Phonon propagation with isotope scattering and spontaneous anharmonic decay

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Calculations of the propagation of high-energy phonons introduced into a crystal at $T=0$ K are presented. The phonons are assumed to undergo elastic scattering and anharmonic decay processes. We show that in addition to the "quasidiffusion" component to the phonon propagation (discussed previously by Levinson and co-workers) there is also a contribution that arrives at close to the time expected for ballistic propagation. This latter mode of propagation continues to be of substantial importance even when the elastic scattering is much stronger than the anharmonic processes.

I. INTRODUCTION

In this paper we consider what happens when high-energy phonons (with energy comparable to $k\theta_D$, θ_D = Debye temperature) are introduced into a crystal. The crystal is assumed to be at a sufficiently low temperature that the thermal phonons can be neglected. In most crystals, high-energy phonons have a very short mean free path, even at $T=0$ K. Two classes of scattering processes are important.

(1) Elastic scattering arising from the distribution of isotopic masses, and from other defects.

(2) Spontaneous anharmonic decay in which the original phonon splits into two lower-energy phonons.

We write the decay times and mean free paths for these two processes as τ_I, τ_A, and Λ_I, Λ_A, respectively. The phonon lifetimes τ_I and τ_A are, in most cases, rapidly decreasing functions of phonon energy E. Thus, a high-energy phonon will diffuse very slowly and will stay close to the point where it was generated. However, after several anharmonic decays the average energy of the phonons present will have decreased to the point that Λ_I and Λ_A are much larger and the diffusion is then rapid. Thus, to calculate what happens when a high-energy phonon is introduced into a crystal, it is necessary to understand the way in which the decay products of the original phonon cascade down to lower energies, and how the rapid increase of the mean free path affects their spatial distribution.

This type of situation occurs in several types of phonon experiments. In conventional heat-pulse experiments[1] a thin metal film evaporated onto one face of a crystal is heated electrically and radiates phonons. These phonons are detected by a bolometer film on the other side of the crystal. If the metal film is heated strongly for a short time, a fairly small number of high-energy phonons are injected into the crystal. The decay products of these phonons are what is detected by the bolometer. In a variant of this technique a light pulse is absorbed at the surface of a semiconductor.[2] Electrons and holes are produced, and as these relax towards the band edge, high-energy phonons are generated. Finally, there are experiments in which an elementary particle is scattered in a crystal, and the part of the recoil energy appearing as phonons is detected.[13] In this process the phonons initially produced are always of high energy and must relax before they can reach the detector.

An elegant theory of phonon propagation under these conditions has been proposed by Kazakovtsev and Levinson.[4,5] It is argued that after several anharmonic decays have occurred the phonon distribution as a function of space and time assumes a scaling form. In addition to this work, several groups have performed Monte Carlo simulations of phonon propagation in particular systems.[6–9] The relation between the scaling approach and the results of the Monte Carlo calculations has not been established, and the range of validity of the scaling theory is not clear. In this paper we try to resolve these issues. We will show that for most real crystals of size in the normal experimental range the scaling theory is a poor approximation. It should apply, however, for systems with stronger elastic scattering such as mixed crystals. We demonstrate these results both by physical arguments and by comparison of the scaling theory with Monte Carlo simulations. In addition, we will present numerical results for specific crystals of experimental interest.

II. MODEL FOR PHONON RELAXATION

The rate of phonon scattering from the distribution of isotopic masses is given by

$$\frac{1}{\tau_I}=\frac{\pi v_0 g_2 \omega^2 D(\omega)}{6}\ , \tag{1}$$

where v_0 is the volume per atom and $D(\omega)$ is the phonon density of states per unit volume. g_2 measures the isotopic mass disorder and is given by

$$g_2 \equiv \sum_i x_i \left[\frac{\Delta M_i}{\overline{M}}\right]^2\ , \tag{2}$$

where x_i is the concentration of isotope i whose mass differs from the mean atomic mass $\overline{M}$ by the amount ΔM_i. Equation (1) holds for cubic Bravais crystals, and also for diamond-structure crystals as shown by Tamura.[10] For low frequencies ($\omega \ll \omega_D$), Eq. (1) reduces to

$$\frac{1}{\tau_I}=\frac{v_0 g_2 \omega^4}{4\pi v_D^3} , \tag{3}$$

where v_D is the Debye velocity defined by

$$\frac{1}{v_D^3}=\tfrac{1}{3}\sum_j \left\langle \frac{1}{v_j^3} \right\rangle . \tag{4}$$

In this equation v_j is the velocity of the jth polarization acoustic phonon, and $\langle\ \rangle$ denotes an angular average.

The anharmonic decay rate cannot be expressed in a simple way and varies with propagation direction and phonon polarization. For longitudinal acoustic phonons with $\omega \ll \omega_D$, and for some of the transverse phonons one has[11]

$$\frac{1}{\tau_A} \propto \omega^5 . \tag{5}$$

For the slower branch transverse phonons the decay rate is zero.[12,13] At frequencies comparable to ω_D one cannot derive a simple expression for the frequency dependence of τ_A. One can argue, however, that it is not necessary to know the details of the phonon lifetimes at higher frequencies. The point is that in most crystals τ_I and τ_A are very short (10–100 psec) when ω is comparable to ω_D,[14] and so the distance that phonons can diffuse before they decay into the frequency range $\omega \ll \omega_D$ is very small. This argument fails for crystals such as NaF or solid He for which there is no isotope scattering, since then there are always some high-energy transverse phonons that are completely stable which will propagate ballistically away from the point of generation. For the majority of crystals (in which isotope scattering is reasonably strong) the diffusivity of *all* high-frequency phonons can be considered to be very small. Thus, the argument is that there is no need to keep track of the phonon scattering and decay processes until the phonon frequency has fallen into the range well below ω_D. (We will test this idea to some extent in the computer simulations.)

In the low-frequency range we can write

$$\frac{1}{\tau_I}=BE^4 , \tag{6}$$

where E is the energy in degrees Kelvin. For anharmonic decay let the decay rate averaged over phonon polarizations and **k** directions be

$$\frac{1}{\tau_A}=AE^5 . \tag{7}$$

We will consider the values of A and B for some specific crystals in Sec. V.

III. SCALING THEORY

Consider first the relaxation of a spatially homogeneous distribution of phonons under the action of anharmonic decay processes. Let the number of phonons in the energy range E, δE, be $n(E)\delta E$. Then

$$\frac{dn(E)}{dt}=-AE^5 n(E)+\int_E^\infty dE'\, AE'^5 P(E',E) n(E') , \tag{8}$$

where $P(E',E)\delta E$ is the probability that in the decay of a phonon of energy E' a phonon with energy in the range $E,\delta E$ is produced. Initially, only high-energy phonons are present. We are interested in the solution of this equation after several decays have occurred.

We write

$$n(E,t)\equiv\frac{g(u,t)}{E^2} , \tag{9}$$

$$u\equiv AE^5 t . \tag{10}$$

For $P(E',E)$ we choose

$$P(E',E)=\frac{60E^2(E'-E)^2}{(E')^5} . \tag{11}$$

This is the correct form of P in the simplest type of phonon decay, namely the relaxation of phonons of a single polarization by collinear decay processes as occurs in liquid ^{4}He.[15] Then we obtain

$$\frac{dg}{d\ln t}=-u\frac{dg}{du}-ug+12\int_u^\infty du'\, g(u')\left[\frac{u}{u'}\right]^{4/5}\times\left[1-\left[\frac{u}{u'}\right]^{1/5}\right]^2 . \tag{12}$$

Since $\ln t$ appears only on the left-hand side of this equation, we may look for solutions of the form

$$g(u,t)=\sum_n A_n\phi_n(u)e^{-\lambda_n \ln t}=\sum_n \frac{A_n\phi_n(u)}{t_n^\lambda} , \tag{13}$$

where ϕ_n and λ_n are eigenfunctions and eigenvalues from the equation

$$\lambda_n\phi_n=u\phi_n+u\frac{d\phi_n}{du}-12\int_u^\infty du'\,\phi_n(u')\left[\frac{u}{u'}\right]^{4/5}\times\left[1-\left[\frac{u}{u'}\right]^{1/5}\right]^2 . \tag{14}$$

The spectrum of eigenvalues determines the behavior of the solution. Let us assume that there are no negative eigenvalues which would lead to an instability.

From energy conservation,

$$\int_0^\infty du\, g(u,t)/u=\text{const} . \tag{15}$$

Therefore, there must be an eigenfunction $\phi_0(u)$ with eigenvalue zero, and so for large t

$$g(u,t)\approx A_0\phi_0(u) \tag{16}$$

and

$$n(E,t)\approx\frac{A_0\phi_0(AE^5t)}{E^2} . \tag{17}$$

This is the scaling solution.[5] The rate at which the phonon distribution function approaches this limiting form is determined by the spectrum of the eigenvalues λ_n which is not yet known.

It is straightforward to calculate ϕ_0 numerically and the result[16] for $n(E)$ from (17) is shown in Fig. 1. Kazakovtsev and Levinson[4] have shown that the asymptotic forms of $n(E,t)$ for large and small E have given by[17]

$$n(E,t) \propto E^2 t^{4/5} \quad \text{for} \quad AE^5 t \ll 1 \; ; \tag{18a}$$

$$n(E,t) \propto E^{-2} \exp(-AE^5 t) \quad \text{for} \quad AE^5 t \gg 1 \; . \tag{18b}$$

We have used the scaling solution to calculate the rate at which the average energy of the phonons decreases. The result is

$$\langle E \rangle = \frac{\int En(E,t)dE}{\int n(E,t)dE} = \frac{1}{(At)^{1/5}} \frac{\int du\, \phi_0(u)/u}{\int du\, \phi_0(u)/u^{6/5}} = \frac{0.57}{(At)^{1/5}} \; . \tag{19}$$

Note that in the scaling approximation the average phonon energy after time t is independent of the original energy of the injected phonons. The presence of elastic-scattering processes has no effect on the phonon distribution function.

Now suppose that the initial condition is that there are some high-energy phonons in the region near to the origin. Kazakovtsev and Levinson (KL) (Refs. 4 and 5) consider what happens when the condition

$$\tau_I \ll \tau_A, t_b \tag{20}$$

is satisfied, where t_b is the time it would take the phonons to reach the detector if they traveled ballistically, i.e., $t_b = L/v_D$. Since τ_I and τ_A depend strongly on energy, one has to specify what phonon energy is involved in Eq. (20). One certainly needs (20) to hold at the *initial* phonon energy. In Sec. IV we will discuss this condition in more detail.

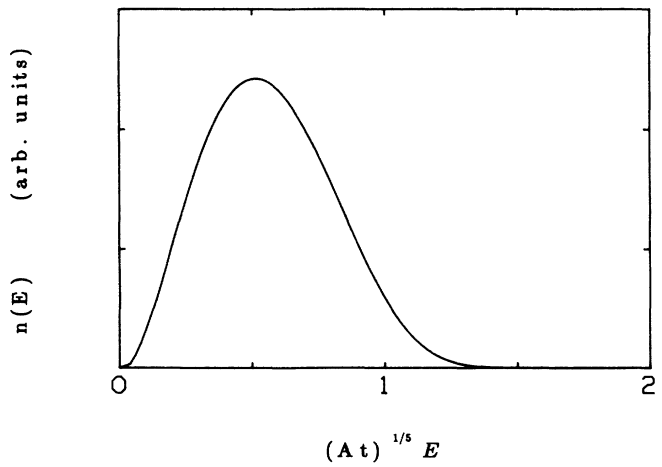

FIG. 1. Dependence of phonon distribution function $n(E)$ on energy in the scaling regime.

Given Eq. (20), KL (Refs. 4 and 5) argue that one should be able to describe phonon transport by a diffusion equation of the form

$$\frac{\partial}{\partial t} n(E,\mathbf{r},t) = D(E)\nabla^2 n(E,\mathbf{r},t) + \hat{S}n(E,\mathbf{r},t) \; , \tag{21}$$

where

$$D(E) = 1/3 v_D^2 \tau_I(E) \tag{22}$$

and $\hat{S}n(E)$ represents the rate at which n is changed by phonon decays, i.e., the right-hand side of Eq. (8). Based on (21), KL find a scaling solution for the phonon distribution function that should hold for long times, i.e., after the initially injected phonons have undergone several decays. The main characteristics of this solution are the following.

(1) At time t the energy of a typical phonon is proportional to $(At)^{-1/5}$.

(2) The distance from the origin that the phonon distribution has spread to after time t is of order

$$r \sim v_D [\tau_A(\tilde{E})\tau_I(\tilde{E})]^{1/2} [t/\tau_A(\tilde{E})]^{9/10} \; , \tag{23}$$

where $\tilde{E}$ is the original phonon energy. Using the explicit forms for τ_A and τ_I [Eqs. (6) and (7)] we obtain

$$r \sim v_D \frac{A^{2/5}}{B^{1/2}} t^{9/10} \; . \tag{24}$$

Thus, the distribution spreads more slowly than in ballistic propagation, but faster than in normal diffusion. [Note that although r in Eq. (23) is written in terms of $\tilde{E}$, r is in fact independent of this quantity.]

(3) The phonon distribution at distance r and time t has the scaling form

$$n(E,r,t) \propto E^{\alpha} f(\xi,\eta) \tag{25}$$

with α a constant, and

$$\xi \equiv \frac{r}{v_D[\tau_I(E)\tau_A(E)]^{1/2}} \; , \tag{26}$$

$$\eta \equiv \frac{t}{\tau_A(E)} \; . \tag{27}$$

IV. VALIDITY OF SCALING THEORY

The scaling theory is based on the assumption that Eq. (20) holds, but it is not clear how much greater τ_A and t_b must be than τ_I in order for the theory to be a good approximation. It is easy to see that there are some difficulties with the diffusion picture associated with the low-energy part of the phonon spectrum. Consider, the value of r^2 averaged over the phonon distribution and weighted by energy, i.e.,

$$\langle r^2 \rangle = \int dE \int d^3r\, r^2 n(E,r,t) E / E_{\text{tot}} \; , \tag{28}$$

where E_{tot} is the total energy of the phonons. $\langle r^2 \rangle$ will increase with time as n changes according to Eq. (21).

The change in n resulting from the operation $\hat{S}$ does not lead to any change in $\langle r^2 \rangle$ since it simply changes the energy distribution of the phonons at a particular space point. The diffusion term in (21) leads to the following result:

$$\frac{d}{dt}\langle r^2 \rangle = 6 \int dE \int d^3r \, n(E,r,t) E D(E)/E_{\text{tot}} \ . \quad (29)$$

Now the scaling solution for n is $\propto E^2$ at low E and $D(E) \propto E^{-4}$. Hence, the integral over E is logarithmically divergent at low energy, and so the diffusion theory gives

$$\frac{d}{dt}\langle r^2 \rangle = \infty \ , \quad (30)$$

which is clearly impossible since the sound velocity sets an upper limit on the velocity of propagation.

The difficulty arises because there are always some phonons of low enough energy that the diffusion approximation breaks down. We can look at the problem in an alternate way by dividing up the range of phonon energies into distinct regions as follows. Suppose we are interested in the propagation of phonons a distance L from the source. A phonon of energy E will remain at this energy for a time of about $\tau_A = (AE^5)^{-1}$. In this time the phonon can diffuse a distance

$$l(E) = [D(E)\tau_A(E)]^{1/2} = v_D[\tau_I(E)\tau_A(E)/3]^{1/2} \ . \quad (31)$$

Let the energy at which $l(E)$ becomes equal to L be E_1. Hence, phonons with $E > E_1$ are unlikely to reach the detector without further down conversion. Phonons with $E < E_1$ stand a good chance of reaching the detector before undergoing further anharmonic decay. However, these phonons are in two distinct classes. If E is less than a critical energy E_2 they will be able to travel *directly* to the detector without further isotope scattering. E_2 is such that

$$L = v_D \tau_I(E_2) \ . \quad (32)$$

Between E_1 and E_2 the phonons will *diffuse* to the detector, i.e., they will undergo a series of isotope scatterings. The energies E_1 and E_2 are

$$E_1 = (v_D^2/3L^2AB)^{1/9} \ , \quad (33)$$

$$E_2 = (v_D/LB)^{1/4} \ . \quad (34)$$

The situation is summarized in Fig. 2. Initially, the phonons are in region α with high energy and an extremely small diffusion coefficient. They cannot reach the detector before down converting. In β they can diffuse to the detector without further down conversion and in γ they are able to reach the detector by ballistic propagation.

In the KL theory, which is based upon the diffusion equation, there is the implicit assumption that all the phonons reaching the detector do so while in the energy range β. This assumption is implicit because phonons in γ have such a long mean free path that they cannot be described by a diffusion coefficient. Recall now that when phonons down convert the probability distribution for the energy of the decay products is given by $P(E',E)$

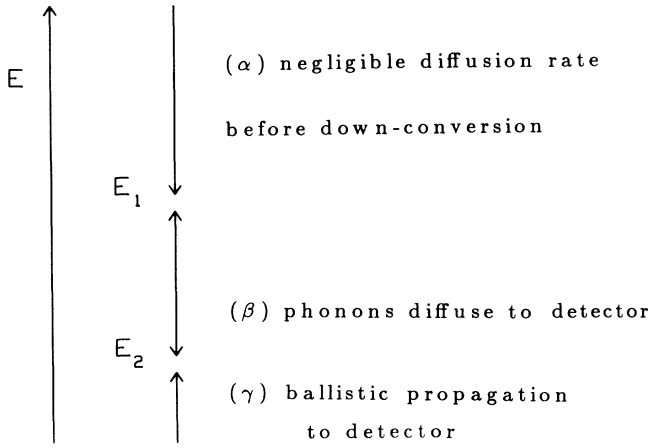

FIG. 2. Diagram showing the different nature of phonon propagation in different energy ranges.

[Eq. (11)]. Although this probability is peaked at $E = E'/2$, it is a broad distribution, and there will always be a certain number of phonons which pass directly in one decay from the region α to the region γ. These phonons will arrive at the detector at a time close to the ballistic time t_b, and are not accounted for by the scaling theory.

One can see, therefore, that for the scaling picture to be a good approximation one needs the condition

$$N \equiv \frac{E_1}{E_2} = \left[\frac{B^5 L}{81 A^4 v_D} \right]^{1/36} \gg 1 \ . \quad (35)$$

If, for example, we wanted to have N greater than some critical value N_c, we then need to have the isotope scattering strength B larger than a critical value B_c given by

$$B_c = \left[\frac{81 A^4 v_D}{L} \right]^{1/5} N_c^{36/5} \ . \quad (36)$$

Thus, to have N_c modestly large, e.g., 5, the isotope scattering rate must be larger than the anharmonic decay rate by a huge factor. Hence, we expect that the KL theory, while formally correct in the extreme limit of strong isotope scattering, may be an inadequate approximation in many practical cases. In the next section we investigate this by numerical methods.

Equation (35) gives the interesting result that $N \propto L^{1/36}$. Thus, the KL theory should become a better approximation as the distance from the source to the detector increases. However, the extremely small power of L means that the accuracy of the approximation changes very slowly as L is varied. Hence, for practical purposes the extent to which phonon propagation in a particular crystal is well described by the scaling theory is almost unaffected by the crystal size.

V. COMPUTER CALCULATIONS

We carried out Monte Carlo calculations in which a sequence of phonons of energy E_0 were generated at the

origin with wave vector in a random direction. The phonon speed was always v_D. Isotope scatterings and anharmonic decays occurred at the rates given by Eqs. (6) and (7), and the probability distribution of the decay products was governed by Eq. (11). After an isotopic scattering the phonon wave vector was assigned a random direction, and after anharmonic decay the wave vector was left unchanged. The propagation was continued until the phonon first reached a distance L from the origin. Thus, the simulation represents the energy received at a detector on the surface of a spherical sample of radius L, the surface of the sample being considered as a perfect absorber of phonons. In the simulation histograms were constructed for the phonon energy flux at the detector as a function of time, and for the number of arriving phonons as a function of energy.[9,18]

In most of the simulations about 30 000 phonons were allowed to obtain the histograms.

A. Test of scaling theory

If the initial E_0 of the phonons is sufficiently high the results are expected to be independent of this quantity, regardless of whether or not scaling theory holds. Phonon propagation is then described by the parameters B, A, v_D, and L from which one dimensionless quantity can be formed. This can be chosen to be N [Eq. (35)] or

$$B^* \equiv B/(A^4 v_D/L)^{1/5}, \tag{37}$$

which can be regarded approximately as the ratio of the strength of isotope scattering to anharmonic decay. Thus, all systems with the same N or B^* should have the same time dependence of the received signal when expressed in reduced time $t^* \equiv t/t_b$. This time dependence for a representative set of values of B^* is shown in Fig. 3. One can see that even for $B^* = 3000$ the scaling theory is only a rough approximation, since there is still a significant signal that arrives at close to the time expected for ballistic phonons. For $B^* = 3000$, $N = 2.69$. Thus, the energy range $E_1 - E_2$ of the β region in Fig. 2 is still only 1.69 times the width of the γ region. Hence, even for isotope scattering this strong there is a fairly large probability that a phonon will pass directly from the α region to γ and then arrive at the detector nearly at the ballistic arrival time.

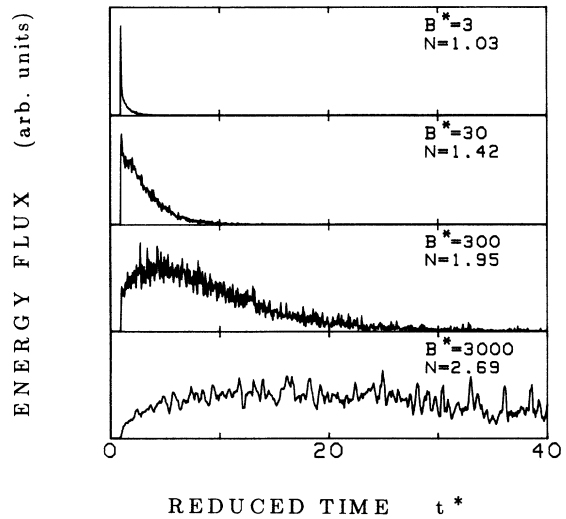

FIG. 3. Energy flux arriving at a detector as a function of the reduced time $t^* \equiv t/t_b$, where t_b is the ballistic arrival time L/v_D.

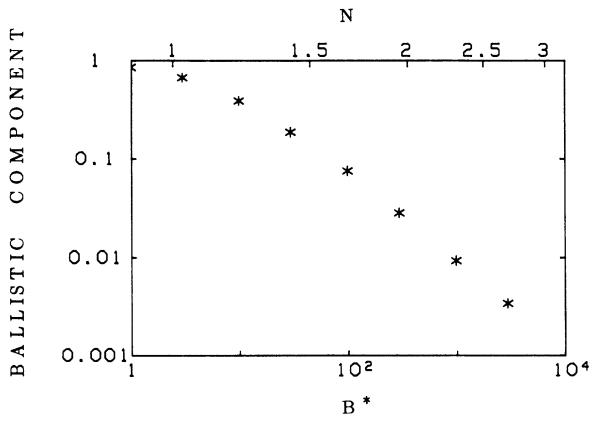

FIG. 4. Ballistic component to the signal as a function of B^* and N. The ballistic component is defined as the part of the energy flux that arrives no later than 1.2 times the time t_b it takes ballistic phonons to reach the detector. B^* and N are defined in Eqs. (37) and (35), and measure the relative strengths of the isotopic and anharmonic scattering.

In Fig. 4 we show the strength of the "ballistic" signal versus N and B^*. The ballistic signal is defined as those phonons arriving no later than 1.2 times the ballistic arrival time L/v_D.

In Figs. 5 and 6 we show the average energy $\langle E \rangle$ and the average arrival time $\langle t \rangle$ as a function of N and B^*. The energy is divided by E^*, defined by

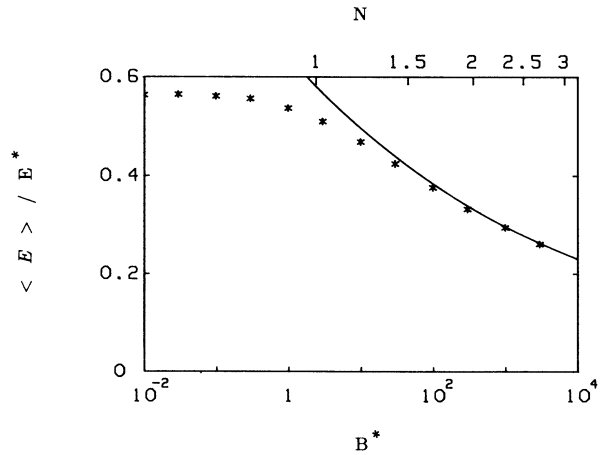

FIG. 5. Average energy $\langle E \rangle$ of the phonons arriving at the detector as a function of B^* and N. E^*, B^*, and N are defined in Eqs. (38), (37), and (35), respectively. The solid line shows $\langle E \rangle$ proportional to $B^{*-1/9}$ as predicted by scaling theory [Eq. (40)].

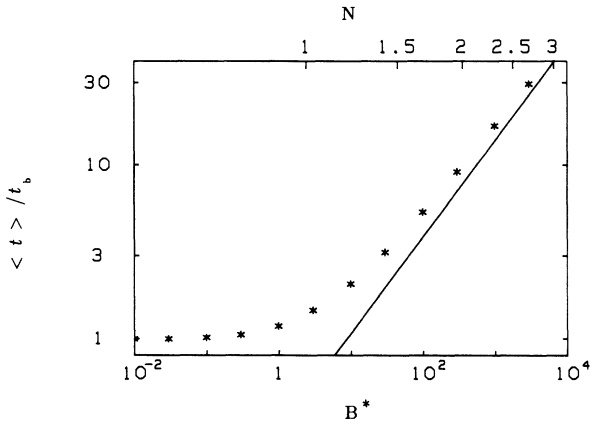

FIG. 6. Average arrival time $\langle t \rangle$ (*) of phonons at the detector as a function of B^* and N. t_b is the arrival time for phonons propagating ballistically, and B^* and N are defined in Eqs. (37) and (35). The solid line shows $\langle t \rangle$ proportional to $B^{5/9}$ as predicted by scaling theory [Eq. (39)].

$$E^* \equiv (v_D / AL)^{1/5} . \tag{38}$$

From the scaling theory (Sec. III) the arrival time of the phonons should vary as

$$\langle t \rangle \propto \frac{L^{10/9} B^{5/9}}{v_D^{10/9} A^{4/9}} \propto \frac{L}{v_D} B^{*5/9} \tag{39}$$

and so

$$\langle E \rangle \propto \frac{v_D^{2/9}}{L^{2/9} A^{1/9} B^{1/9}} \propto \frac{E^*}{B^{*1/9}} . \tag{40}$$

For large enough B, $\langle t \rangle$ and $\langle E \rangle$ do depend on B approximately as predicted by Eqs. (39) and (40) (see Figs. 5 and 6).

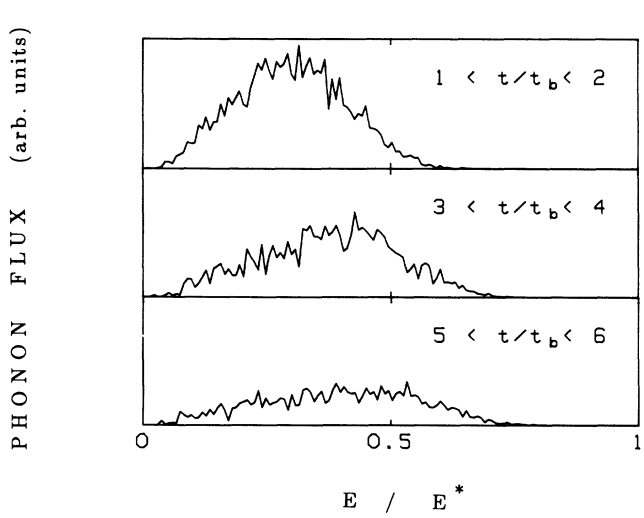

FIG. 7. Energy distribution of phonons arriving at the detector in different time intervals as indicated. E^* is defined in Eq. (38). t_b is the arrival time for ballistic phonons. The calculation is for $B^* = 100$.

These numerical results were obtained by starting with phonons of energy $10E^*$. The results are unaffected if the initial energy is increased above this value. The spectrum of phonons arriving at the detector is in all cases similar to the scaling form shown in Fig. 1. However as expected from the discussion we have given, the phonons that arrive at close to the ballistic arrival time have a significantly lower energy than those arriving later. In Fig. 7 we show this effect for $B^* = 100$. It is interesting that the later-arriving phonons have higher energies even though they have, of course, had more time to down convert.

B. Calculations for Si and Ge

As an example of these effects, we have carried out calculations for Si and Ge. Of course, since we consider just one phonon polarization and neglect phonon focusing and other anisotropic effects, the results are only qualitative. The spontaneous decay rates of acoustic phonons have been calculated by Berke, Mayer, and Wehner (BMW).[19] Using the same elastic constants as they do, we obtain the Debye velocities listed in Table I. From the known isotopic abundances we then obtain values of B, in essential agreement with other workers.[10,20] To obtain a reasonable value of A is more complicated because A depends on propagation direction and polarization. BMW present their results in terms of the quantity $a \equiv \Gamma / q^5$, where Γ is the decay rate of the phonon amplitude ($\tau = 1/2\Gamma$) and q is the phonon wave vector. In Si the average over directions of a is 5.5×10^{-30} cgs for longitudinal phonons and 2×10^{-31} cgs for fast transverse phonons. For slow transverse phonons, a is believed to be small[19] provided that there is some phonon dispersion to eliminate collinear processes. From these results we obtain average decay rates for longitudinal phonons of

$$\frac{1}{\tau} = A_L E^4 \tag{41}$$

with $A_L = 7.0 \times 10^{-4}$ sec^{-1}K^4. For fast transverse phonons the corresponding formula has $A_{FT} = 2.6 \times 10^{-4}$ sec^{-1}K^{-4}. To obtain an average of A over the whole spectrum we weight the A's for different polarizations by the number density of phonons in the different branches (proportional to v^{-3}), and obtain the final value[21] for Si shown in Table I.

For Ge, BMW calculate the decay rates for longitudinal phonons only. This rate (expressed in terms of a) averaged over the (100), (110), and (111) directions is 0.35

TABLE I. Debye velocities obtained using the elastic constants of Berke, Mayer, and Wehner (Ref. 19).

	Si	Ge
Debye velocity v_D (10^5 cm sec^{-1})	5.91	3.55
A (sec^{-1}K^5)	1.6×10^{-4}	7.4×10^{-4}
B (sec^{-1}K^{-4})	0.46	7.0

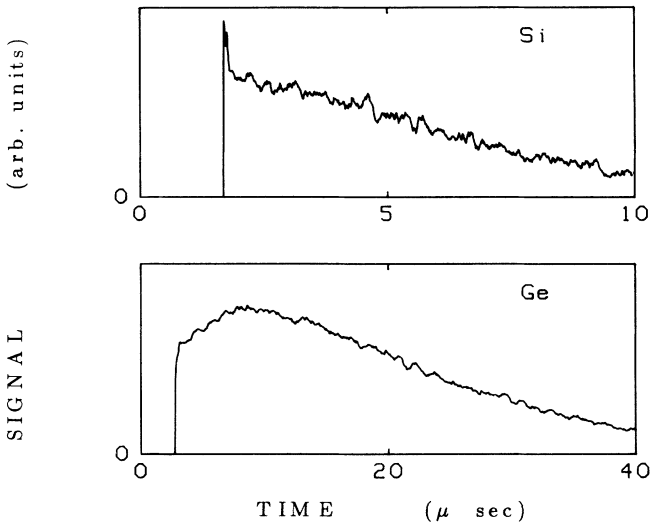

FIG. 8. Energy flux as a function of time for propagation in Si and Ge. The detector distance is taken to be 1 cm.

times the rate for Si. To estimate A for Ge we have therefore taken the A for Si, multiplied it by 0.35, and divided by the ratio of v_D^5 in the two crystals.

Given values of v_D, A, and B, we can calculate the quantity B^* for a given distance L to the detector. The result is $B^*=35$ for Si and 174 for Ge. L is taken to be 1 cm, but as noted earlier the dependence of B^* on L is very weak. In Fig. 8 we show the results of a computer simulation of phonon propagation in these crystals. The energy of the injected phonons was $10E^*$, which is 820 and 540 K for Si and Ge, respectively. The average energy of the phonons arriving at the detector is 34 and 19 K for Si and Ge, respectively. Thus the average number of decays was between 4 and 5. Of course, without anharmonic decays the signal would have no component arriving at close to the ballistic time and would have an average arrival time much later.

It is clear from the results of Fig. 8 that one cannot expect scaling theory to apply to these crystals, at least for sample sizes in the normal range. To have the propagation reasonably well described by the scaling theory one needs $B^* \gtrsim 3000$. There are probably no crystals in which the scattering due to the naturally occurring mix of isotopes gives this large value of B^*. Thus, the scaling theory is only likely to apply well to alloy crystals (e.g., Si-Ge alloys) or artificial isotopic mixtures such as ^{3}He-^{4}He crystals.

VI. SUMMARY

We have given an analysis of the way in which high-energy phonons injected into a crystal at $T=0$ K propagate under the influence of isotope scattering (elastic scattering) and anharmonic decays. The principal result is that the scaling theory of Kazakovtzev and Levinson is a reasonable approximation only when the elastic scattering is extremely strong, and that as a consequence the theory is probably not useful for crystals where the scattering arises only from the different masses of the naturally occurring isotopes.

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