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Transport phenomena near the Mott transition

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We consider the transport properties of a strongly (antiferromagnetically) correlated electron system in the temperature regime where the Fermi-liquid coherence ceases to exist. We find that the resistivity is linear in temperature, the thermal conductivity is almost temperature independent obeying approximately the Wiedemann-Franz law, while the Hall coefficient acquires a temperature dependence. The sign of the thermopower and Hall coefficient are hole-like. We calculate the residual resistivity caused by a random potential using the slave-boson technique. The disorder changes the slope of the temperature-dependent resistivity, but the Fermi surface remains relatively sharp.

I. INTRODUCTION

The thermodynamic and transport properties of strongly correlated systems have received renewed interest in connection with the anomalous physical properties of the transition metal oxides. The t-J model is one of the simplest Hamiltonians used to describe these systems. It is defined by the Hamiltonian:

$$H = \sum_{i,j,\alpha} (-t_{ij} c_{i\alpha}^{+} c_{j\alpha} + J_{ij} S_i S_j) \tag{1}$$

subject to the single occupancy constraint $c_{i\alpha}^{+} c_{i\alpha} \leq 1$. This inequality constraint can be converted into a holonomic constraint

$$f_{i\alpha}^{+} f_{i\alpha} + b_i^{+} b_i = 1 \tag{2}$$

by introducing a Bose field b_i which keeps track of the empty sites, and a Fermi field $f_{i\alpha}$, which carries the spin quantum number,

$$c_{i\sigma}^{+} = f_{i\sigma}^{+} b_i \ . \tag{3}$$

The description of the nature of the ground state, the elementary excitations, and the low-temperature thermodynamics of this model is a long-standing problem going back as far as Pomeranchuk[1] and Landau, who discussed the thermal transport in the insulating limit of (1) in terms of fermionic excitations.

It is becoming increasingly clear that at half-filling the model is insulating and exhibits some form of magnetic long-range order. Very far away from half-filling the low-energy physics is described by Fermi-liquid theory. The transition between these two regimes is up to now an unsolved problem. In this paper we would like to study the transport properties of a phase, in which the Fermi-liquid coherence is lost above some characteristic temperature, $T_{\rm coh}$, which is much lower than the spin correlation energy J, which locks the spins into singlets. We cannot prove rigorously that this situation occurs in some region of parameter space of (1). However, assuming that this situation is realized, it is possible to derive an effective Lagrangian describing the low-energy physics of this phase, and one can study in detail the thermodynamics and the transport properties. This study is the main goal of this paper.

The main ideas of the derivation of the effective Lagrangian were introduced in the resonating valence bond theory.[2,3] For earlier references to the study of transport using this technique see Refs. 4–6. In this approach one introduces auxiliary fields which describe the fluctuations of the bond variables $\Delta_{ij} \approx c_i^{+} c_j$ to decouple the exchange term in (1). If the amplitude fluctuations of these fields around a value $\Delta_{ij} = \Delta$ uniform in space is small, the effective Hamiltonian which describes the low-energy physics of the problem is given by

$$H = -\frac{1}{2m_f} f^{+} \left[\nabla - i \left[\frac{e}{c\hbar} A + a \right] \right]^2 f - \frac{1}{2m_b} b^{+} (\nabla - ia)^2 b + \lambda (b^{+} b + f^{+} f - 1) \ , \tag{4}$$

where a is the continuum limit of the phase of the bond variable $\Delta_{i,j} = \Delta \exp(i a_{ij})$. A is the external electromagnetic field and e is the charge of the electron.

For a neutral system $e=0$ and the coupling to the external vector potential vanishes. The coupling to the internal gauge field a, however, is not changed since it describes the effects of the constraint (i.e., the infinite U Hubbard repulsion). λ is a Lagrange multiplier that plays the role of a longitudinal scalar potential. It acquires an expectation value and fluctuations around this expectation value are short ranged. The effective Hamiltonian (4) describes the subsystem of fermions and bosons interacting with a gauge field. This description was introduced by Baskaran, Zou, and Anderson.[7] The interaction with the gauge field is very important. It enforces the constraint $j_B + j_F = 0$ and expresses the physical resistivity as the sum of the resistivities of the Fermi and Bose subsystem:[8]

$$R = R_b + R_f \ . \tag{5}$$

The transverse gauge field describes an overdamped collective mode which strongly scatters the Bose and Fermi particles. The singular effect of a diffusive spectrum $\omega = ik^3$ was discovered by Reizer[9] and applied to the strong correlation problem by Lee.[10]

The assumptions leading to the effective Lagrangian (4) are not likely to be valid very close to half-filling, because of the onset of dimerization, spontaneous formation of flux, or other forms of magnetic long-range order. Dimerization occurs in a large-N limit of (1) close to half-filling,[11,12] in a large-S generalization of (1),[13] and in a spin-one-half frustrated antiferromagnet.[14] At half-filling, the uniform bond variable is also unstable against a phase modulation which indicates the spontaneous formation of flux.[11,15] However, a small concentration of holes destroys the magnetic long-range order. This happens in the quantum dimer model,[16] in a large-N limit of (1),[17] and in the semiclassical treatment of the doped quantum antiferromagnet.[18]

The effective Lagrangian (4) allows a very transparent description of the transition between the Fermi liquid and a non-Fermi-liquid regime. At zero temperature the bosons are condensed and (1) describes a Fermi liquid with strong antiferromagnetic correlations, close to a Mott transition.[19] There is a temperature scale $T_{\rm coh}$ below which the system displays Fermi-liquid behavior. $T_{\rm coh}$ corresponds to the Bose condensation transition temperature in the slave-boson approach. If $T_{\rm coh} \ll J$ there will be an intermediate temperature range $T_{\rm coh} \ll T \ll J$ where the spins are partially frozen out but there is no Fermi-liquid coherence. In this case we will show that the transport is dominated by the holes.

In this paper we will study the transport properties in this regime. We derive a quantum Boltzmann equation from the effective Lagrangian (4), and use it to calculate the residual resistivity, the optical conductivity, the thermal resistivity, and the thermopower. We conclude with a critical discussion of the applicability of this idea to the description of experimental results in the transition-metal oxide systems.

We note that same form of the effective action (but with different parameters) appears also in the slave-fermion[20] Schwinger-boson approach to the strongly correlated electron system if we assume[21] a phase where the Schwinger bosons are neither condensed nor pair condensed while the slave fermion forms a degenerate Fermi liquid. Physically it would correspond to a phase with free local magnetic moments but coherent charge transport. This physical picture is very different from the one that we advocate for the transition-metal oxides. However, the results of this paper, which are based on the effective action (4), could be used in this context as well.

II. THE KINETIC EQUATION

The kinetic equation is derived using self-consistent perturbation theory for the Keldysch Green functions of the fermions, the bosons, and the gauge field:

$$\begin{aligned} G^f_{12} &= -i\langle f_1 f_2^+ - f_2^+ f_1\rangle \ , \\ G^b_{12} &= -i\langle b_1 b_2^+ + b_2^+ b_1\rangle \ , \\ D_{12} &= -i\langle a_1 a_2^+ + a_2^+ a_1\rangle \ , \end{aligned} \tag{6}$$

and the retarded Green functions:

$$\begin{aligned} G^f_{R\,12} &= -i\langle f_1 f_2^+ + f_2^+ f_1\rangle \Theta(t_1 - t_2) \ , \\ R^b_{R\,12} &= -i\langle b_1 b_2^+ - b_2^+ b_1\rangle \Theta(t_1 - t_2) \ , \\ D_{R\,12} &= -i\langle a_1 a_2^+ - a_2^+ a_1\rangle \Theta(t_1 - t_2) \ . \end{aligned} \tag{7}$$

The quantum distribution functions $S^f(\varepsilon,p), S^b(\varepsilon,p), S^a(\varepsilon,p)$ are defined by

$$G^u(\varepsilon,p) = S^u(\varepsilon,p)[G^u_R(\varepsilon,p) - G^u_A(\varepsilon,p)] \ , \tag{8}$$

where u stands for f, b, or a. When the imaginary part of the retarded Green function is sharply peaked the transport theory can be formulated entirely in terms of the semiclassical distribution function $n(p)$: $S^f(p) = 1 - 2n^f(p)$, $S^b(p) = 1 + 2n^b(p)$. In this problem we will be able to perform this reduction from distribution functions in momenta and energy to distribution functions in the momenta alone for the fermions and the bosons but not for the gauge field. Calculating the self-energy diagrams in Fig. 1 one obtains a system of equations for the distribution functions $S^f(p), S^b(p)$ of the fermion and boson fields:

$$\begin{aligned} &\left[\frac{d}{dt} + E'\frac{d}{dp} + v\frac{d}{dr}\right] S(p) = I(p) \ , \\ &I = \int dk \, {\rm Im} D_A(\xi_p - \xi_{p+k}, k) \left[v^2 - \frac{(vk)^2}{k^2}\right] \\ &\qquad \times [1 - S^a(\xi_p - \xi_{p+k}, k)(S_p - S_{p+k}) - S_p S_{p+k}] \ . \end{aligned} \tag{9}$$

The equations for the Bose and the Fermi functions $S^f(p), S^b(p)$ have the same form. Equation (9) describes the scattering of the electrons and the bosons by the gauge field. The electrical field E' acting on the fermions and bosons is different: $E'_f = E + e, E'_b = e$ where E is external electromagnetic field and e is the field produced by the gauge field: $v = (p + k/2)/m_u$.

In this derivation we neglected the higher-order corrections to the self-energy diagrams, Fig. 2, such as renormalization of the particle-hole vertex or corrections to the Green function inside the self-energy diagram. These corrections are nonsingular for both Bose and Fermi fields; for the Bose field they contain parameter m_f/m_b and become very small in the limit $m_b \gg m_f$, whereas corrections to the Fermi functions are of the order of unity. Thus, the results obtained for the Bose distribution function are quantitative, whereas the fermion properties can be only estimated. However, as we show below, almost all physical properties are governed by the Bose

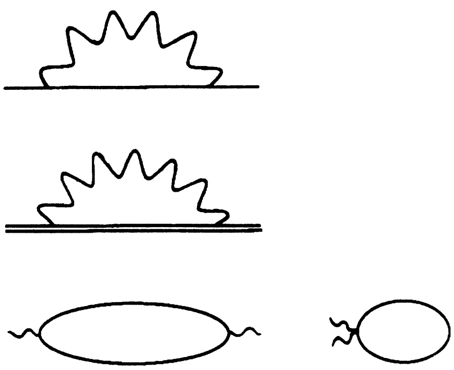

FIG. 1. Self-energy graphs for inelastic scattering included in the kinetic equations. The straight, double, and wavy lines denote the Fermi-, Bose-, and gauge-field propagator, respectively.

subsystem. The gauge field acquires a diffusive spectrum, so that its quantum distribution function cannot be replaced by a function of p only. The equation for it follows from the Dyson equation, Fig. 1. The effective Lagrangian (4) does not contain the usual $F_{\mu\nu}^2$ terms, because the field a has no intrinsic dynamics. The photon Green function and the distribution function can be expressed directly through the fermion and boson Green functions without solving the kinetic equation:

$$D_R^{-1}=\Pi_R^f+\Pi_R^b;\quad S^a=\frac{\Pi_c^f+\Pi_c^b}{D_R^{-1}-D_A^{-1}}\ . \tag{10}$$

Here Π denotes the polarization matrix of the photon which in one-loop approximation becomes

$$\Pi_{R\mu\eta}(\omega,k)=\int d\varepsilon\int dp\ v_\mu v_\eta[G^R(\varepsilon,p)G(\varepsilon+\omega,p+k)+G^A(\varepsilon+\omega,p+k)G(\varepsilon,p)]\ , \tag{11}$$

$$\Pi_{C\mu\eta}(\omega,k)=\int d\varepsilon\int dp\ v_\mu v_\eta[2G^R(\varepsilon,p)G^A(\varepsilon+\omega,p+k)+G(\varepsilon+\omega,p+k)G(\varepsilon,p)]\ .$$

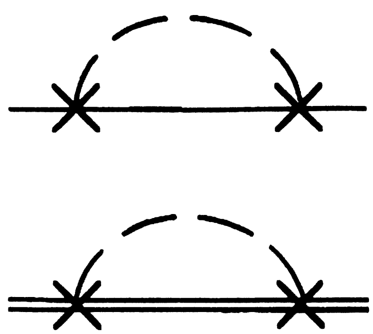

FIG. 2. Self-energy graphs for impurity scattering.

Since the longitudinal field is screened, only the transverse part of the gauge field is responsible for the long-range interactions between fermions and bosons. Thus we keep only the transverse part of the polarization Π. If both Fermi and Bose subsystems are at equilibrium the gauge field distribution function becomes $S^a(\varepsilon,p)=\coth(\varepsilon/2T)$. Equation (10) expresses the fact that the deviations away from equilibrium in the Bose or Fermi subsystems drive the photons away from equilibrium. As we show below, the contribution of the Fermi field to the polarization (10) dominates over the Bose one, therefore photons tend to be dragged by the fermion current. This is analogous to the phonon drag effect in normal metals. There is a fundamental difference between these two situations. In normal metals the phonon drag effect is very important and dominates the low-temperature transport properties of metals with closed Fermi surfaces. In our case the bosons govern the physical properties of the system and the drag effect is not essential.

The kinetic equations (9)–(11) describe a composite system of low-density bosons and high-density fermions, driven by external fields E_f, E_b, in a media of overdamped photons. The total currents of fermions and bosons obey the constraint $j_f+j_b=0$ which can be formally derived considering the stability condition $\delta L/\delta a=0$ where δa are macroscopic variations of the gauge potential. This constraint has a simple meaning: only electrons can move eventually in a system. By definition the electron operator is a product of Fermi and anti-Bose operators, thus the motion of real electron implies the motion of fermion in the same direction and boson in the reverse: $j_{\rm el}=j_f=-j_b$.

Thus, in a state with electric current, fermions and bosons drift in opposite directions. Since the density of bosons is low the average drift velocity of the fermion gas is less than the drift velocity of the boson gas. Thus, in the first approximation we can neglect the effect of the fermion drift on the properties of the gauge field. Since the properties of the gauge field are governed mainly by the fermion system, in the leading approximation we can regard photons as being in equilibrium and use Eq. (11) to estimate the effects of the drag.

We evaluate now the photon Green function in the equilibrium. The photon transverse polarization operator at small frequencies and momenta has a general form:

$$\Pi_R(q,\omega)=\chi q^2-i\Gamma(q)\omega\ ,\qquad \chi=\chi_f+\chi_b,\quad \Gamma(q)=\Gamma_f(q)+\Gamma_b(q)\ , \tag{12}$$

where χ_f, χ_b are diamagnetic susceptibilities of the Fermi and Bose systems, $\Gamma_f(q)$, $\Gamma_b(q)$ are their damping coefficients. The susceptibilities of the Bose and Fermi systems should be viewed as parameters that determine the low-energy physical properties. They can be estimated by their values in noninteracting systems:

$$\chi_f=\frac{1}{24\pi^2\nu_f(0)},\quad \chi_b=\frac{1}{48\pi^2\nu_b(0)}n_b(0)\ , \tag{13}$$

where $\nu_b(\varepsilon)$ is the density of states of bosons and $\nu_f(\varepsilon)$ is the density of states of a single species of fermions. This estimate neglects Fermi-liquid corrections which are present because of the interaction with the longitudinal and transverse gauge fields. At large temperatures the Bose distribution function $n_b(0)$ is small: $n_b(0)\sim\delta/(m_b T)$. At lower temperatures $T\leq\delta m_b^{-1}$ the interaction between bosons becomes important and we can no longer estimate $n_b(0)$ using the free-boson approximation. We do not have a theory describing this regime. However, we will assume that Bose condensation does not take place and therefore $n_b(0)\leq 1$ in all temperature ranges $T_{\rm coh}\leq T\leq m_b^{-1}$. If $m_b \gg m_f$, χ_b is always less than χ_f, whereas if $m_b\sim m_f$, χ_b becomes of the order of χ_f only at a temperature $T\sim\delta m_b^{-1}$. As a first approximation we will neglect a weak temperature dependence of the polarization operator (12) caused by χ_b. However, the temperature dependence of χ_b becomes important in the discussion of the Hall effect (Sec. V).

The susceptibility of the Fermi system is governed by states which are very far from the Fermi surface. It is weakly renormalized by the interaction with the gauge field and is almost temperature independent. This was checked directly by evaluation of the next order corrections to the bubble diagram of Fig. 1.

Since the main contribution to the damping coefficients comes from photons with $q\ll 1$ and $\omega/(vq)\ll 1$, $\Gamma_f(q),\Gamma_b(q)$ depend strongly on q and on the presence of scattering in the system. In the presence of scattering $\Gamma_{f,b}$ are approximately constant for $\tau_{f,b}$, $v_{f,b}q\ll 1$. $\tau_{f,b}$ is the fermion and boson transport scattering time and is discussed further in Sec. III. These limiting values are proportional to the residual conductivities of the Bose and Fermi subsystem. As we show below, the conductivity of the Fermi system is larger, so that $\Gamma_f(q)\gg\Gamma_b(q)$ in this region. At larger momenta $(vq)\gg\tau_{f,b}^{-1}$ the damping coefficients Γ_f and Γ_b are inversely proportional to q, as a result of Landau damping. In this region

$$\Gamma_f(q)=\gamma p_f/\pi q, \quad \Gamma_b(q)\sim\delta^{1/2}/q \ , \tag{14}$$

so that $\Gamma_f\gg\Gamma_b$ in this region also. γ is a number equal to 1 to lowest order in the perturbation theory in the interaction with the gauge field. Evaluation of higher-order corrections to the bubble (Fig. 1) do not show any singular contributions. Therefore it does not acquire singular temperature or momentum dependence. Thus we conclude that the main contribution to the photon propagator $D(q,\omega)$ comes from the Fermi subsystem only.

To calculate transport coefficients we linearize the kinetic equation (9) describing the scattering of fermions and bosons by photons. We consider the Bose system first. Linearizing it around the equilibrium solution $S=S_0+S_1$ we get the collision operator

$$I(S_1^b)=-\int dk\,\frac{\Gamma(k)(\xi_p-\xi_{p+k})}{\Gamma^2(k)(\xi_p-\xi_{p+k})^2+\chi^2k^4}\left[v^2-\frac{(vk)^2}{k^2}\right]$$
$$\times\left[[S_1^b(p)-S_1^b(p+k)]\coth\left[\frac{\xi_p-\xi_{p+k}}{2T}\right]+S_1^b(p)\coth\left[\frac{\xi_{p+k}}{2T}\right]+S_1^b(p+k)\coth\left[\frac{\xi_p}{2T}\right]\right] . \tag{15}$$

The main contribution to this integral comes from the low-energy processes with

$$|\xi_p-\xi_{p+k}|\sim\chi k^2/\Gamma(k)\sim(m_bT)^{3/2}/m_f\ll T \ .$$

In this energy range we can keep only the first term in braces in the previous equation and replace $\coth(x)$ in it by x^{-1}. The typical distribution function changes on the energy scale of T, so that its variation as a function of angle θ_p becomes more important [for all transport properties $S_1(p)$ contains factor $\cos(\theta_p)$]. Thus we neglect its energy dependence and keep only its dependence on angle $S_1^b(p)=2\cos(\theta_p)\phi(p)$. In this approximation $S_1^b(p)$ becomes the eigenfunction of the collision operator:

$$I(\cos(\theta_p)\phi_b(p))=-\tau_{b\rm tr}^{-1}(p)\cos(\theta_p)\phi_b(p) \ , \tag{16}$$

$$\tau_{b\rm tr}^{-1}=2T\int dk\frac{\Gamma(k)}{\Gamma^2(k)(\xi_p-\xi_{p+k})^2+\chi^2k^4}\left[v^2-\frac{(vk)^2}{k^2}\right](1-\cos\theta_{p,p+k}) \ . \tag{17}$$

The form of the function $\phi(p)$ depends on the transport property considered. However, for any transport process solution (16) serves as a good starting approximation.

Now we consider fermions. Linearizing their collision integral we obtain

$$I(S_1^f)=\int dk\frac{\Gamma(k)(\xi_p-\xi_{p+k})}{\Gamma^2(k)(\xi_p-\xi_{p+k})^2+\chi^2k^4}\left[v^2-\frac{(vk)^2}{k^2}\right] ,$$
$$\times\left[[S_1^f(p)-S_1^f(p+k)]\coth\left[\frac{\xi_p-\xi_{p+k}}{2T}\right]+S_1^f(p)\tan\left[\frac{\xi_{p+k}}{2T}\right]+S_1^f(p+k)\tan\left[\frac{\xi_p}{2T}\right]\right] . \tag{18}$$

The main contribution to this integral comes from the energy transfer $|\xi_p - \xi_{p+k}| \sim T$ which coincides with the energy scale of their distribution function $S_1^f(p)$ peaking around the Fermi surface. Thus, in this case we cannot neglect the energy transfer in the scattering process. In contrast to the Bose scattering integral the momentum transferred in a typical collision is small: $k^2 \sim T\Gamma(k) \ll p_F^2$. We cannot guess the general form of the solution of (18) for the fermions. We can only estimate scattering rates entering different transport properties.

III. ELECTRICAL CONDUCTIVITY

In this section we discuss the electrical response of the combined Bose Fermi system. The zero-frequency electrical conductivity of the Bose system is calculated by linearizing the kinetic equation (9). Using the property (16) it follows that

$$S_1^B = -\tau_{btr} \mathbf{E}_B \cdot \nabla_p S_0^B \tag{19}$$

is a solution with τ_{btr} given by (17). The main contribution to this integral comes from the processes with small energy transfer, so that $|p+k| \approx |p|$. Choosing the variables $|p+k|$ and θ_{p+k} and neglecting a weak dependence of angular factors in (16) on $|p+k|$ we perform integration over $|p+k|$ and then over θ_{p+k}:

$$\tau_{btr0}^{-1} = \frac{T}{4m_b\chi} . \tag{20}$$

The result (20) does not depend on the damping constant and, thus, is not sensitive to the impurities and inelastic scattering which change the damping coefficient at low momenta. To estimate the corrections to (20) which are due to these mechanisms we insert the solution (19) into collision integral (16) and evaluate the corrections to the scattering rate due to the actual $\Gamma(q)$ dependence:

$$\tau_{btr}^{-1} = \tau_{btr0}^{-1}(1+\Delta) ,$$

$$\Delta \approx m_f \tau_f^{-1}, \quad T \ll \frac{m_f^2}{\tau_f^2 m_b} , \tag{21}$$

$$\Delta \approx m_f^2 \tau_f^{-2} \left[\frac{1}{m_B T} \right]^{1/2}, \quad T \gg \frac{m_f^2}{\tau_f^2 m_b} .$$

The fact that the inelastic lifetime of the bosons is proportional to the temperature was first pointed out by Nagaosa and Lee.[22] Equation (21) shows that this result is unsensitive to the modification of the propagator of the gauge field by impurities or inelastic effects. The presence of impurities only increases the scattering rate by an amount proportional to the concentration of impurities. The resistivity of the Fermi system is small $R_f \approx (Tm_f)^{4/3}$.[10] Using formula (5) we conclude that the physical resistivity is governed by the Bose subsystem:

$$R = \frac{T}{4\delta\chi}(1+\Delta) . \tag{22}$$

In three dimensions, the infrared singularities are not so severe and the estimate of (20) in the clean case gives

$$\tau_{btr}^{-1} \simeq T^{3/2}\sqrt{m_f} . \tag{23}$$

The physical conductivity is given by the Bose conductivity in the regime where the number of bosons is small.

We now consider large frequencies. In this case we cannot use the kinetic equation and we have to resort to an approximate evaluation of the Kubo formula. The expression for the Bose conductivity in terms of the exact Green functions and vertex functions of the system is

$$\sigma_b(\omega) = \int dp \int dx \left[\frac{n_b(x) - n_b(x+\omega)}{\omega} \right] v_p^2 \gamma_p^2(x, x+\omega) \mathrm{Im} G_A(x,p) \mathrm{Im} G_A(x+\omega, p) , \tag{24}$$

where we introduced a vertex function γ_p describing the renormalization of the vertex p by the interaction at short length scales. At high temperatures $T \gtrsim \delta/m_b$ this renormalization is not important: $\gamma_p \approx 1$. The Green functions are approximated by inserting a pure imaginary, temperature, momentum and frequency dependent self energy:

$$\mathrm{Im}\Sigma_b(\omega,p^2,T) = \int dq \, \frac{\Gamma(q)(\omega_p - \xi_{p+q})}{\Gamma^2(q)(\omega_p - \xi_{p+q})^2 + \chi^2 q^4} \left[v^2 - \frac{(vq)^2}{q^2} \right] \left[\coth\left[\frac{\xi_{p+q}}{2T}\right] + \coth\left[\frac{\omega - \xi_{p+q}}{2T}\right] \right] . \tag{25}$$

We now estimate the frequency dependence of the Bose self-energy in various limits. When $\omega = \xi_p \gg T$,

$$\Sigma_b(\xi_p, p^2) \simeq \sum_q \frac{q_\perp^4 \Gamma_q \omega}{q^2[(\omega\Gamma_q)^2 + q_\perp^4]} . \tag{26}$$

The dominant contribution comes from large momenta and gives, with $\varepsilon = \omega = \xi_p$ and $q_\perp^2 = q^2 - (\mathbf{q}\cdot\mathbf{p})/p^2$,

$$\Sigma_b(\varepsilon) \simeq \varepsilon^{3/2} m_b^{1/2} \tag{27}$$

except for the very dirty limit $m_f \tau^{-1} > \delta^{1/2}$ when (27) crosses over to

$$\Sigma_b(\varepsilon) \simeq (\tau^{-1} m_f)\varepsilon . \tag{28}$$

The crossover frequency above which (27) holds is $\omega_c^B = T^{2/3} m_b^{-1/3}$. It is also useful to have some estimates of limiting values of the off-shell self-energy:

$$\mathrm{Im}\Sigma_b(\varepsilon,p^2) \simeq p^2 \frac{\sqrt{\varepsilon}}{\sqrt{m_F}}, \quad \varepsilon \gg p^2 \tag{29}$$

$$\mathrm{Im}\Sigma_b(\varepsilon,p^2) \simeq \frac{\varepsilon^3}{p^5 \chi^2}, \quad \varepsilon \ll p^2 . \tag{30}$$

In the limit of large frequencies the direct interaction of the bosons with the spinons via the longitudinal part of the gauge field becomes important; it results in the off-shell self-energy

$$\mathrm{Im}\Sigma_{bl}(\omega,p^2)=\begin{cases}\dfrac{\omega^2 m_f}{p}, & \omega \ll p^2/m_f \quad (31)\\ \omega^{3/2}m^{1/2}f, & \omega \gg p^2/m_f\,. \quad (32)\end{cases}$$

(32) represents the physical processes in which a boson excites a fermionic particle hole pair when it propagates.

With this information we can estimate the Bose conductivity using (24). In Eq. (24) $n_b(x+\omega)$ is small unless $x \le -\omega$ but at these energies $\mathrm{Im}G_A(p,x)$ becomes very small, thus $n_B(x+\omega)$ yields negligible contribution to the integral. The main contribution to the integral over x comes from $0 \le x \le \omega$. In this frequency range $\mathrm{Im}G_A(p,x)\mathrm{Im}G_A(p,x+\omega)$ considered as a function of ω is peaked around $\xi_p \approx x$ and $\xi_p \approx x+\omega$. We perform integration over p in these regions separately and get

$$\sigma_b(\omega)=v_B m_b^{-1}\int dx \frac{n_B(x)}{\omega^2}\left[\mathrm{Im}\Sigma^A(x,\sqrt{2m_b\omega})+\mathrm{Im}\Sigma^A(\omega,\sqrt{2m_b(x+\mu_b)})\frac{(x+\mu_b)}{\omega m_b}\right]. \quad (33)$$

Equation (33) expresses the Bose high-frequency conductivity in terms of the off-shell self-energy in different limits $p^2/(m_b\varepsilon) \gg 1$ and $p^2/(m_b\varepsilon) \ll 1$. At very large frequencies the second limit dominates $\mathrm{Im}\Sigma^A(\omega,x) \gg \omega\,\mathrm{Im}\Sigma^A(x,\omega)$.

Combining (33) and (32) we conclude that the Bose contribution to the optical conductivity decays slower than the Drude model with a frequency-independent relaxation rate:

$$\sigma_B(\omega)=\delta T^2\omega^{-1/2}\,. \quad (34)$$

We now turn to the Fermi contribution to the optical conductivity:

$$\sigma_F(\omega)=\int dp\int dx\left[\frac{n_F(x)-n_F(x+\omega)}{\omega}\right]v_p^2\gamma_p^2(x,x+\omega)\mathrm{Im}G_A(p,x)\mathrm{Im}G_A(p,x+\omega)\,. \quad (35)$$

In this case (35) is simply related to the Fermi transport time τ_F calculated using the kinetic equation:

$$\tau_f^{-1}(\xi_p,T)=\int dq\frac{(\xi_{p+q}-\xi_p)\Gamma(q)}{\{[(\xi_{p+q}-\xi_p)\Gamma(q)]^2+q_\perp^4\}}\frac{q_\perp^4}{q^2}\left[\coth\left[\frac{\xi_p-\xi_{p+q}}{2T}\right]-\tanh\left[\frac{\xi_{p+q}}{2T}\right]\right] \quad (36)$$

At finite temperatures and zero frequency it gives the known[10] result

$$\frac{1}{\tau_F}=T^{4/3}m_F^{1/3}\,. \quad (37)$$

At zero temperature and low but finite frequency we find

$$\tau_f^{-1}\simeq\begin{cases}\omega^{3/2}\tau^{1/2}, & \omega \ll \omega_c^F \quad (38)\\ \omega^{4/3}m_f^{1/3}, & \omega \gg \omega_c^F\,. \quad (39)\end{cases}$$

The crossover frequency between the two regimes is given by $\omega_c^F=m_f^2/\tau^3$.

The Fermi contribution to the high-frequency optical conductivity is given by

$$\sigma_F(\omega,T)\simeq\frac{1}{m_f\omega^2\tau_F(\omega,T)}\,. \quad (40)$$

Combining (34) and (40) we conclude that at high frequencies the Bose conductivity dominates the optical conductivity. The conductivity falls off slower than the prediction of a standard Drude theory. The processes in which a hole scatters off-spin excitations to recombine into a bare electron are responsible for the slow falloff of $\sigma(\omega)$.

IV. THERMAL PROPERTIES

A temperature gradient and an electric field induce an electric current J and a heat current U. For small disturbances the response is proportional to the fields and defines the thermoelectric coefficients

$$J=e^2K_0(E-\nabla\mu)+K_1\left[\frac{-\nabla T}{T}\right],$$
$$U=eK_1(E-\nabla\mu)+K_2\left[\frac{-\nabla T}{T}\right]. \quad (41)$$

The thermal conductivity is measured when the electric current is zero and it is given by

$$\kappa=\frac{1}{T}\left[K_2-\frac{K_1^2}{K_0}\right]. \quad (42)$$

The electromotive force induced by a thermal gradient when the electric current is zero defines the thermopower coefficient

$$Q=\frac{K_0^{-1}K_1}{eT}\,. \quad (43)$$

We calculate K^1 and K^2 by considering the response of the Bose and Fermi system to electric fields E_B, E_F, respectively, and a thermal gradient ∇T. E_B and E_F are the effective fields acting on the bosons and fermions, respectively; their difference gives the total field

$$E_F - E_B = Ee \tag{44}$$

while they are adjusted to obey the constraint

$$J = J_F = -J_B \tag{45}$$

The analog of Eq. (41) for the Fermi and Bose subsystem, the index α stands for fermions (F) or bosons (B).

$$J_\alpha = K_{0\alpha}(E_\alpha - \nabla\mu_\alpha) + K_{1\alpha}\left[\frac{-\nabla T}{T}\right],$$
$$U_\alpha = K_{1\alpha}(E_\alpha - \nabla\mu_\alpha) + K_{2\alpha}\left[\frac{-\nabla T}{T}\right]. \tag{46}$$

The chemical potential of the electrons is given by the difference of the Fermi and the Bose chemical potentials

$$\mu = \mu_F - \mu_B \tag{47}$$

which allows us to solve (44) and (45) for the effective fields $\tilde{E}_\alpha = E_\alpha - \nabla\mu_\alpha$ acting on the fermion and the boson system

$$\tilde{E}_F = \frac{K_B^0}{K_F^0 + K_B^0}\tilde{E} + \frac{K_F^1 + K_B^1}{K_F^0 + K_B^0}\frac{\nabla T}{T},$$
$$\tilde{E}_B = \frac{K_F^0}{K_F^0 + K_B^0}\tilde{E} + \frac{K_F^1 + K_B^1}{K_F^0 + K_B^0}\frac{\nabla T}{T}. \tag{48}$$

The electric current J can be expressed either in terms of the Bose or the Fermi current J_B, J_F.

$$J = \frac{e^2 K_f^0 K_b^0}{K_f^0 + K_b^0}E + \frac{e(K_f^0 K_b^1 - K_f^1 K_b^0)}{K_f^0 + K_b^0}\frac{\nabla T}{T}. \tag{49}$$

The first term is nothing but the conductivity and just gives the formula[8] (5):

$$K^0 = \frac{K_F^0 K_B^0}{K_F^0 + K_B^0}. \tag{50}$$

The second term is K_1 of the electron system

$$K^1 = \frac{K_F^1 K_B^0 - K_F^0 K_B^1}{K_F^0 + K_B^0}. \tag{51}$$

This is a new result. The heat current of the electrons is given by

$$U = U_F + U_B. \tag{52}$$

We insert (48) in (52) and compare with Eq. (41) to identify

$$K^2 = K_f^2 + K_b^2 - \frac{(K_f^1 + K_b^1)^2}{K_f^0 + K_b^0}, \tag{53}$$

which expresses the thermal conductivity of the electrons in terms of the thermoelectric coefficients of the Bose and the Fermi subsystem.

We now estimate these coefficients when the scattering is dominated by inelastic scattering off the gauge field. To evaluate the effect of this scattering we again use the linearized quantum kinetic equation. We consider the processes of energy and momentum relaxation separately.

We start with the Bose system. In this case the collision operator has the set of eigenfunctions of the general form $S_1(p) = 2\cos(\theta_p)\phi(p)$ with eigenvalue τ_{btr}^{-1} which correspond to the processes of momentum relaxation. The rate of the energy relaxation is described by the eigenvalues of the collision operator on the functions of the general form $S_1(p) = 2\phi(p)$. To estimate these eigenvalues we insert in the collision operator (16) the trial $\phi(p)$, evaluate the result, and compare it with $\phi(p)$. We get

$$\tau_{ben}^{-1} \approx \frac{T}{\Gamma(\sqrt{m_B T})}. \tag{54}$$

The change of the distribution function resulting from the temperature gradient has a general form $\cos(\theta_p)\phi(p)$. Comparing (54) with the transport rate (20) we see that the energy relaxation is much slower than the relaxation of the momentum, and can be neglected for all fluctuations of this form. The physical reason for this is that the transport rate is enhanced by the scattering off an anomalously large number of low-energy photons. This enhancement is more suppressed in the calculation of the energy relaxation rate because the energy relaxation has an additional factor of $\omega \sim q_\perp^3 \ll q_\perp^2$. Thus, any such distribution function is the approximate eigenfunction of the collision integral with eigenvalue τ_{btr}^{-1}, and the kinetic equation for bosons acquires a simple form:

$$v_k^b\left[(E^b - \nabla\mu_b) - \xi_k^b\frac{\nabla T}{T}\right]\frac{\partial n_b}{\partial \xi_k} = -\tau_{btr}^{-1}\delta n_b. \tag{55}$$

In order to evaluate the thermal or electrical current induced by δn_b in (55) we need the form of the Bose distribution function. We use the distribution function n_b of free bosons and get

$$K_b^2 = \tau_{btr} v_b T^2 \beta m_b{}^{-1}[3\Phi_3(\beta) - 4\Phi_2(\beta)\ln\beta + \Phi_1(\beta)\ln^2\beta], \tag{56}$$

$$K_b^1 = \tau_{btr} v_b T^2 \beta m_b{}^{-1}[2\Phi_2(\beta) - \Phi_1(\beta)\ln\beta],$$

where $\beta = \exp(\mu/T)$ and

$$\Phi_s(\beta) = \int_0^\infty dx\frac{x^{s-1}}{\exp x - \beta}. \tag{57}$$

At large temperatures $v_b T \gg \delta$ the result (56) simplifies:

$$K_b^2 = \tau_{btr}\delta T m_b{}^{-1}[6 - \ln(\delta/v_b T) + \ln^2(\delta/v_b T)],$$
$$K_b^1 = \tau_{btr}\delta T m_b{}^{-1}[2 - \ln(\delta/v_b T)]. \tag{58}$$

Now we consider the Fermi system. We estimate the eigenvalues of the collision operator which describe the relaxation of the energy and the electrical current.

The deviations for equilibrium in the Fermi system induced by an electric field are given by $\delta f_2^T \simeq (-\partial f/\partial e)v_K^F\cos\theta$ while a thermal gradient produces a distribution $\delta f_2^T \simeq (-\partial f/\partial e)v_K^f\cos\theta(\xi_k/T)$. Inserting these trial eigenfunctions into collision integral (18) we estimate the corresponding eigenvalues:

$$\tau_{f\mathrm{tr}}^{-1} \simeq T^{4/3} m_f^{1/3}, \quad \tau_{f\mathrm{en}}^{-1} \simeq T^{2/3} m_f^{-1/3} . \tag{59}$$

In the Fermi system energy relaxation proceeds much faster than current relaxation because the main contribution to the relaxation comes from processes with very small momentum transfer which dissipate energy very efficiently but cannot dissipate momentum. This allows us to estimate the thermal conductivity of the Fermi system

$$K_f^2 \simeq T^{4/3} m_f^{-2/3} \tag{60}$$

The thermopower of the Fermi system is small $K_f^1 \approx m_f^{-1}$ and its estimation is complicated by drag effects. Since its contribution to the thermoelectric response (51) is small compared with the Bose subsystem (56) it will not be estimated here.

Combining (51), (53), (58), and (60) we finally get thermoconductivity and thermopower of the total system at large temperatures:

$$\kappa \approx 8\chi\delta + T^{1/3} m_f^{-2/3} , \tag{61}$$

$$S \approx 2 + \ln(\nu_b T/\delta) . \tag{62}$$

For temperatures $Tm_f \ll \delta^3$ the Wiedeman-Franz law is obeyed, with the Wiedemann-Franz ratio being 2, while for $Tm_f > \delta^3$ the heat transport is dominated by the spin degrees of freedom and is the same as in the insulator. The thermopower S is anomalously large compared to the Fermi-liquid result.

V. MAGNETIC SUSCEPTIBILITY AND HALL COEFFICIENT

An external magnetic field B induces screening currents in the Fermi and Bose systems. Therefore the Fermi and Bose systems feel effective magnetic fields B_F and B_B. They can be determined by minimizing the free energy

$$F = \tfrac{1}{2}(\chi_B B_B^2 + \chi_F B_F^2) \tag{63}$$

subject to the constraint $B_F - B_B = \tilde{B} \equiv (e/c\hbar)B$. The result is

$$B_F = \frac{\chi_B}{\chi_f + \chi_B} \tilde{B} , \tag{64}$$

$$B_b = \frac{-\chi_F}{\chi_F + \chi_B} \tilde{B} . \tag{65}$$

The minimized free energy is

$$F = \frac{\chi_F \chi_B}{2(\chi_F + \chi_B)} \tilde{B}^2 , \tag{66}$$

which identifies the total diamagnetic susceptibility of the system as

$$\chi_d = \frac{\chi_B \chi_F}{\chi_B + \chi_F} \left[\frac{e}{c\hbar} \right]^2 . \tag{67}$$

In the following we will drop the factor $e/c\hbar$ in the susceptibility. It will be reinstated in the discussion of the numerical values of the effective Lagrangian parameters in Sec. VII. At high temperatures the Bose susceptibility is small and decreases with temperature: $\chi_b \sim \delta/(Tm_b^2)$. To describe it more quantitatively we introduce the phenomenological parameter T^* by $\chi_b = (T^*/T)\chi_f$ at $T \gg T_{\mathrm{coh}}, T^*$. At low temperatures $T \le T_{\mathrm{coh}}, T^*$ the temperature dependence of χ_b ceases. If $T^* \ge T_{\mathrm{coh}}$ the Bose susceptibility becomes comparable to the Fermi one in the temperature range $T_{\mathrm{coh}} \le T \le T^*$. We will use T^* instead of m_b in the discussion of the properties of real cuprates. In a simple picture of noninteracting fermions and bosons, $T_{\mathrm{coh}} \approx \delta/m_B$ is approximately equal to $T^* \approx \delta/m_F$ if one assumes $m_F \approx m_B$. However it is very likely that the fluctuations of the gauge field substantially depress the coherence temperature, in which case one could have two intermediate asymptotic regimes $T_{\mathrm{coh}} \le T \le T^*$ and $T^* \le T \le 1/m_f$.

The measured susceptibility is the sum of the paramagnetic susceptibility of the fermions χ_p^F and χ_d. In the noninteracting Fermi gas the paramagnetic susceptibility is simply related with diamagnetic one: $\chi_p = 3\chi_d$. As is well known in the theory of the Fermi liquid this relation is renormalized by the interactions. We have checked that the interaction with the gauge field, in spite of being nonlocal, does not result in any singular contributions to the paramagnetic or the diamagnetic susceptibility in the leading and next to leading orders in the perturbation theory, and results in

$$\chi_p = 3\eta\chi_d \tag{68}$$

with $\eta \ge 1$. We will use (68) for the estimate of the total susceptibility, which becomes:

$$\chi_{\mathrm{tot}} = \chi_f \left[3\eta - \frac{\chi_b}{\chi_f + \chi_b} \right] . \tag{69}$$

Provided that $T \gg T^*$ (69) can be estimated as

$$\chi = \chi_F \left[3\eta - \frac{T^*}{(T + T^*)} \right] . \tag{70}$$

Having determined the effective magnetic fields felt by the bosons and the fermions one can easily calculate the Hall coefficient in the framework of the kinetic equation. Replacing the driving term E' in Eq. (9) by $E' + v \times B'$ with B' standing B_b, B_f for bosons and fermions, respectively, and following the steps leading to (5) we find that the xy component of the resistivity tensor is given as the sum of the xy component of the resistivity tensor of the Fermi and Bose subsystem:

$$R_{xy} = R_{xy}^B + R_{xy}^F . \tag{71}$$

The Hall resistivity of the bosons can be reliably estimated as

$$R_{xy}^B = -\frac{\chi_B B}{(\chi_F + \chi_B)\delta} . \tag{72}$$

The Hall resistivity of fermions R_{xy}^F cannot be reliably estimated in this framework. The continuum Lagrangian

(4) cannot capture the details of the curvature of the Fermi surface which are crucial for the determination of the Hall coefficient. In other words, R^F_{xy} calculated from the lattice model (1) and the continuum limit (4) are going to differ significantly except in the limit of very large doping. Because of this limitation we are going to parametrize $R^F_{xy}=R_{\rm Hf}B_f$. We will treat $R_{\rm Hf}$ as a parameter which can be estimated from more detailed band structure calculations. Combining (71) and (72) we find for the Hall number $n_H=B/R_{xy}$

$$\frac{1}{n_H}=\frac{R_{\rm Hf}\chi_B-\delta^{-1}\chi_F}{\chi_F+\chi_B} . \tag{73}$$

At high temperatures (73) becomes

$$\frac{1}{n_H}=\left[R_{\rm Hf}+\frac{1}{\delta}\right]\frac{T^*}{T}-\frac{1}{\delta},\quad T^*\le T . \tag{74}$$

Estimates based on renormalized band structure calculations[23] show that for reasonable values of the next-nearest-neighbor hopping amplitude in the effective fermionic Hamiltonian $R_{\rm Hf}$ is negative (hole-like), large ($R_{\rm Hf}\sim 4$), and depends weakly on doping at small δ. Thus, the temperature-dependent correction to the Hall coefficient is positive at very small doping and negative when $\delta>R_{\rm Hf}^{-1}$. At low temperatures (73) becomes

$$n_H=R_{\rm Hf}^{-1}\left[1+\left[1+\frac{1}{\delta R_{\rm Hf}}\right]\frac{T}{T^*}\right],\quad T\le T^* . \tag{75}$$

VI. EFFECT OF IMPURITIES IN THE SLAVE-BOSON TECHNIQUES: ELASTIC SCATTERING

At very low temperatures inelastic effects are rare and the dominant scattering mechanism is due to impurities. It is therefore important to analyze the effect of a weak random potential in the slave-boson technique. We consider diagonal disorder

$$H_D=\sum_i v_i c^+_{i\sigma}c_{i\sigma} \tag{76}$$

with

$$\overline{v_iv_j}=w^2\delta_{ij} . \tag{77}$$

This term modifies the coupling to the density of the Fermi Bose system in Eq. (4)

$$\delta L=(v_i+i\lambda_i)f^+_{i\sigma}f_{i\sigma}+\sum_i i\lambda_i b_i^+b_i . \tag{78}$$

When the random potential is absent the saddle-point solution for $i\lambda_i$ is uniform, $i\lambda_i=\lambda_0$. λ_0 is fixed from the constraint equation (2).

The random potential v_i makes the saddle-point solution $i\lambda_i\equiv\lambda^0+\delta\lambda_i$ nonuniform in space. We can interpret $v_i^B\equiv\delta\lambda_i$ as the random potential seen by the bosons and $v_i+\delta\lambda_i\equiv v_i^F$ as the random potential seen by the fermions. λ_i^0 and hence v_i^F and v_i^B are easily determined in terms of the random potential v_i when the random potential is a small perturbation. Defining the compressibility kernels:

$$\kappa^B_{ij}=\left[\frac{\partial n_i^B}{\partial v_j^B}\right]_{v^B=0},\quad \kappa^F_{ij}=\left[\frac{\partial n_i^F}{\partial v_j^F}\right]_{v^F=0} . \tag{79}$$

We find

$$v_i^F=\frac{1}{(\kappa^F+\kappa^B)^{ij}}\kappa^B_{jk}v_k , \tag{80}$$

$$v_i^B=\frac{1}{(\kappa^F+\kappa^B)^{ij}}\kappa^F_{jk}v_k . \tag{81}$$

Now we estimate the compressibility matrices for Fermi and Bose systems. The density of fermions is large; their density correlations are short ranged (their Debye correlation length is of the order of lattice spacing):

$$\kappa^F_{ij}=m_f\delta_{ij} . \tag{82}$$

At high temperatures the uniform ($q=0$) compressibility of the Bose system coincides with the compressibility of the ideal Bose gas δ/T. At very low temperatures bosons condense and form a superfluid with compressibility $m_b/\ln\delta$. As was explained in the Introduction, we do not understand the regime $T\approx\delta/m_b$ where strong fluctuations of the gauge field presumably suppress the transition temperature. However, we believe that matching the compressibilities in these two regimes we get the reasonable estimate of the compressibility for all temperatures: $\kappa(q=0)=\delta/(T+\delta/m_b)$.

Repeating the same arguments for the temperature dependence of the Debye screening length we arrive at $l_B^{-2}=(Tm_B+\delta)$, which interpolate between the high-temperature regime where l_B coincide with thermal wavelength and low-temperature regime where l_B becomes interparticle spacing. Combining these estimates we get the estimate for compressibility at all temperatures and momenta which becomes exact at very low ($T\le\delta/m_b$) and high ($T\gg\delta/m_B$) temperatures.

$$\kappa^B(q)=\frac{1}{1+(l_Bq)^2}\frac{\delta m_B}{Tm_B+\delta} . \tag{83}$$

Comparing (83) and (82) we see that $\kappa^B\ll\kappa^F$ unless we consider the region of low temperatures $T\le\delta/m_b$ and small momentum $(ql_B)\ll 1$ where they become comparable, $\kappa^B\sim\kappa^F$. Therefore estimating the random potential as seen by fermions and holes (6) and (7) we can neglect κ^B in the denominator compared with κ^F, and get the variance of the random potential

$$\langle v_q^Bv_{-q}^B\rangle=w^2 ,\qquad \langle v_q^Fv_{-q}^F\rangle=\left[\frac{1}{(l_Bq)^2+1}\frac{\delta}{Tm_b+\delta}\right]^2w^2 . \tag{84}$$

These formulas mean that the bosons feel an almost unscreened random potential, whereas the random potential seen by the fermions becomes a very smooth function of coordinates. The smooth potential cannot scatter fermions effectively so that their transport relaxation time remains large. The estimates of the variance of the random potential of the bosons and the fermions can be used for their residual resistivity. We replace diagrams Fig. 1

in the derivation of the kinetic equation by the standard impurity diagrams (Fig. 2) (the crossed line stands for the variance of the random potential) and follow the procedure outlined in Sec. II. The random potential seen by the fermion system is smooth and causes scattering events at angles less than $\theta_c \approx \sqrt{\delta + T m_B}$. The residual resistivities of the Fermi subsystem can then be estimated as

$$\rho_F \propto \frac{m_b^2 w^2 \theta_c^3 \kappa_B^2}{(\kappa_B + \kappa_F)^2} . \tag{85}$$

In all regimes the residual resistivity of fermions is small and the residual resistivity is governed by the Bose system: $\rho \approx \rho_B$,

$$\rho_B \propto \frac{m_b^2 w^2 \kappa_F^2}{\delta(\kappa_F + \kappa_B)^2} . \tag{86}$$

The residual resistivity in Eq. (86) can be estimated in the different temperature ranges:

$$\rho_B \propto \begin{cases} \left[\dfrac{m_b w m_f}{m_b + m_f} \right]^2 \delta^{-1}, & T \le T_{\rm coh} \\ (m_b m_f w T)^2/\delta^3, & T_{\rm coh} \le T \le T^* \\ (m_b w)^2/\delta, & T^* \le T \end{cases} \tag{87}$$

For weak disorder the coefficient of proportionality between the residual resistivity and the variance of the random potential (87) is much larger than in Fermi-liquid theory. For stronger disorder we expect even more dramatic effects since the bosons are very easily localized if $m_b \gg m_f$. Therefore, even weak disorder can drive a metal insulator transition in this system. The Fermi surface can remain relatively sharp, even when the resistivity is large, because the physical resistivity is governed by the Bose subsystem while the sharpness of the Fermi surface is governed by the inelastic scattering rate of the fermions which is much smaller than the physical resistivity.

VII. CONCLUSION

The slave-boson formalism[24,25] provides a suitable language for describing the transition between a Fermi-liquid and a non-Fermi-liquid regime.[26] When the bosons are condensed the fermionic quasiparticles are the only relevant low-energy degrees of freedom and one recovers Fermi-liquid theory. When the slave bosons are not condensed the adiabatic continuity between the noninteracting system and the system with interactions is broken, and the Fermi-liquid picture is no longer a good one.

It is useful to recollect how this transition happens in the heavy-fermion problem.[27,28] At low temperatures when the bosons are condensed the system behaves as a Fermi liquid which can be obtained by turning on U in the Anderson model. At high temperatures when the bosons are not condensed the system is better thought of as a two fluid of conduction electrons and local moments, the local moments being degrees of freedom which are not present in the $U=0$ limit of the Anderson lattice. The analogies between the high-temperature superconductors and the heavy-fermion systems and the relevance of the scale $T_{\rm coh}$ for the phenomenological description of both systems have been discussed in Ref. 29 by Levin and collaborators, and in Ref. 30.

Turning to the one-band $t-J$ model at $T=0$, Eq. (1) describes a Fermi liquid, with a Luttinger Fermi surface.[19] However, in many respects this Fermi liquid behaves as a collection of a small numbers of holes.[17] At high temperatures the bosons are not condensed. If we insist on describing this situation in terms of quasiparticles one would say that the lack of Bose condensation signals the loss of Fermi-liquid coherence, and the quasiparticles form an incoherent quantum fluid [IQF]. There is a very significant difference with the heavy-fermion problem. Here the antiferromagnetic exchange energy is still much larger than any temperature of interest and therefore the spin degrees of freedom are still quenched. From the point of view of the spin response the quasiparticles still form a degenerate Fermi liquid. The incoherence is revealed only when we investigate the charge transport. A second fundamental difference with the heavy fermion is that at high temperatures we still have a one-band model. The fermions and the bosons still describe the same charge degrees of freedom, since they are linked inextricably by the constraint (3) even at high temperatures. However, the transport properties can be understood in a simple way if we think of a system of weakly interacting bosons. This suggests that in this regime one can understand the transport properties by focusing on the motion of a few holes in a magnetic background. One could refer to this situation as partial charge spin separation (PCSS).

We now compare the predictions of our analysis with some experiments in the copper oxides. In carrying out this comparison it is important to bear in mind that the predictions for the transport properties which are dominated by the Bose subsystem become quantitative only at high temperatures. At very low temperatures when the susceptibility of the bosons is comparable to the susceptibility of the fermions, the simple interacting Bose-gas picture is not applicable. That is why the predictions that do not involve the exact form of the Bose distribution are more robust than the ones which involve the exact details of this distribution. Also, we consider only the transport properties within the *ab* plane.

The resistivity is linear in temperature in most high-T_c compounds.[31] To explain this fact in this framework we need to assume that either the contribution of the Bose susceptibility to the resistivity [see Eq. (22)] is small or that it is temperature independent. So, we expect linear resistivity only when $T \gg T^*$ or when $T \ll T^*$. We propose to view the cuprate superconductors in the parameter range $T_{\rm coh} \approx T^* \le T \le T_F$. The slope of the resistivity in good quality single crystals[32] is about 0.5 $\mu\Omega$ cm/K in 1:2:3 and 2.0 $\mu\Omega$ cm/K in 2:1:4.[33] We use these slopes and Eq. (22) to estimate χ. We find $\chi \simeq 200$ K for the 2:1:4 and $\chi \simeq 400$ K for the 1:2:3 compound.

We can also estimate χ comparing Eq. (69) with the measured magnetic susceptibility: $\chi_{\rm tot} \sim 1.5 \times 10^{-4}$

emu/mol Cu (Ref. 32) in the 1:2:3 compound and $\chi_{tot} \sim 0.5\times10^{-4}$ emu/mol Cu (Ref. 34) in the 2:1:4 compound. Neglecting χ_b in Eq. (69) and converting to dimensionless units we find $\eta\chi_f \sim 800$ K and $\eta\chi_f \sim 250$ K for the 1:2:3 and 2:1:4 compounds, respectively, which is not inconsistent with our previous estimate. The estimate of the Fermi susceptibility based on the free-fermion equation (13) is also consistent with the values deduced from the slope of the resistivity. The Bose contribution to (69) makes the susceptibility an increasing function of temperature.

The transport properties of $La_{2-x}Sr_xCuO$ have been extensively investigated. It is antiferromagnetic when $x=0$ and is superconducting in the range of compositions $0.05 \leq x \leq 0.25$. Recent muon spin resonance,[35] however, has indicated that $x=0.15$ is the only metallic composition which is microscopically homogeneous and therefore we will compare our results with this composition only. The Hall coefficient in the *ab* plane[33] is relatively large, and increases with temperature. Converting to our dimensionless units gives a Hall number per copper $n_H \approx 0.3$. The sign and the temperature dependence of the Hall number can be accounted for by Eq. (74), with $\delta \approx 0.3$ provided that R_{Hf} is hole-like and larger than $1/\delta$. The consistency of this assumption can be tested experimentally once the full shape of the Fermi surface is mapped out.

The thermopower has been measured in (Ref. 36) and is given by about 70 μV/K which when converted to our dimensionless units, $S \approx 0.7$. This is much larger than what one would predict for a Fermi liquid but smaller than the prediction of (62).

The thermal conductivity has been measured in $YBa_2Cu_3O_7$.[37] It is almost temperature independent or increases slightly with temperature. The Wiedemann-Franz ratio observed was around 5.5 in our units. Equation (61) predicts weak temperature dependence of the thermal conductivity with the Wiedemann-Franz ratio being ~ 4 in reasonable agreement with experiment.[37]

The Hall coefficient in the $YBa_2Cu_3O_7$ compound displays more complex behavior. The Hall number is very small and remarkably linear[38] in a wide temperature range. We could attempt to explain this effect in this framework if we assume that in the 1:2:3 compound T^* is large and therefore one is in the regime $T \leq T^*$ where (75) applies, and R_{Hf} is anomalously large. However, we do not have a qualitative understanding of this regime, where the bosons interact strongly with the fermions and themselves.

The analysis of Sec. VI predicts a residual resistivity which is proportional to the concentration of impurities with a very large proportionality coefficient (which scales with the inverse hole concentration). We expect this phase to be much more sensitive to impurities than the Fermi-liquid phase. The large sensitivity of the dc resistivity to small changes in the impurity concentration seems to be a general feature of transition-metal oxides.[39,40]

We have not addressed here the problem of estimating semiquantitatively the energy scales T_{coh} and T^* and the parameters which appear in the low-energy Lagrangian. It is particularly important to understand the mechanisms that make T_{coh} so low. While various mean-field theories[41] predict superconductivity in the *t-J* model, more sophisticated treatments (such as exact diagonalization of finite systems or theories which take into account the next orders of the perturbation theory) all indicate that one needs to go beyond the two-dimensional model (1) to explain high-temperature superconductivity. Therefore to describe superconductivity additional residual interactions should be added to the effective Lagrangian (4). These are important questions outside the scope of this paper.

We conclude with a discussion of some physical effects which have not yet been measured and are natural consequences of the theory. Equation (47) indicates that the chemical potential of the electrons in the regime $T \gg T^*$ is strongly temperature dependent. $\mu \approx \text{const} - \ln\delta/[m_B T]$. This should be contrasted with the conventional Fermi-liquid result. In the regime where the bosons are not condensed but when their thermal wavelength is comparable with the interparticle spacing Eq. (83) predicts a constant chemical potential which is indistinguishable from the Fermi-liquid result. The anomalous dependence of the chemical potential on temperature could be observed by measuring the temperature dependence of the contact potential between a normal metal and a high-temperature superconductor in its normal state.

The observation of "low-temperature superconductivity"[42] in a single-layer bismuth compound "Bi 2:2:0:1" will allow magnetoresistance and specific-heat measurements at low temperatures in the normal state. We expect negative magnetoresistance in the metallic phase because in the presence of a magnetic field the dynamics of the gauge field become propagating instead of diffusive at low frequencies and, in addition, a Chern-Simons term appearing in the effective action of the gauge field decreases substantially its fluctuations. Both effects decrease the scattering by the gauge field which is responsible for the linear resistivity. Hence a magnetic field should reduce the resistivity.

The same collective mode which is responsible for the linear resistivity should give the dominant contribution to the electronic specific heat $c_v \approx (T/\chi)^{2/3}$. This contribution to the specific heat should also decrease in the presence of a magnetic field.

Towards the end of the work reported here, we became aware of the work by Nagaosa and Lee[22] who considered the problem of transport in the model (4) in a more qualitative manner. They pointed out the importance of the region $\omega \ll T$ in the evaluation of the Bose conductivity. We agree with their conclusions on the electric transport. Application of similar ideas to the calculation of the Bose energy relaxation rate allows us to conclude, however, that the thermal transport obeys approximately the Wiedemann-Franz law.

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