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Various approaches to the linear response in the near-asymptotic regime

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We demonstrate to $O(q^{-2})$ the correspondence between the response in the Gersch and the multiple-scattering approach. In the course we establish a rigorous requirement for the preference of one scaling variable over another. We also show that the reduced response, when computed by means of a particle-hole (spectral function) approach, does not lead to a systematic $1/q$ expansion. For liquid ^{4}He and nuclear matter we calculate components of the dominant final-state interaction. The asymptotic region for liquid ^{4}He is reached, for not too large y, when $q \geq 10$ Å^{-1}. Results for nuclear matter obtained by means of spectral functions indicate that even for momentum transfers as high as $q \sim 9$ fm^{-1} the convergence is slow, except for the smallest y values.

I. INTRODUCTION

Modern studies of the dominant final-state-interaction (FSI) part of the linear response of a nonrelativistic system for large momentum transfer q have followed various paths. Historically, there is first the direct and rigorous approach of Gersch and co-workers.[1] For systems with an interaction v possessing a finite Fourier transform, these authors derived an exact series for the (incoherent part of the) response in powers of $1/q$. The coefficients are functions of a kinematic variable y_w, which is a given function of the energy transfer ω and q. For not too strong v and not too large densities, that series converges and the first terms thus determine the dominant parts of the FSI.

In a second approach one expands the response in an alternative series, corresponding to a multiple scatterings of the knocked-on particle with the medium. Versions exist which are valid for regular forces of the type above, as well as for others which are strong and/or singular.[2] In both cases, the coefficients of the resulting series depend on a different variable $y_0 = y_0(q\omega)$. The latter is not entirely of kinematical origin and contains some average separation energy which, in principle, relates to dynamics. (A seemingly different theory by Silver[3] has been shown to lead to results which are closely related to those obtained in the multiple-scattering approach.[4])

In the past few years increased efforts have been directed towards an accurate calculation of the response of nonrelativistic fermion systems, specifically of nuclear matter. In one approach in this class, emphasis is on the determination of ground and excited states which build the response. Starting from some unperturbed basis, one modifies ground and excited states by one and the same correlation function thus constructing so-called orthogonalized-correlated-basis (OCB) states. The parameters in the correlation function are variationally determined by minimizing the ground-state energy.[5,6] These calculations presently incorporate what, in the language of perturbation theory, is called 1p-1h and (approximately) 2p-2h correlations over and above those present in the zeroth-order states (see Ref. 7 for an alternative discussion on the influence of 2p-2h states). Another approach is based on the observation that the response is in essence the exact interacting *p-h* propagator (IPHP), to which there exist both nonperturbative and perturbative approximations.[8] In a popular approximation one links the response there to the spectral function (SF) for removing a particle ("hole spectral function").[9–11] The latter are measured in single-particle knock-out reactions [cf. $(e,e'p)$] which is just the retained level of approximation to the fully inclusive cross section.

There is no doubt that, independent of the course followed, in the end all exact theories produce the same answer. There then remains the question to what extent this also holds for selected parts or approximations. All theories above appear to have the same asymptotic limit ($q \rightarrow \infty$ at fixed y). However, except for potential models,[12] we do not know of attempts to demonstrate the measure of correspondence for terms or contributions beyond the limit above. Part of the following note just explores that relation for many-body systems.

In Sec. II we establish the relation between the dominant $1/q$ FSI contribution to the response, calculated by the Gersch and multiple-scattering series, and the spectral function approach. Other theories for the response, e.g., OCB and (interacting) particle-hole propagator [(I)PHP] theories are shown not to lead to a systematic $1/q$ expansion. In Sec. III we report on numerical results for FSI relevant to nuclear matter and liquid ^{4}He. Emphasis is on the relative magnitude of various components of the dominant FSI and the rate of convergence of $1/q$ expansions.

II. DOMINANT PARTS OF THE LINEAR RESPONSE

We address the linear longitudinal response of an A-particle system described by a nonrelativistic Hamiltonian H_A with wave functions and energies Φ_A and ε_A. The dominant incoherent part of the response per particle may be written as

$$S^{\text{incoh}}(q\omega)=\sum_n |\langle \Phi_A^0|e^{-i\mathbf{q}\cdot\mathbf{r}_1}|\Phi_A^n\rangle|^2\delta(\omega+\varepsilon_A^0-\varepsilon_A^n)$$
$$=-\pi^{-1}\text{Im}\langle \Phi_A^0|e^{-i\mathbf{q}\cdot\mathbf{r}_1}\times(\omega+\varepsilon_A^0-H_A+i\eta)^{-1}e^{i\mathbf{q}\cdot\mathbf{r}_1}|\Phi_A^0\rangle \;. \tag{2.1}$$

We now compare in three different approaches the leading parts of the reduced (incoherent) response $\phi(qy)\equiv(q/m)S(qy)$ with m the mass of the constituent particles and $y=y(q\omega)$, some scaling variable replacing ω.

A. The Gersch series

Assuming only the existence of the Fourier transform of the elementary interaction v, it has been shown that the reduced response permits the asymptotic expansion[1]

$$\phi(qy_w)=F_0(y_w)+(m/q)[F_1(y_w)+F_{1,0}(y_w)]+O(q^{-2}) \tag{2.2}$$

with coefficients F, dependent on the Gersch-West variable [1,13]

$$y_w=-(q/2)(1-2m\omega/q^2) \;. \tag{2.3}$$

The first two coefficients in (2.2) are

$$F_0(y_w)=(2\pi)^{-2}\int_{|y_w|}^{\infty} n(p)p\,dp \tag{2.4}$$

and for an infinite medium with density ρ (Ref. 1),

$$F_1(y_w)=(2i\pi\rho)^{-1}\int_{-\infty}^{\infty}[\exp(iy_w s)]ds \times\int \rho_2(\mathbf{r}-s\hat{\mathbf{q}},0;\mathbf{r},0)d\mathbf{r} \times\int_0^s v(\mathbf{r}-\sigma\hat{\mathbf{q}})d\sigma \;. \tag{2.5}$$

Their calculations requires the single-particle momentum distribution $n(p)$ [normalized as $(2\pi)^{-3}\int n(p)d\mathbf{p}=1$] and the partially nondiagonal two-particle density matrix ρ_2 of the fully interacting system.

The term $F_{1,0}$ in (2.2) (in Ref. 1 included in the definition of F_1) reads ($p_z=\mathbf{p}\cdot\hat{\mathbf{q}}$)

$$F_{1,0}(y_w)=i(2\pi\rho)^{-1}\int_{-\infty}^{\infty}[\exp(iy_w s)]s\,ds\int\rho_2(\mathbf{r}-s\hat{\mathbf{q}},0;\mathbf{r},0)v(r)d\mathbf{r}$$
$$=(2\pi)^{-3}\frac{d}{dy_w}\int d\mathbf{p}\,\delta(p_z-y_w)\int d\mathbf{r}_2\cdots\int d\mathbf{r}_A[\Phi_A^{0*}(\mathbf{p},\mathbf{r}_2,\ldots,\mathbf{r}_A)V_1(\mathbf{r}_2,..,\mathbf{r}_A)\Phi_A^0(\mathbf{p},\mathbf{r}_2,\ldots,\mathbf{r}_A)] \tag{2.6a}$$

with

$$V_1=\sum_{j\geq 2}v_{1j}=H_A-H_{A-1}-h_1 \;. \tag{2.7}$$

The average of V_1, the interaction of particle 1 with the medium as required in Eq. (2.6a), is easily evaluated by insertion of a complete set of eigenstates of $H_{A-1}+h_1$, the latter being the kinetic-energy operator of particle 1. Introducing the spectroscopic amplitude

$$\Gamma_n(p)=\langle\Phi_A^0|a_{\mathbf{p}}^\dagger\Phi_{A-1}^n\rangle \;,$$

with $a_{\mathbf{p}}^\dagger$ a creation operator for a particle with momentum $\mathbf{p}$, one may rewrite Eq. (2.6a) as $\{\mu_A=[(A-1)/A]m\}$

$$F_{1,0}(y_w)=(2\pi)^{-3}\frac{d}{dy_w}\times\sum_n\int d\mathbf{p}\left[\Delta_n-\frac{p^2}{2\mu_A}\right]|\Gamma_n(p)|^2\delta(p_z-y_w) \;. \tag{2.6b}$$

$\Delta_n=\varepsilon_A^0-\varepsilon_{A-1}^n$ is the particle separation energy when the residual $A-1$ particle system is left in the excited state n.

Equation (2.6b) may be evaluated by means of the properly normalized spectral function for the removal of a particle

$$P(pE)=\sum_n|\Gamma_n(p)|^2\delta(E+\Delta_n) \;,\qquad \int P(pE)dE=n(p) \;. \tag{2.8}$$

We now define

$$X(p)\equiv\sum_n\Delta_n|\Gamma_n(p)|^2 \tag{2.9a}$$
$$=\frac{p^2}{2m}n(p)+\langle\Phi_A^0|a_{\mathbf{p}}^\dagger[a_{\mathbf{p}},W]|\Phi_A^0\rangle \;, \tag{2.9b}$$
$$=\int dE(-E)P(pE) \;, \tag{2.9c}$$

with W the total potential-energy operator.[14,15]

Using (2.4) and the fact that X, Γ_n, and the single-particle momentum distribution n are functions of $|\mathbf{p}|$, one then finds from (2.6b)

$$F_{1,0}(y_w)=(4\pi^2)^{-1}y_w n(y_w)\left[\frac{y_w^2}{2\mu_A}-\Delta(y_w)\right],$$
$$\Delta(y)\equiv X(y)/n(y)\,. \tag{2.6c}$$

It will be useful to rewrite (2.6c). We thus introduce an as yet undefined average separation energy $\bar{\Delta}$ in

$$\overline{\delta y}(q)\equiv(m/q)(y_w^2/2\mu_A-\bar{\Delta})+O(q^{-2})\,. \tag{2.10}$$

With [cf. (2.6c)]

$$\bar{D}(y)\equiv y/(4\pi^2)n(y)[\Delta(y)-\bar{\Delta}]\,, \tag{2.11}$$

one subsequently finds

$$F_{1,0}(y_w)=-\overline{\delta y}(q)(q/m)F_0'(y_w)-\bar{D}(y_w)\,. \tag{2.6d}$$

Equation (2.6d) can now be used to reach an alternative expression for the first terms in the Gersch series (2.2)

$$\phi(qy_w)=F_0(y_w)-\overline{\delta y}(q)F_0'(y_w)+(m/q)[F_1(y_w)-\bar{D}(y_w)]+O(q^{-2})\,. \tag{2.12}$$

B. The multiple-scattering series approach

In the multiple-scattering series (MSS) approach one writes the full Hamiltonian as $H_A=H_A^0+V_1$ and expands the corresponding propagator

$$G(\omega+\varepsilon_A^0)\equiv(\omega+\varepsilon_A^0-H_A^0-V_1+i\eta)^{-1}$$

in (2.1) by means of

$$G_0(z)=(z-H_{A-1}-h_1)^{-1}\,.$$

Thus,

$$G(z)=G_0(z)\sum_{n\geq 0}[V_1G_0(z)]^n\,. \tag{2.13}$$

Using eigenstates of $H_{A-1}^0+h_1$ as in (2.7), the lowest-order ($n=0$) term in (2.13) gives, for the reduced response including recoil[9]

$$\phi^{\rm MSS}(q\omega)\approx(2\pi)^{-3}\sum_n\int d\mathbf{p}\,|\Gamma_n(p)|^2\delta(\omega+\Delta_n-[A/(A-1)]e_0(\mathbf{p}+\mathbf{q}))$$
$$=(2\pi)^{-3}\int dE\int d\mathbf{p}\,P(pE)\delta(\omega-E-[A/(A-1)]e_0(\mathbf{p}+\mathbf{q}))\,, \tag{2.14}$$

where $e_0(p)=p^2/2m$ is the energy of a free particle, and $P(pE)$ is the single-particle spectral function (2.8).

The appearance of state-dependent separation energies Δ_n in the argument of the δ function above presents an exact conversion of the expression (2.14) by means of a purely kinematic scaling variable. This is only possible if Δ_n is replaced by an average $\bar{\Delta}$. Introducing the customary impulse-approximation (IA) scaling variable

$$\bar{y}_0^A=-\frac{A-1}{A}q+\left[\frac{A-1}{A}\left[2m(\omega+\bar{\Delta})-\frac{q^2}{A}\right]\right]^{1/2}$$
$$=y_w-\overline{\delta y}(q)\,, \tag{2.15}$$

one establishes $\overline{\delta y}(q)$, Eq. (2.10), as the difference of the two y variables discussed. One may then expand (2.14) and obtain for the reduced response (we write $\bar{y}_0^A\to\bar{y}$),

$$\phi^{[v]}(q\bar{y})=F_0(\bar{y})+(m/q)[F_1(\bar{y},[v])-\bar{D}(\bar{y})]+O(q^{-2})\,. \tag{2.16}$$

Contrary to the Gersch expansion (2.12), apparently $\bar{y}$, and not y_w, is the natural variable of the coefficients $F_n(y)$ in MSS expansions.

In (2.16) we explicitly mention $[v]$ as a reminder, that we use the v expansion in (2.13), thereby assuming the existence of the Fourier transform of v. We now relax this restriction and expand the full propagator G, Eq. (2.13), in the so-called Watson series, where a singular or strong v (in the momentum representations) is replaced by the corresponding scattering matrix $v\to v^{\rm eff}\equiv t$. One easily generalizes[2]

$$\phi^{[t]}(q\bar{y})=F_0(\bar{y})+(m/q)\{F_1(\bar{y},[t(q)])-\bar{D}(\bar{y})\}+O(q^{-2})\,,$$
$$\phi^{[t]}(qy_w)=F_0(y_w)-\overline{\delta y}(q)F_0'(y_w)+(m/q)\{F_1(y_w,[t(q)])-\bar{D}(y_w)\}+O(q^{-2})\,. \tag{2.17}$$

The same $\bar{D}$ appears whether the MSS (2.16) in v or the Watson analogue (2.17) in t is used in the expression for F_1 in the momentum representation.[4] Notice from Eq. (2.6a) that, even for a singular "bare" v, $\bar{D}$ is finite.

Although one cannot directly derive a Gersch series in t^2, we give in the second equation above the MSS result when (2.15) is substituted in the MSS result in $\bar{y}$. Since [cf. (2.15)] $\overline{\delta y}(q)=O(q^{-1})$, the two expressions are formally identical to order q^{-2}.

In spite of this expected result, we do not know of a previous derivation which, even to $O(q^{-2})$, appears not to be trivial. In fact, in the MSS expansions (2.16) and (2.17) for the reduced response,[2,16] the correction $\bar{D}$ has been overlooked.[17] The same holds for Silver's expression for the response,[3] which, we recall, can be derived from the MSS expansion.[4]

The formal identity of the two series discussed is not the last word on this issue. Consider, for instance, the expressions (2.12) and (2.16) to $O(q^{-2})$ at fixed large q. Defining the ratio

$$\overline{C_0}(q,y)\equiv\overline{\delta y}(q)F_0'(y)/F_0(y)\ , \tag{2.18}$$

it is conceivable that, locally in y, $|\overline{C_0}(q,y)|\gtrsim 1$.

In that case, two series for the reduced response in q^{-1} have numerically dissimilar lowest-order parts and the question arises which series should be used, or equivalently, which scaling variable is "best." There clearly is a simple and sharp criterion. Define

$$C_1(q,y)=(m/q)F_1(y)/F_0(y)\ ,\qquad \overline{C_D}(q,y)=(m/q)\overline{D}(y)/F_0(y)\ . \tag{2.19}$$

The variable for which the relative correction $|C_1(q,y)-\overline{C_D}(q,y)|$ is smallest is [in the region where (2.17) holds] manifestly the preferred one.

C. Spectral-function approaches, interacting particle-hole propagator, and orthogonalized-correlated-basis theories

1. SF

It has been shown in Sec. II B above that the lowest-order term of the MSS expansion (2.14) can be written in a form which features the single-particle spectral function and a δ function, containing the energy of a bare single particle. The two dominant $1/q$ expansions terms have been given in (2.16) and, in principle, expressions for higher-order terms in the MSS can be derived.

Unrelated to the above, spectral functions have been used to compute the response without reference to either a systematic MSS or a $1/q$ expansion. In order to indicate how these developments emerge, we return to the lowest-order so-called plane-wave-impulse-approximation (PWIA) term (2.14). Disregarding recoil one has

$$\phi^{\rm SF}(q\omega)=(2\pi)^{-3}\int dE\int d\mathbf{p}\,P(pE)\delta(\omega-E-e_0(\mathbf{p}+\mathbf{q}))\ . \tag{2.20}$$

One now observes that the last two factors in the integrand in (2.20) relate to the spectral function for a free particle above the Fermi level. In general,

$$\begin{aligned}P_p(pE)&=-\pi^{-1}{\rm Im}\{[E-e_0(p)-\Sigma(p,E)]\}^{-1}\ , & E>\varepsilon^n_{A+1}-\varepsilon^0_A\ ,\\ P(pE)&=+\pi^{-1}{\rm Im}\{[E+e_0(p)+\Sigma(p,E)]^{-1}\}\ , & E>\varepsilon^n_{A-1}-\varepsilon^0_A\end{aligned} \tag{2.21}$$

with Σ the self-energy of the particle, and on thus obtains

$$S(q\omega)=(2\pi)^{-3}\int d^3p\int dE\ P(pE)P_p(\mathbf{p}+\mathbf{q},\ \omega-E)\ . \tag{2.22}$$

Equation (2.22) doe not appear to be related to either theory discussed above. However, it emerges as a well-defined approximation in nonperturbative approaches.[8] Notice, at this level, the symmetric appearance of the knocked-on particle before and after absorption of the momentum transfer q ("hole" and "particle" in perturbation theory).

Next we explore a few simple approximations to the particle propagator in (2.22) and consider first the quasiparticle approximation to the spectral function, which is the probability for adding a particle to the A-particle ground state:

$$S(q\omega)\approx(2\pi)^{-3}\int d^3p\int dE\ P(pE)\frac{\pi^{-1}Z^2(\mathbf{p}+\mathbf{q})\gamma(\mathbf{p}+\mathbf{q})}{[\omega-E-e(\mathbf{p}+\mathbf{q})]^2+[Z(\mathbf{p}+\mathbf{q})\gamma(\mathbf{p}+\mathbf{q})]^2} \tag{2.23a}$$

$$\approx(2\pi)^{-3}\int d^3p\int dE\ P(pE)Z(\mathbf{p}+\mathbf{q})\delta(\omega-E-e(\mathbf{p}+\mathbf{q}))\ . \tag{2.23b}$$

Here

$$\begin{aligned}e(p)&=e_0(p)+\mathcal{V}(p)\ ,\\ \mathcal{V}(p)&={\rm Re}\Sigma(p,e(p))\ ,\\ Z(p)&=\{1-[\partial\,{\rm Re}\Sigma(p,E)/\partial E]_{E=e(p)}\}^{-1}\ ,\\ \gamma(p)&={\rm Im}\Sigma(p,e(p))\ ,\end{aligned} \tag{2.24}$$

with $e(p)$ the pole position of the spectral function (2.21) with ${\rm Im}\Sigma\to 0$. Equation (2.23b) is obtained by using

$$\lim_{\gamma\to 0}\left[\pi^{-1}\frac{Z^2(p)\gamma(p)}{[\omega-e(p)]^2+[Z(p)\gamma(p)]^2}\right]=Z(p)\delta(\omega-e_0(p))\ , \tag{2.25}$$

which is the correct limit for large p. Equation (2.23b) is similar to (2.20) but has, in the δ-function argument, dressed single-particle energies and features in addition to the quasiparticle strength Z.[8]

Consider now the high-q behavior of IA (2.23b), sometimes called the impulse approximation.[18] Introducing, for instance, the West variable, one obtains

$$\phi^{\rm SF}(qy_w)\approx(2\pi)^{-4}\int d^3p\int_{-\infty}^{\infty}ds\int dE\ P(pE)Z(\mathbf{p}+\mathbf{q})\exp(is\{y_w-\mathbf{p}\cdot\hat{\mathbf{q}}-(m/q)[E+e_0(p)+\mathcal{V}(\mathbf{p}+\mathbf{q})]\})\ . \tag{2.26}$$

First we recall that for $k\to\infty$, the real part of the self-energy $\mathcal{V}(k)\to 0$ and, as a consequence, IA and PWIA are identical to order q^{-2}.

Taking the limit above, one may perform the E integral by means of (2.8) and (2.9). The same algebraic manipulation used there leads to

$$\phi^{SF}(qy_w)=(2\pi)^{-4}\int d^3p\int_{-\infty}^{\infty} ds\,\exp[is(y_w-\mathbf{p}\cdot\hat{\mathbf{q}})]\int dE\,P(pE)\{1-is(m/q)[E+e_0(p)]\}+O(q^{-2})$$
$$=F_0(y_w)[1-\overline{C_0}(q,y_w)]=(m/q)\overline{D}(y_w)+O(q^{-2})\,, \quad (2.27)$$
$$\phi^{SF}(q\overline{y})=F_0(\overline{y})-(m/q)\overline{D}(\overline{y})+O(q^{-2})\,.$$

Comparison with Eq. (2.16) shows that the response computed in either the IA (2.23b) or the PWIA (2.20) misses the term $(m/q)F_1(y)$ which is generated by the MSS, while both expressions do contain some higher-order terms. The missing term can therefore only be generated by interacting p-h propagators, which is, of course, precisely what emerges from Sec. II B. The same is the case for generalizations of (2.20) using relativistic kinematics.[9]

2. *IPHP*

Expression (2.22) appears in nonperturbative theories for the response expressed in terms of the interacting dressed particle-hole propagator in which only the noninteracting part is retained. As such, it is equivalent to the Lindhard function calculated with dressed propagators.[11]

The widely used random phase incorporates some interaction between particles and holes, but the propagators there usually refer to bare particles.[19,20] Moreover, the specific interactions retained in the random-phase approximation (RPA) emphasize long-range correlations, and, hence, describe small- and not large-q behavior of the response.

Reference 21 is an example of an ancient and rather primitive attempt to incorporate some dressing as well as interaction effects. At present there are no realistic results available for the interacting p-h propagator beyond the content of (2.22). Yet, as is the case for noninteracting propagators, those theories do not produce *systematic* $1/q$ expansions.

3. *OCB*

A number of recent calculations address directly the response by means of ground and excited OCB states.[5,6] Those are obtained by orthogonalizing the results of the application of a common correlation function to the uncorrelated basis states. The correlation function is determined by a variational calculation of the ground-state energy.

The procedure described may be sufficient for some purposes, but is obviously not exact. However, the claim is made that, in the language of perturbation theory, 1p-1h excitations are treated exactly and 2p-2h states approximately. In the MSS formalism this translates to exact retention of all terms in v and of some in v^2. Differently stated, the reduced response, expanded in powers of $1/q$, should be exact to order q^{-2}. It would be of great interest to check that claim, to extract the corresponding expansion coefficients and to compare the two lowest-order ones with their counterparts discussed in Secs. II A and II B above. Failure to do so for an ideally exact calculation would directly call into question the basic tenet of the OCB model for the response, namely, the sufficiency of one correlation function for the ground and all excited states.

III. NUMERICAL RESULTS

Equation (2.6a) and Eqs. (2.6c) and (2.9) show two ways to calculate the correction $F_{1,0}(y_w)$. Those require knowledge of either the half-nondiagonal two-particle density matrix ρ_2, or the single-particle spectral function $P(pE)$.[22] None depends on the average separation energy $\overline{\Delta}$.

This is not the case for the reduced response $\phi(q\overline{y})$ as function of the IA variable $\overline{y}$, Eqs. (2.16) or (2.17). We recall that, for sufficiently strong or singular v, there is no way to avoid that approach. One is then led to the scaling variable $\overline{y}$ and to split $F_{1,0}$ as in Eq. (2.6d). As a result, $\overline{y}$ and the component $\overline{D}$ of the correction above depend on the average separation energy $\overline{\Delta}$ as do expressions for ϕ to any finite order.

Since, in practice, approximations will be involved in each calculation, a comparison of the outcome would serve as a test. Unfortunately this does not appear to be possible. To our knowledge, for no system are both sources of information simultaneously available.

We proceed to the average separation energy $\overline{\Delta}$, which is not uniquely defined: every choice defines a corresponding $\overline{y}$ and $\overline{\delta y}(q)$ and in the end a $\overline{\Delta}$ dependent $\overline{D}$ as appearing in the reduced response (2.16) or (2.17). It is, in particular, impossible to cause vanishing of the function $\overline{D}(y)$, Eq. (2.11), by choice of a constant. However, one may care instead that $\overline{D}$ vanishes in the *mean*, i.e.,

$$(2\pi)^{-3}\int d\mathbf{p}\,n(p)[\Delta(p)-\overline{\Delta}]=0$$

or

$$\overline{\Delta}=(2\pi)^{-3}\int d\mathbf{p}\,X(p)\,. \quad (3.1)$$

Using (2.9c), one infers that the so-defined $\overline{\Delta}$ is just minus the average removal energy as defined by Koltun.[15] One can then use his sum rule for $\overline{\Delta}$ in terms of the kinetic and total energy per particle. Assuming the interactions to be of purely two-body nature, one has

$$\overline{\Delta}=2\langle E_A\rangle-\langle T_A\rangle\,. \quad (3.2)$$

In addition to (3.2), one may evaluate the right-hand side of (3.1) with $X(p)$ computed by Eq. (2.9c). We proceed to illustrate the content of the remarks above for liquid ^{4}He and nuclear matter.

A. Liquid ^{4}He

$\bar{\Delta}$: With $\langle T_A \rangle$, $\langle E_A \rangle = 14.82$, respectively, -21.58 K,[23] $2m_{\rm He}\bar{\Delta}$ is estimated to be ~ 0.17 Å^{-2}, which, in (2.10), is small compared to typical y^2 values ~ 1–4 Å^{-2}.

C_0: Approximating the single-atom momentum distribution $n(p)$ by a Gaussian with width $p_0 \sim 1.28$ Å^{-1}, one has $C_0(q,y) \sim y^3/qp_0^2$. For $q > 10$ Å^{-1}, values $C_0(q,y) \geq 1$ only occur in the very tails of actually measured responses, and there the difference between y_w and $\bar{y}$ matters indeed. In the same region the coefficient functions $F_n(y)$, $n \geq 2$ may not be negligible compared to lower-order terms retained.

FSI: The exact expression for the dominant, large-q FSI [Eqs. (2.5) and (2.6a)] requires knowledge of the half-nondiagonal two-particle density matrix ρ_2. Unfortunately, there are, as yet, no available accurate values in a compact form.[24] We shall therefore continue to use previously proposed approximations,[2–4] which all are of the form

$$\rho_2(\mathbf{r}-s\hat{\mathbf{q}}, 0{:}\mathbf{r},0) \sim \rho\rho_1(0,s)\chi([g]) \ . \qquad (3.3)$$

ρ is the number density and $\rho_1(0,s)/\rho$ is the (reduced) off-diagonal single-particle density matrix, which is the Fourier transform of the single-particle momentum distribution. Finally, χ is some functional of the pair distribution function $g(r)$. We have used, in particular,[1,2]

$$\rho_2(\mathbf{r}-s\hat{\mathbf{q}}, 0;\mathbf{r},0) \approx \rho\rho_1(0,s) g(|\mathbf{r}-s\hat{\mathbf{q}}/2|) \qquad (3.4a)$$

$$\approx \rho\rho_1(0,s)[g(r)g(|\mathbf{r}-s\hat{\mathbf{q}}|)]^{1/2} \ . \qquad (3.4b)$$

In order to avoid, for the present purpose, irrelevant complications due to a condensate, we shall present results computed for $T=4$ K, i.e., above the transition temperature $T_c = 2.17$ K. Analyses for $T \leq T_c$ with the erroneous omission of $\bar{D}$ in (2.12) can be found in Ref. 16 (see Ref. 2 for the same using Silver's formalism). A reanalysis, using the correct expression (2.17) as well as new predictions for $q=13$ and 18 Å^{-1}, can be found in Ref. 25.

In Fig. 1 we present, for a typical $q=13$ Å^{-1} and a standard y range, the C ratio (2.18) and (2.19), computed with the model of Ref. 2 and the required input elements of Ref. 26; the total dominant FSI relative to F_0 is given by $C_1 - \bar{C}_0 - \bar{C}_D$. For small y it starts out with negative values and then changes sign. It remains relatively small except, again, in the extreme wings and, indeed, ^{4}He for medium- and large-q values is ideally suited to study dominant FSI.

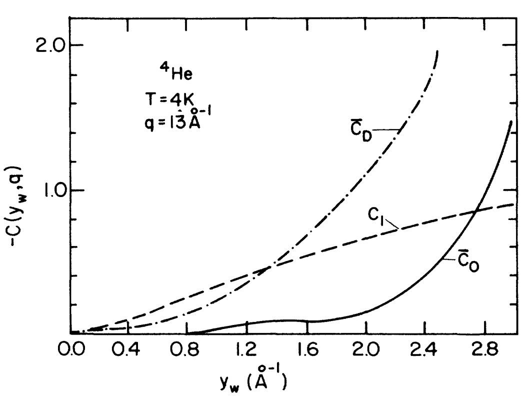

FIG. 1. Ratios $\overline{C_0}$, Eq. (2.18); C_1, $\overline{C_D}$, Eq. (2.19), for liquid ^{4}He at $T=4$ K and $q=13$ Å^{-1}.

$1/q$ expansion: Table II in Ref. 16 contains ample material demonstrating that the convergence is satisfactory for $q > 10$ Å^{-1} and $y < 2$–3 Å^{-1} which is also inferred from the C ratios.

B. Nuclear matter

We shall restrict ourselves in the following to results for the response (2.23b) obtained with spectral functions. Sources of information on the former are on the OCB calculations reported in Ref. 10 for the Urbana v_{14} interaction, and the results obtained by means of a self-consistent Green's-function method.[27] In the last reference, the central part of the $^3S_1$-$^3D_1$ channel of the Reid soft-core potential is used only in $L=0$ states.

$\bar{\Delta}$: The result, which can be derived from Ref. 10, is $\bar{\Delta} = -36.5$ MeV and thus $2m_N\bar{\Delta} \approx -0.069$ GeV^2, which is relatively small compared to typical y^2 values 0.4–0.6 GeV^2 in present-day large-q experiments. For substantially lower y, one can no longer neglect $\bar{\Delta}$ in $\bar{D}$. For completeness we also mention Koltun's results for finite nuclei: $2m_N\bar{\Delta} \approx -0.055$ GeV^2 (Ref. 15) (Capitani *et al.* cite slightly smaller values[28]).

Since the interaction used in Ref. 27 does not bind, only (3.2) provides an estimate. With a quasiparticle approximation for the spectral function, one obtains $\bar{\Delta} = -32.9$ MeV, reasonably close to the outcome above.

$\Delta(p)$: In Fig. 2 we show the ratio $\Delta(p)$, Eq. (2.6c) obtained with the spectral functions from Refs. 10 and 27. Since the Fermi momenta in the two references differ ($p_F = 1.33$, respectively, 1.4 fm^{-1}) a comparison is most naturally made for an abscissa p/p_F. The difference in $\Delta(p)$ is due to the missing partial-wave components in the interaction v used in Ref. 26. We also show the comput-

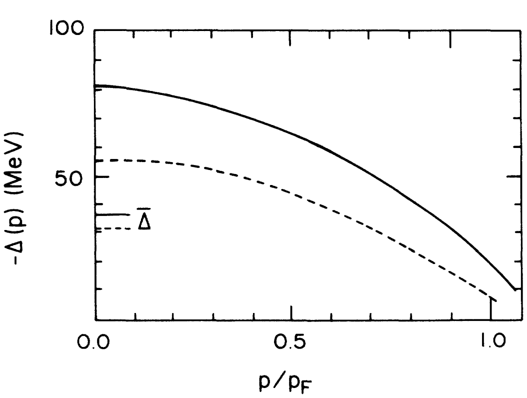

FIG. 2. $\Delta(p)$, Eq. (2.6), for nuclear matter. Solid and dashed lines correspond to results for spectral functions from Ref. 10, respectively, 26. The mean values $\bar{\Delta}$ are indicated by a horizontal bar.

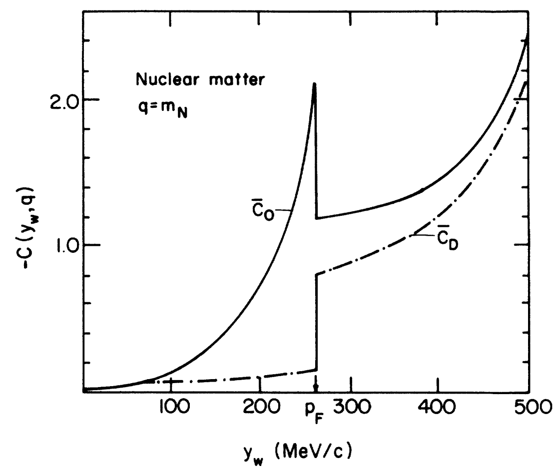

FIG. 3. Same as Fig. 1 for nuclear matter for $q = m_N$ GeV.

ed average $\bar{\Delta}$: The smoothness of $\Delta(p)$ shows that the estimate (3.1) is a plausible one.

C_0: For nuclei probes at $q \approx 1$ GeV, $\overline{C_0}(q,y) \leq 1$ only for quite small $y \leq 0.2$ MeV.

FSI: At present, no results exist on the "true" interaction part F_1, Eq. (2.5) in (2.2) or (2.12). We therefore only entered in Fig. 3 for $q/m_N = 1$ (m_N is the nucleon mass) $(q/m)\overline{C_0}$, $\overline{C_D}$, Eqs. (2.18) and (2.19). Notice the typical discontinuities which reflect the same in $n(p)$. The different behavior of the two ratio's is due to the y^3 increase of C_0.

$1/q$ expansion: We considered the numerical outcome of calculations (2.23b), i.e., the response computed by means of spectral functions. The corresponding reduced responses, when written in terms of the West, respectively, the IA scaling variables were considered to be "data" and were expanded in a $1/q$ expansion. Since the response spreads over orders of magnitude, it is hard to represent the limited input by stable expansion functions. It is therefore somewhat surprising, but gratifying, to find that, for $y \leq 0.3$ GeV, the extracted two lowest-order coefficients agree with the theoretical expressions of $\phi^{\rm SF}(qy)$, Eq. (2.27), for the model. Higher-order coefficient functions appear poorly determined by the set of data and their extraction awaits more accurate results.

IV. CONCLUSION

We have been concerned above with the description of the approach to the asymptotic limit of the (reduced) linear response for a nonrelativistic system $[y = y(q\omega)]$

$$\phi(qy) = (q/m)S(q,\omega) \ .$$

One of our main results is a proof that the dominant final-state interaction between the knocked-on constituent and the material is formally the same, whether the underling description is the theory of Gersch and his coworkers in terms of the Gersch-West scaling variable y_w, or a version of the standard multiple-scattering theory in terms of the IA variable $\bar{y}$. The two approaches above differ only in the scaling variable y used and which replaces the frequency variable ω (the energy loss in an inclusive scattering experiment). Incidentally, in the development we encountered a term in the second series of order m/q, which has been overlooked in previous publications.

The equivalence above has been called "formal," indicating that the two series as functions of q are identical to order $O(q^{-2})$. However, since the coefficients of the series are functions of scaling variables with a difference $\propto q^{-1}$, these may, locally in y show quite appreciable numerical differences. Under those circumstances, a series in one y variable may represent the response $S(q\omega)$ to given order in q^{-1} better than a series in an alternative variable to the same order in q^{-1}. The question then arises which scaling variable is preferable and we discussed a simple criterion.

A somewhat undesirable feature of the multiple-scattering series or IA scaling variable is the appearance of some average energy, on which all terms in the series and the ultimate result depend. Yet we could suggest a motivated and simple choice.

We further investigated a third approach which has been applied for fermion systems. There, one links the response to the interacting particle-hole propagator which, in an approximation, relates to the single-particle (hole) spectral function. The latter theory has the correct asymptotic limit, but appears to contain only part of the exact dominant final-state interaction. Those are only recovered if interactions are included between the particle and hole propagators.

The various approaches have been tested on liquid ^{4}He and nuclear matter, each in a q,y regime which suggest an approach to the asymptotic region. For both forms of matter, we determined parameters and functions related to, and relevant for, the dominant FSI. These indicate that, for ^{4}He, one is indeed near the asymptotic region for not too large y and also that there the particular choice of this scaling variable hardly matters.

The situation is apparently different for nuclear matter. We considered the response computed by means of spectral functions as "data" and performed a series expansion in powers of $1/q$. The extracted first two coefficients in the series are indeed close to their theoretical values (2.27) but the "data" are not smooth enough to reliably extract higher-order coefficients. The order of magnitude of their size indicates, that, even for q as large as 9 fm^{-1}, convergence is only found for rather small y.

This last, still somewhat incomplete, result is yet of relevance for the quest of scaling of actual data. That is the possibility to represent those by $F_0(y)$, the asymptotic part of the reduced response, function only of a scaling variable. With FSI effects, which are of the same order as $F_0(y)$, it seems not too meaningful to try to extract from even lower-q data information, like the single-particle momentum distribution present in F_0.

There is clearly a need for an accurate determination, both theoretically and experimentally, of systematic FSI effects as discussed above. In the meantime it would be of great interest to obtain maximum information from OCB calculations of the response. Albeit not giving a systematic series in $1/q$, their information might very well be useful. Clearly, the study of the approach of the response to its asymptotic limit discloses most valuable information and its gathering will continue to be rewarding.

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This work grew out of a discussion with Shmuel Gurvitz, who correctly pointed out that, in previous MSS derivations of the dominant FSI term, a term of $O(1/q)$ had been omitted. We acknowledge the stimulus which his remark provided. We further thank Omar Benhar for graciously putting at our disposal the results that he and his co-workers obtained for the spectral functions and the response of nuclear matter. J. Besprosvany has been helpful in executing the $1/q$ expansion for nuclear matter. This research was supported in part by the Condensed Matter Theory Program of the Division of Materials Research of the U.S. National Science Foundation under Grant No. DMR-85-19077 (at Washington University in Saint Louis).

[1]H. A. Gersch, L. J. Rodriguez, and Phil N. Smith, Phys. Rev. A **5**, 1547 (1972).

[2]A. S. Rinat, Phys. Rev. B **40**, 6625 (1989).

[3]R. N. Silver, Conden. Matter Theory **3**, 131 (1987); Phys. Rev. B **37**, 3794 (1988); **38**, 2283 (1988).

[4]A. S. Rinat and M. N. Butler, Phys. Rev. B **41**, 11 149 (1990).

[5]S. Fantoni and V. R. Pandharipande, Nucl. Phys. **A473**, 234 (1987).

[6]A. Fabrocini and S. Fantoni, Nucl. Phys. **A503**, 375 (1989).

[7]Y. Horikawa, F. Lenz, and N. C. Mukhopadyay [Phys. Rev. C **22**, 1680 (1980)] derived the equivalent of expression (2.23a) for a finite nucleus in the shelf-mode representation.

[8]Some representative references are J. H. Hubbard, Proc. R. Soc. London, Ser. A **240**, 539 (1957); V. M. Galitskii and A. B. Migdal, Zh. Eksp. Teor. Fiz. **34**, 139 (1958) [Sov. Phys.—JETP **7**, 96 (1958]; P. C. Martin and J. Schwinger, Phys. Rev. **115**, 1342 (1959); P. Nozières, *Theory of Interacting Fermion Systems* (Benjamin, New York, 1966).

[9]A. E. L. Dieperink, T. De Forest, Jr., I. Sick, and R. A. Brandenburg, Phys. Lett. **63B**, 261 (1976); H. Meyer-Hadjuk, Ch. Hadjuk, P. U. Sauer, and W. Theis, Nucl. Phys. **A395**, 332 (1983); C. Ciofi degli Atti, E. Pace, and G. Salme, Phys. Lett. **126B**, 303 (1983); C. Ciofi degli Atti, Nuovo Cimento **76**, 330 (1983).

[10]O. Benhar, A. Fabrocini, and S. Fantoni, Nucl. Phys. **A505**, 267 (1989).

[11]A. Ramos, A. Polls, and W. H. Dickhoff, *Proceedings EPIC, 1989*, edited by O. Benhar and A. Fabrocini (Giardini, Pisa, in press).

[12]A. S. Rinat and R. Rosenfelder, Phys. Lett. B **193**, 411 (1987).

[13]G. B. West, Phys. Rep. C **18**, 264 (1975).

[14]A. S. Reiner-Rinat, Phys. Rev. **133**, B1105 (1964). [k^2 in Eq. (33) there should read $k^2n(k)$.]

[15]D. S. Koltun, Phys. Rev. Lett. **28**, 182 (1972).

[16]A. S. Rinat and M. F. Taragin, Phys. Rev. B **41**, 4247 (1990).

[17]We thank S. A. Gurvitz for drawing our attention to this point.

[18]The use of the term PWIA for (2.20) and (2.14) is somewhat of a misnomer. In the context of the MSS theory, it is just the lowest-order term. The addition PW refers to the use of bare single-particle energies, which are the only ones which naturally occur in that series. The term PWIA can therefore only be understood in models where the single particle is dressed. (See below).

[19]See, for instance, G. E. Brown, *Many-Body Problems* (North-Holland, Amsterdam, 1972); K. Goeke and J. Speth, Annu. Rev. Nucl. Part. Sci. **32**, 65 (1985).

[20]B. Tanatar, E. F. Talbot, and H. R. Glyde, Phys. Rev. B **36**, 8376 (1987).

[21]A. S. Reiner-Rinat, Phys. Rev. **138**, B389 (1965).

[22]The actual identity of the right-hand sides of Eqs. (2.6a) and (2.6c) imply a relation between integrals over the spectral function and the density matrix ρ_2, which can also directly be proved by evaluation of the commutator in (2.9b) Refs. 15 and 21).

[23]E. Manuosakis, V. R. Pandharipande, and Q. N. Usmani, Phys. Rev. B **31**, 7022 (1985); P. A. Whitlock and R. M. Panoff, Can. J. Phys. **65**, 1409 (1987).

[24]M. L. Ristig and J. W. Clark, Phys. Rev. B **40**, 4355 (1989); M. L. Ristig (private communication).

[25]A. S. Rinat, Phys. Rev. B (to be published).

[26]$g(r)$: E. C. Svensson, V. F. Sears, A. D. B. Woods, and P. Martel, Phys. Rev. B **21**, 3638 (1982); $n(p)$, $\rho_1(0,r)$: P. A. Whitlock and R. M. Panoff, Can. J. Phys. **65**, 1409 (1987); D. M. Ceperley and E. L. Pollock, *ibid.* **65**, 1416 (1987); (private communication).

[27]A. Ramos, A. Polls, and W. H. Dickoff, Nucl. Phys. A **503**, 1 (1989).

[28]G. P. Capitani *et al.* (unpublished).