\},\\&\qquad h_0=0,\quad t_0<0,\\&=\kappa_{0,\mathrm{reg}}+\frac{n}{8(4-n)u_0}\\&\times\left\{Q_+^{(4-n)/(n+8)}\left[\left[1-\frac{\epsilon}{2}\frac{6}{n+8}[1-(1-\tilde{u})Q_+^{-1}]\right]\Bigg/\left[1-\frac{\epsilon}{2}\frac{n+2}{n+8}[1-(1-\tilde{u})Q_+^{-1}]\right]\right]-1\right\},\\&\qquad h_0=0,\quad t_0>0,\end{aligned} \tag{5.23}$$

where

$$\kappa_{0,\mathrm{reg}}=(nK_4/4)[\ln(1+r_0)+r_0/(1+r_0)]. \tag{5.24}$$

It is worth commenting that (5.23) yields the universal specific-heat amplitude ratio[11] correct only to *zeroth* order in ϵ. This is because the exponent

$$\alpha=\frac{\epsilon}{2}\frac{4-n}{n+8}\Bigg/\left[1-\frac{\epsilon}{2}\frac{n+2}{n+8}\right]$$

is $O(\epsilon)$. One needs the specific heat correct to $O(\epsilon^2)$ to obtain the universal ratio correctly to $O(\epsilon)$.

C. Specific heat

The specific heat at fixed X is given by

$$C=T\left[\frac{\partial S}{\partial T}\right]_X=-T\left[\frac{\partial}{\partial T}\left[\frac{\partial F}{\partial T}\right]_H\right]_X, \tag{5.25}$$

where X represents some thermodynamic constraint, for example, fixed density ρ. Evaluation of (5.25) requires knowledge of the implicit T dependence of t_0, u_0, etc., which depends on the particular path taken to arrive at the effective S^4 model, as well as the precise nature of the constraint X. We shall exhibit the calculation for the case of the dilute Bose gas where the constraint is that of fixed density (Sec. V B) and the temperature dependence enters via

$$r_0=-R_d\beta\mu,\quad u_0=U_d\beta v_0/\Lambda_T^d, k_\Lambda=\Gamma_d/\Lambda_T \tag{5.26}$$

[see Eqs. (5.25), (5.27), and (6.4) in Ref. 5]. Here

$$\Gamma_d=2\sqrt{\pi}\left[\tfrac{1}{2}(d-2)\Gamma\left[\frac{d}{2}\right]\zeta\left[\frac{d}{2}\right]\right]^{1/(d-2)},$$

$R_d=4\pi/\Gamma_d^2$, and $U_d=8\pi^2/\Gamma_d^\epsilon$ are dimensionless constants, $\beta=1/k_BT$, $\Lambda_T=h/(2\pi mk_BT)^{1/2}$ is the thermal de Broglie wavelength, and m, v_0 are ^{4}He atomic parameters. Finally, the number density is

$$\rho=(8\pi/n\Gamma_d^2)k_\Lambda^d\rho_0 , \tag{5.27}$$

where $\rho_0=(\partial A/\partial r_0)$ was calculated in Sec. V B. The free energy $A\equiv A_{\rm Bose}$ which appears in (5.25) differs also by a temperature-dependent factor and additive term from $A\equiv A_{\rm spin}$ calculated in Sec. IV due to spin and space rescaling: One finds

$$A_{\rm Bose}=\frac{1}{\beta}k_\Lambda^d\left[A_{\rm spin}-\frac{nK_d}{2d}\ln(R_d)\right] . \tag{5.28}$$

For completeness we also exhibit the relation between the Bose and spin order parameters and conjugate fields—determined by the spin and volume rescaling factors [Eq. (6.2) in Ref. 5]:

$$\begin{aligned}\beta H_{\rm Bose}&=k_\Lambda^{d/2}R_d^{-1/2}h_0 ,\\ M_{\rm Bose}&=k_\Lambda^{d/2}R_d^{1/2}M_0 .\end{aligned} \tag{5.29}$$

Setting $h_0=0$, we calculate the specific heat $C_\pm$ for $T\gtrless T_c$. As mentioned earlier, only the first two terms in (4.17) contribute for $T<T_c$, while only the first, second, and fourth contribute for $T>T_c$. One finds

$$S_\pm=k_\Lambda^dk_B\left[\rho_0[t_0-d(n+2)K_du_0]+\frac{(d-2)/t_0}{4u_0}(\rho_0-\rho_{0,\rm reg})+\frac{(d-2)t_0^2}{8(n+8)u_0^2}Q_\pm^{(4-n)/(n+8)}(1-Q_\pm^{-1})-\frac{d+2}{2}\left[A_{\rm spin}-\frac{nK_d}{2d}\ln(R_d)\right]\right] . \tag{5.30}$$

The constraint may be put in the form[5]

$$\frac{n}{2(d-2)}K_dn\bar t=\rho_0-\rho_{0,\rm reg},\quad \bar t\equiv\left[\frac{T_c}{T}\right]^{d/2}-1 , \tag{5.31}$$

where $T_c(\rho)\approx T_c^0(\rho)$ is the transition temperature at given density ρ, and $T_c^0(\rho)$ is the ideal gas transition temperature defined by $\rho\Lambda_{T_c}^d=\zeta(\frac{1}{2}d)$. It is easy to see that for $t_0\to0$, (5.31) yields $t_0\sim\bar t^{1/(1-\alpha)}$, so long as $\alpha>0$ (to order ϵ, this requires $n<4$, which we henceforth assume). One sees therefore that the most singular parts of the entropy at constant density are the terms *linear* in t_0: $S_\pm=S_{\rm reg}+S_{\pm,\rm sing}$, with

$$\begin{aligned}S_{\pm,\rm sing}&=\frac{nd\Gamma_d^2}{16\pi}k_B\rho t_0+O\left[\frac{t_0^{2-\alpha}}{u_0},\frac{t_0^2}{u_0},t_0\bar t,\ldots\right] ,\\ S_{\rm reg}&=\frac{nK_d}{2d}\frac{d+2}{2}k_\Lambda^dk_B\left[\frac{2}{d}+\ln(R_d)+O(u_0)\right] .\end{aligned} \tag{5.32}$$

One finds then $C_\pm=C_{\rm reg}+C_{\pm\rm sing}$, with

$$\begin{aligned}C_{\pm,\rm sing}&=-\frac{[(nd)^2\Gamma_d^2K_d/8\pi(d-2)]k_Bu_0\rho[1+O(\bar t,\bar t^{1/(1-\alpha)})]}{D_\pm(Q_\pm)} ,\\ C_{\rm reg}&=\frac{d}{2}S_{\rm reg}=\frac{n(d^2-4)\Gamma_d^2}{32\pi}k_B\rho\left[\frac{2}{d}+\ln(R_d)+O(u_0,\bar t)\right] ,\end{aligned} \tag{5.33}$$

and

$$\begin{aligned}D_+(Q_+)&=\frac{n}{4-n}\left[Q_+^{(4-n)/(n+8)}\left[1-\frac{\epsilon}{2}\frac{6}{n+8}[1-(1-\tilde u)Q_+^{-1}]\right]\Big/\left[1-\frac{\epsilon}{2}\frac{n+2}{n+8}[1-(1-\tilde u)Q_+^{-1}]\right]-\frac{n}{4}\right] ,\\ D_-(Q_-)&=\frac{4}{4-n}\left[Q_-^{(4-n)/(n+8)}\left[1-\frac{\epsilon}{2}\frac{6}{n+8}[1-(1-\tilde u)Q_-^{-1}]\right]\Big/\left[1-\frac{\epsilon}{2}\frac{n+2}{n+8}[1-(1-\tilde u)Q_-^{-1}]\right]-1\right] .\end{aligned} \tag{5.34}$$

Note the resemblance to the *inverse* of the unconstrained specific heat (5.23). The functions $Q_\pm(\bar t)$ are determined via the constraint equation (5.3).[5] It is easy to see that (5.33) yields the usual Fisher-renormalized[12] specific-heat exponent $\alpha'=-\alpha/(1-\alpha)$. Similarly, the universal amplitude ratio $r_c=C_+/C_-$ is renormalized via $r_c'=r_c^{-1/(1-\alpha)}$.

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APPENDIX A: DETAILS OF RECURSION RELATION SOLUTIONS

We outline here in somewhat more detail the solutions to the recursion relations (2.22)–(2.26). The solution for $u(l)$ is elementary and is given by (2.29). The solution for $w(l)$ follows immediately via (2.27) and (2.28). The solutions for $r_T(l)$ and $r_L(l)$ are more complicated. Following Rudnick and Nelson,[1] we begin by analyzing the *simplified* recursion relations, valid for $r,u\le O(\epsilon)$:

$$\frac{dr_L}{dl}=(2-12K_4u)r_L-4(n-1)K_4ur_T+4(n+2)K_4u\ , \tag{A1}$$

$$\frac{dr_T}{dl}=[2-4(n+1)K_4u]r_T-4K_4ur_L+4(n+2)K_4u\ . \tag{A2}$$

Diagonalization of the first two terms in each equation yields two eigencombinations $r_1=(1/n)[r_L+(n-1)r_T]$ and $r_2=(1/n)(r_L-r_T)$ with solutions

$$r_1(l)=r_1(0)e^{2l}/Q(l)^{(n+2)/(n+8)}\ , \tag{A3}$$

$$r_2(l)=r_2(0)e^{2l}/Q(l)^{2/(n+8)}\ . \tag{A4}$$

These are now used to generate the full solutions. The first step involves converting the recursion relations to integral equations. One finds, in a straightforward way,

$$\begin{aligned} r_1(l)=&r_1(0)e^{2l}/Q(l)^{(n+2)/(n+8)}+[e^{2l}/Q(l)^{(n+2)/(n+8)}]\int_0^l dl'e^{-2l'}Q(l')^{(n+2)/(n+8)}\\ &\times\{4(n+2)K_4u(l')+[4(n+2)K_4/n]u(l')r_L(l')^2/[1+r_L(l')]\\ &+[4(n-1)(n+2)K_4/n]u(l')r_T(l')^2/[1+r_T(l')]\\ &-(18K_4/n)w(l')^2/[1+r_L(l')]^2-[2(n-1)K_4/n]w(l')^2/[1+r_T(l')]^2\\ &-[4(n-1)K_4/n]w(l')^2/[1+r_L(l')][1+r_T(l')]\}\ , \end{aligned} \tag{A5}$$

$$\begin{aligned} r_2(l)=&r_2(0)e^{2l}/Q(l)^{2/(n+8)}+[e^{2l}/Q(l)^{2/(n+8)}]\int_0^l dl'e^{-2l'}Q(l')^{2/(n+8)}\\ &\times\{(8K_4/n)u(l')r_T(l')^2/[1+r_T(l')]-(8K_4/n)u(l')r_L(l')^2/[1+r_L(l')]\\ &-(4K_4/n)w(l')^2/[1+r_L(l')][1+r_T(l')]+(18K_4/n)w(l')^2/[1+r_L(l')]^2\\ &+[2(n-1)K_4/n]w(l')^2/[1+r_T(l')]^2\}\ . \end{aligned} \tag{A6}$$

The basic technique used to evaluate the remaining integrals is to divide each term into a slowly varying piece, a function only of $e^{\epsilon l}$, and a rapidly varying piece. An integration by parts is then performed, putting the derivative on the slowly varying piece, which then becomes smaller by a factor of ϵ. The remaining integral can then be dropped. One also must take into consideration which region of integration contributes most to the integral. For example, $r_L(l)$ and $r_T(l)$ are small, of order ϵ, for most of the interval $0\le l\le l^*$, becoming large, of order unity, only over the last part of the interval, during which slowly varying functions, such as $u(l)$, change only by $O(\epsilon^2)$. It was precisely arguments such as these that led to the reduced set of recursion relations (2.22)–(2.26) and must be used here again to further simplify the analysis. Finally, if the entire integrand is slowly varying, the integral is performed exactly: Usually such terms involve only rational functions of $e^{\epsilon l}$.

To illustrate, the combinations $e^{-2l'}r_L(l')$, $e^{-2l'}r_T(l')$, and $e^{-2l'}w^2(l')$ are slowly varying, as are $Q(l')$ and $u(l')$. Thus, for example,

$$\int_0^l dl'[e^{-2l'}u(l')Q(l')^{(n+2)/(n+8)}r_L(l')]\frac{r_L(l')}{1+r_L(l')}\approx\left[\int^{l'}\frac{r_L(l'')}{1+r_L(l'')}dl''\right]e^{-2l'}u(l')Q(l')^{(n+2)/(n+8)}r_L(l')\Bigg|_0^l\ , \tag{A7}$$

where the integral remaining after the integration by parts, with the derivative on the slowly varying part, has been dropped. The integral of the function $r_L/(1+r_L)$ is performed by realizing that the important contribution comes from the region $r_L\gg\epsilon$. In this region we may approximate $r_L(l'')\approx r_L(l')e^{-2(l'-l'')}$, $l''<l'$. This yields

$$\int^{l'}\frac{r_L(l'')}{1+r_L(l'')}dl''\approx\frac{1}{2}\ln[1+r_L(l')]+c\ , \tag{A8}$$

where c is an arbitrary constant of integration, which we take to be zero. The result of (A7) is then

$$e^{-2l}u(l)Q(l)^{(n+2)/(n+8)}r_L(l)\ln[1+r_L(l)]+O(u^2,\epsilon u)\ . \tag{A9}$$

Similarly, we have

$$\int_0^1 dl'[e^{-2l}u(l')Q(l')^{(n+2)/(n+8)}r_T(l')]\frac{r_T(l')}{1+r_T(l')}=e^{-2l}u(l)Q(l)^{(n+2)/(n+8)}r_T(l)\ln[1+r_T(l)]+O(u^2,\epsilon u)\ . \tag{A10}$$

The w^2 integrals are evaluated by first ignoring the $1/(1+r)^2$ denominators, yielding a slowly varying integrand which can then be treated exactly. The remainder is then evaluated via integration by parts: Once again, the major contribution comes from the region $r_L \gg \epsilon$, and the same approximations are made that led to (A8). The result is

$$\int_0^l e^{-2l'} Q(l')^{(n+2)/n+8)} w(l')^2 \left[1 - \frac{r_L(2+r_L)}{(1+r_L)^2}\right]$$

$$= \frac{2}{3K_4} e^{-2l} Q(l)^{(n+2)/(n+8)} [u(l)M(l)^2 - e^{2l} Q(l)^{-(n+2)/(n+8)} u_0 M_0^2]$$

$$- \frac{1}{2} w(l)^2 e^{-2l} Q(l)^{(n+2)/(n+8)} \left[\ln(1+r_L) + \frac{r_L}{1+r_L}\right] + O(u^2, \epsilon u) . \quad \text{(A11)}$$

Analysis of all other terms is essentially the same. We quote only the final results for $r_1(l)$ and $r_2(l)$:

$$r_1(l) = e^{2l} Q(l)^{-(n+2)/(n+8)} \left[r_1(0) + \frac{4(n+2)}{n} u_0 M_0^2 - 2(n+2)K_4 u_0 + O(\epsilon u, u^2)\right]$$

$$- \frac{4(n+2)}{n} u(l) M(l)^2 - 2(n+2)K_4 u(l) + \frac{2(n+2)K_4}{n} u(l) r_L \ln(1+r_L) + \frac{2(n-1)(n+2)K_4}{n} u(l) r_T \ln(1+r_T)$$

$$+ \frac{9K_4}{n} w(l)^2 \left[\ln(1+r_L) + \frac{r_L}{1+r_L}\right] + \frac{(n-1)K_4}{n} w(l)^2 \left[\ln(1+r_T) + \frac{r_T}{1+r_T}\right]$$

$$+ \frac{2(n-1)K_4}{n} w(l)^2 \left[\frac{r_L}{r_L - r_T} \ln(1+r_L) + \frac{r_T}{r_T - r_L} \ln(1+r_T)\right] , \quad \text{(A12)}$$

$$r_2(l) = e^{2l} Q(l)^{-2/(n+8)} \left[r_2(0) + \frac{8}{n} u_0 M_0^2 + O(u^2, \epsilon u)\right] - \frac{8}{n} u(l) M(l)^2 + \frac{4K_4}{n} u(l) [r_T \ln(1+r_T) - r_L \ln(1+r_L)]$$

$$+ \frac{2K_4}{n} w(l)^2 \left[\frac{r_L}{r_L - r_T} \ln(1+r_L) + \frac{r_T}{r_T - r_L} \ln(1+r_T)\right]$$

$$- \frac{9K_4}{n} w(l)^2 \left[\ln(1+r_L) + \frac{r_L}{1+r_L}\right] + \frac{(n-1)K_4}{n} \left[\ln(1+r_T) + \frac{r_T}{1+r_T}\right] . \quad \text{(A13)}$$

These expressions are now used to calculate r_L and r_T via $r_L = r_1 + (n-1)r_2$ and $r_T = r_1 - r_2$. After a number of cancellations, we find

$$r_L(l) = \frac{[r_0 + 2(n+2)K_4 u_0 + O(\epsilon u_0, u_0^2)] e^{2l}}{Q(l)^{(n+2)/(n+8)}} + \frac{(n-1) O(\epsilon u_0, u_0^2) e^{2l}}{Q(l)^{2/(n+8)}}$$

$$+ 12u(l) M(l)^2 - 2(n+2)K_4 u(l) + 6K_4 u(l) r_L \ln(1+r_L) + 2(n-1)K_4 u(l) r_T \ln(1+r_T)$$

$$+ 9K_4 w(l)^2 \left[\ln(1+r_L) + \frac{r_L}{1+r_L}\right] + (n-1)K_4 w(l)^2 \left[\ln(1+r_T) + \frac{r_T}{1+r_T}\right] , \quad \text{(A14)}$$

$$r_T(l) = \frac{[r_0 + 2(n+2)K_4 u_0 + O(\epsilon u_0, u_0^2)] e^{2l}}{Q(l)^{(n+2)/(n+8)}} + \frac{O(\epsilon u_0, u_0^2) e^{2l}}{Q(l)^{2/(n+8)}}$$

$$+ 4u(l) M(l)^2 - 2(n+2)K_4 u(l) + 2K_4 u(l) r_L \ln(1+r_L) + 2(n+1)K_4 u(l) r_T \ln(1+r_T)$$

$$+ 2K_4 w(l)^2 \left[\frac{r_L}{r_L - r_T} \ln(1+r_L) + \frac{r_T}{r_T - r_L} \ln(1+r_T)\right] . \quad \text{(A15)}$$

Defining

$$t(l) = [r_0 + 2(n+2)K_4 u_0 + O(\epsilon u_0, u_0^2)] e^{2l} / Q(l)^{(n+2)/(n+8)} ,$$

$T_L(l) = t(l) + 12u(l)M(l)^2$, and $T_T(l) = t(l) + 4u(l)M(l)^2$, then substituting T_L and T_T for r_L and r_T in the terms of $O(u, w^2)$ on the right-hand sides of (A14), (A15) yields the final results (2.30) and (2.31) [note that $w^2/(T_L - T_T) = 2u$].

The solution for $\tilde{h}(l)$ is now straightforward. The integral equation corresponding to (2.22) is

$$\tilde{h}(l) = \tilde{h}_0 e^{[3-(1/2)\epsilon]l} - e^{[3-(1/2)\epsilon]l} \int_0^l dl' e^{-[3-(1/2)\epsilon]l'} \left[\frac{(n-1)K_4 w(l')}{1+r_T} + \frac{3K_4 w(l')}{1+r_L}\right] . \quad \text{(A16)}$$

By writing $1/(1+r)=1-r+r^2/(1+r)$, we again can isolate the various asymptotic regions. The term linear in r is slowly varying and can be integrated exactly once T is substituted for r. The $r^2/(1+r)$ term is handled in the same way that (A8) was. The final results [Eq. (2.32)] then follow in a straightforward way.

APPENDIX B: VALIDITY OF THE LINEAR SPIN-WAVE APPROXIMATION

Since there is some confusion in the early literature[1,4] on how to handle the vanishing transverse "mass" κ_T on the ordered-phase coexistence curve, we feel it worthwhile to indicate here the region of validity of linear spin-wave theory.

Consider first a model with fixed-length spins $|\mathbf{s}_i|=1$ at temperature T:

$$\overline{H}_1=-\frac{J}{T}\sum_{\langle ij\rangle}|\mathbf{s}_i-\mathbf{s}_j|^2 . \tag{B1}$$

At low temperatures, $J/T\gg 1$, it is appropriate to expand $\mathbf{s}_i$ around the uniform state $\mathbf{s}_i=\hat{\mathbf{M}}$ for all i, where $\hat{\mathbf{M}}$ is a unit vector. One writes $\mathbf{s}_i=\sqrt{1-|\mathbf{s}_i^\perp|^2}\hat{\mathbf{M}}+\mathbf{s}_i^\perp$, where $\mathbf{s}_i^\perp\cdot\hat{\mathbf{M}}=0$. Keeping terms to quadratic order, one finds

$$\overline{H}_1\approx-\frac{J}{T}\sum_{\langle ij\rangle}|\mathbf{s}_i^\perp-\mathbf{s}_i^\perp|^2 . \tag{B2}$$

For small T/J we may treat $\mathbf{s}_i^\perp$ as extended $(n-1)$-dimensional spins, so that (B2) is just a Gaussian model. The change in magnetization is then

$$\Delta M=1-\left\langle\sqrt{1-|\mathbf{s}_i^\perp|^2}\right\rangle\approx\frac{1}{2}\langle|\mathbf{s}_i^\perp|^2\rangle \approx\frac{(n-1)T}{2J}\int_{0<|\mathbf{q}|<1}\frac{1}{q^2} , \tag{B3}$$

which yields $\Delta M\approx[(n-1)K_d/2(d-2)](T/J)$. Self-consistency requires $\Delta M\ll 1$, which is satisfied so long as $d>2$ and $T/J\ll 1$.

The above calculation demonstrates that fluctuations are small, even though $\kappa_T=0$, in the simplest case when $\kappa_L=\infty$. The only requirement is that the coefficient J/T of the gradient-squared term be large.

Let us now include longitudinal fluctuations via a spin weighting term W:

$$\overline{H}_W=-\frac{J}{T}\sum_{\langle ij\rangle}|\mathbf{s}_i-\mathbf{s}_j|^2-\frac{1}{\delta T}\sum_i W(|\mathbf{s}_i|^2-1) . \tag{B4}$$

We assume $W'(0)=0$ and $\frac{1}{2}W''(0)=1$. We will mainly be interested in the case $W(x)=x^2$. Apparently we recover the case of fixed spins when $\delta\to 0$. It seems clear then that we may treat longitudinal fluctuations in the quadratic approximation around the minimum at $|\mathbf{s}_i|=1$ so long as $\delta T\ll 1$.

Consider then the us^4 model:

$$\overline{H}_4=-\frac{R_0^2}{a^2}\sum_{\langle i,j\rangle}|\mathbf{s}_i-\mathbf{s}_j|^2-\sum_i\left[\frac{1}{2}r|\mathbf{s}_i|^2+u|\mathbf{s}_i|^4\right] . \tag{B5}$$

By rescaling the spin via

$$\tilde{\mathbf{s}}_i=(4u/|r|)^{1/2}\mathbf{s}_i , \tag{B6}$$

which serves only to add a constant to the free energy, we find (for $r<0$)

$$\overline{H}_4=-\frac{R_0^2}{a^2}\frac{|r|}{4u}\sum_{\langle ij\rangle}|\tilde{\mathbf{s}}_i-\tilde{\mathbf{s}}_j|^2-\frac{r^2}{16u}\sum_i(|\tilde{\mathbf{s}}_i^2|-1)^2 . \tag{B7}$$

Comparing with (B4), we see that $w(x)=x^2$, $\delta T=16u/r^2$, and $T/J=(a^2/R_0^2)(4u/|r|)$. By the above arguments the linear spin-wave theory will be correct so long as $4a^2u/R_0^2|r|\ll 1$ and $16u/r^2\ll 1$. In particular, if $r=O(1)$ and $a/R_0=O(1)$, we require $u\ll 1$, which is obviously satisfied in our calculation so long as $\epsilon\ll 1$. Alternatively, if we assume $u=O(\epsilon)$, then we require $r,r^2\gg O(\epsilon)$, i.e., $r\gg O(\epsilon^{1/2})$.

APPENDIX C: SPIN-WAVE INTEGRALS

In this appendix we consider the integral

$$\int_q\frac{1}{q^2+r}=\frac{1}{2}K_dk_\Lambda^{d-2}\int_0^1\frac{w^{(d-2)/2}dw}{w+x} \equiv K_dk_\Lambda^{d-2}I_d(x) , \tag{C1}$$

where $x=rk_\Lambda^2$. Of particular interest is the nature of the singularity when $x\to 0$. We assume, as usual, $2<d\le 4$. We write

$$\begin{aligned}I_d(x)&=I_d(0)-x^{(d-2)/2}\int_0^\infty\frac{dw}{1+w}w^{(d-4)/2}+x\int_1^\infty\frac{dw}{x+w}w^{(d-4)/2}\\&=\frac{2}{d-2}-B\left[\frac{d-2}{2},\frac{4-d}{2}\right]x^{(d-2)/2}+\frac{2}{4-d}x-x^2\int_1^\infty\frac{dw}{x+w}w^{(d-6)/2} .\end{aligned} \tag{C2}$$

The last term now has a well-defined Taylor expansion around $x=0$ for all $d<6$. $B(x,y)=\Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the β function. This exhibits the exact $x\to 0$ nonanalyticity:

$$I_d^{\mathrm{sing}}(x)=-\frac{2x}{\epsilon}\left[\frac{\epsilon\pi/2}{\sin(\epsilon\pi/2)}x^{-\epsilon/2}-1\right] , \tag{C3}$$

which is valid for $2<d<6$, and yields the correct $x\ln(x)$ behavior in $d=4$. For small ϵ we may evaluate the remaining terms *in* $d=4$ and approximate $(\epsilon\pi/2)/\sin(\epsilon\pi/2)\simeq 1$:

$$I_d(x)\approx 1-\frac{2}{\epsilon}(x^{-\epsilon/2}-1)-x\ln(1+x),\quad \epsilon\ll 1 . \tag{C4}$$

Furthermore when $x=O(1)$ the singular term may be simplified to yield

$$I_d(x) \approx 1 + x\ln(x) - x\ln(1+x), \quad \epsilon \ll 1, \quad x = O(1) \ . \quad \text{(C5)}$$

Free-energy integrals involve the function

$$\int_q \ln(q^2+r) \approx \frac{K_d}{d} k_\Lambda^d \ln(k_\Lambda^2) + \frac{1}{2} K_d k_\Lambda^d \tilde{I}_d(x) \ , \quad \text{(C6)}$$

where

$$\tilde{I}_d(x) = \int_0^1 w^{(d-2)/2} \ln(w+x)\, dw \ . \quad \text{(C7)}$$

Obviously, $\tilde{I}'_d(x) = I_d(x)$. Using $\tilde{I}_d(0) = -4/d^2$, we may therefore simply integrate (C2) with respect to x to find $\tilde{I}_d(x)$. However, a simpler method is to integrate (C7) by parts to obtain

$$\tilde{I}_d(x) = \frac{2}{d}\ln(1+x) - \frac{2}{d} I_{d+2}(x) \ , \quad \text{(C8)}$$

which yields

$$\tilde{I}_d(x) = \frac{2}{d}\ln(1+x) - \frac{4}{d^2} + \frac{4}{d(d-2)} x - \frac{2}{d} x^3 \int_1^\infty \frac{dw}{x+w} w^{(d-6)/2} + \tilde{I}_d^{\text{sing}}(x) \ , \quad \text{(C9)}$$

where

$$\tilde{I}_d^{\text{sing}}(x) = -\frac{4}{\epsilon d} x^2 \left[\frac{\epsilon\pi/2}{\sin(\epsilon\pi/2)} x^{-\epsilon/2} - 1 \right] . \quad \text{(C10)}$$

The formulas analogous to (C4) and (C5) are

$$\begin{aligned} \tilde{I}_d(x) &\approx \frac{1}{2}(1-x^2)\ln(1+x) - \frac{1}{4} + \frac{1}{2}x - \frac{2}{\epsilon} x^2 (x^{-\epsilon/2} - 1), \quad \epsilon \ll 1 \ , \\ &\approx \tfrac{1}{2}(1-x^2)\ln(1+x) - \tfrac{1}{4} + \tfrac{1}{2}x + x^2 \ln(x), \quad \epsilon \ll 1, \quad x = O(1) \ . \end{aligned} \quad \text{(C11)}$$

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