# Bean's, Kim's, and exponential critical-state models for high-$T_c$ superconductors

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Bean's, Kim's, and the exponential-law critical-state models have been used to calculate the magnetization curves $M(H)$ of hard superconductors assumed to have an infinitely long geometry with rectangular cross section of $2a \times 2b$. Some computed $M(H)$ curves are given to illustrate our analytical results for different $b/a$ and other relevant parameters. These results can be satisfactorily applied to many experimental data, particularly in the study of high-$T_c$ superconductors. A brief analysis on this is also given.

For hard superconductors, when the applied field is above the lower critical field, $H_{c1}$, the supercurrent penetrates from the surface inwards, and follows the critical-state model. This model assumes that penetrated supercurrents flow in every macroscopic region with a density equal to the critical current density $J_c(H_i)$, where $H_i$ is the local internal field.[1,2] The supercurrent penetrated region (either a part or the whole of the sample) with current density $J=J_c(H_i)$ is said to be in a critical state. In the critical state, the flux lattice should be in equilibrium, without flux creep or flux flow. However, in most practical cases, the magnetic field changes so slowly that we can consider the sample to be in a quasiequilibrium state, and still use the critical-state model to calculate the magnetization curves accurately enough.

If $H_{c1}$ is negligible, the magnetization curves $M(H)$ will be dominated by the critical-state model. Since the susceptibility of high-$T_c$ superconductors, recently discovered, approaches $-1$ at very low fields, their $H_{c1}$ can be considered as zero. Therefore, we expect that the critical-state model can be ideal for the $M(H)$ derivation and the $J_c$ determination from magnetic measurements of high-$T_c$ superconductors. The problem is that these materials are granular in nature, and their electromagnetic properties have two contributions, from both high-$H_{c1}$ grains and the matrix or grain-boundary network, which is normal or poorly superconducting.[3] However, for many high-$T_c$ superconductors, the $H_{c1}$ of the grains $H_{c1g}$ is rather high ($>10^4$ A/m) at $T \ll T_c$. If we consider $H < H_{c1g}$, then we shall have a simple case for a partial $M(H)$ curve, which is only determined by the critical-state model. In this case, the magnetization should be expressed as

$$M = fM_g + (1-f)M_m = -fH + (1-f)M_m , \quad (1)$$

where $f$ is the effective volume fraction of the grains, and $M_g$ and $M_m$ are the partial magnetizations of the grains and the matrix (or grain-boundary network), respectively. The real magnetization in the grains is smaller than $-H$ below $H_{c1g}$ due to a flux penetration. In Eq. (1) we use a temperature-dependent $f$ smaller than the real volume fraction of the grains to maintain $M_g = -H$. From Eq. (1) we obtain

$$M_m = (M + fH)/(1-f) . \quad (2)$$

Similarly, the complex susceptibility will be[4]

$$\chi = -f + (1-f)\chi_m , \quad (3)$$

where $\chi_m$ is the partial susceptibility of the matrix, and the susceptibility of the grains has been taken as $-1$, because we consider $H$ below $H_{c1g}$. Its real and imaginary components then are

$$\chi'_m = (\chi' + f)/(1-f) , \quad (4a)$$

$$\chi''_m = \chi''/(1-f) . \quad (4b)$$

From Eqs. (2), (4a), and (4b), if we know $f$, then $M_m$ or $\chi'_m$ and $\chi''_m$ can be derived from the measured $M(H)$ or $\chi'(H_a)$ and $\chi''(H_a)$ curves, where $H_a$ is the amplitude of the applied field. Thus, we can use the critical-state model to fit $M_m(H)$ or $\chi'_m(H_a)$ and $\chi''_m(H_a)$ curves and obtain intergranular $J_c$. To do this, we need analytical solutions of $M(H)$ for different critical-state models in some practical sample shapes. Up to now, for all the critical-state models, the existing $M(H)$ curves were calculated for infinite cylinders or slabs, which may be enough for the explanation of the $M(H)$ curve or a rough estimation of $J_c$, but certainly not sufficient for more accurate quantitative investigation. We[5,6] derived in this work, for the first time, $M(H)$ curves for infinitely long orthorhombic samples based on Bean's, Kim's, and the exponential-law critical-state models.[1,2,7]

For $H > H_{c1g}$, the intergranular $J_c$ is almost zero and our results can be used for the high-field $M(H)$ curves and intragranular $J_c$ determination, although in this case the sample shape is irrelevant, and an unlinear reversible $M(H)$ component has to be considered.

It has been found[8,9] that the high-$T_c$ superconductors show an exponential field dependence of $J_c$ in single crystals. This means that, since there were no explicit calculations of the $M(H)$ loops (only in Ref. 10 were there given some numerical calculations of the loops for simple infinite slabs, although involving some error), our equations can be very useful for fitting a large number of experimental data. The analytical results of $M(H)$ loops for the exponential model have recently been published for an infinite slab and cylinder in Ref. 11.

In this Brief Report, we report our results based on Bean's, Kim's, and the exponential critical-state models. The sample shape chosen for our calculations is an infinitely long column with rectangular cross section $2a\times 2b (b\geq a)$. On deriving $M(H)$ curves, the supercurrent path has been considered to be rectangular with equal distance to the sample sides, which is a direct deduction from the critical-state model.[6]

Bean's, Kim's, and the exponential models assume that $J_c(H_i)$ can be written as

$$J_c(H_i)=k_B\,, \tag{5a}$$

$$J_c(H_i)=k_K/(H_{0K}+|H_i|)\,, \tag{5b}$$

$$J_c(H_i)=k_E\exp(-|H_i|/H_{0E})\,, \tag{5c}$$

respectively. The corresponding full penetration fields will be

$$H_p=k_B a\,, \tag{6a}$$

$$H_p=H_{0K}[(1+p_K^2)^{1/2}-1]\,, \tag{6b}$$

$$H_p=H_{0E}\ln(1+p_E)\,, \tag{6c}$$

where $p_K$ and $p_E$ are the following parameters:

$$p_K=(2k_K a)^{1/2}/H_{0K}\,, \tag{7a}$$

$$p_E=k_E a/H_{0E}\,. \tag{7b}$$

To illustrate our results, we have chosen $b/a=1$, 2, and 100. It can be proved that the results for $b/a=1$ are the same as those for a cylinder of radius $a$. The results for $b/a=100$ can be considered as an infinite slab of thickness $2a$. For comparison among the different models, we have normalized all the $M(H)$ values to $H_p$.

Some computed $M(H)$ curves are shown in Figs. 1-3 for Bean's, Kim's, and exponential models, respectively. Figures 1(a), 2(a), and 3(a) show the initial curves and the loops with different $H_m$. We observe that, in contrast to Bean's model, the other two models present a minimum in their initial curves and the middle part of the loops is much wider than in both sides. Also, the low-$H_m$ loops (that is, for $H_m=0.5H_p$) for Kim's and the exponential models are narrower than for Bean's model.

Figures 1(b), 2(b), and 3(b) show the dependence on the ratio $b/a$ of high-$H_m$ loops for the three models. We observe that the width of the loops increases with increasing the ratio $b/a$. Actually, there is a relationship be-

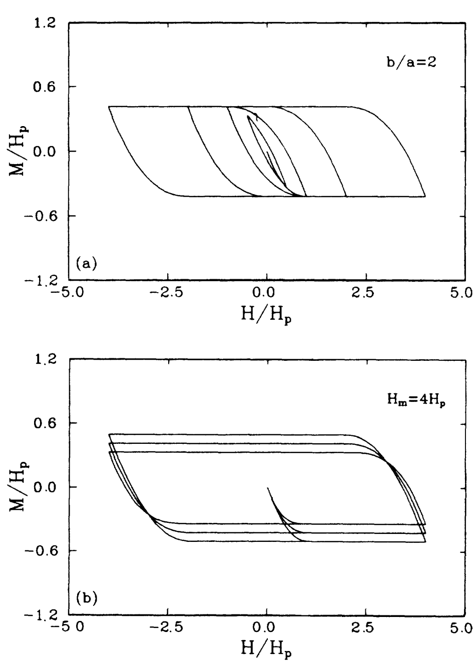


FIG. 1. Computed $M(H)$ curves scaled by $H_p$ for Bean's model. The calculation conditions are (a) $b/a=2$ and $H_m=0.5H_p$, $H_p$, $2H_p$, and $4H_p$; (b) $H_m=4H_p$ and $b/a=1$ (smallest), 2, and 100 (largest).

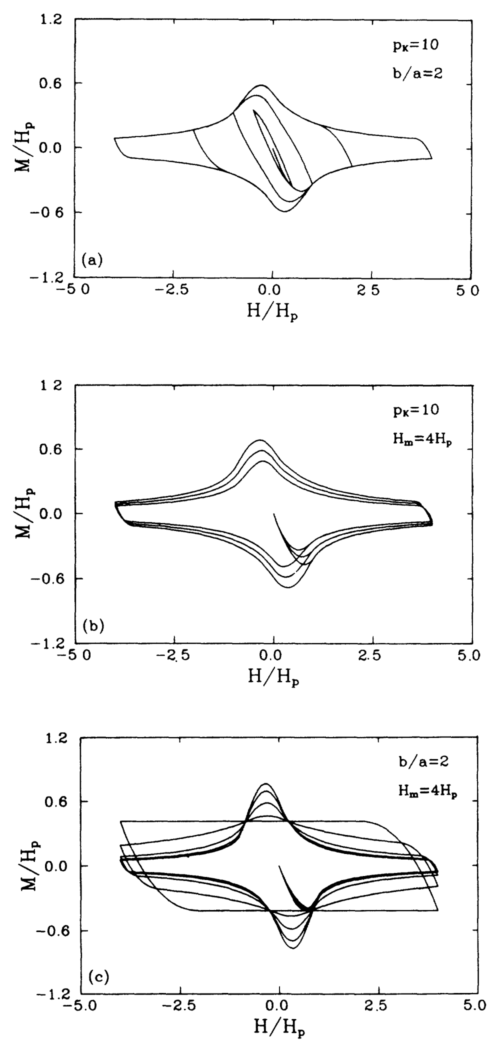


FIG. 2. Computed $M(H)$ curves, scaled by $H_p$, for Kim's model. The calculation conditions are (a) $b/a=2$, $p_K=10$, and $H_m=0.5H_p$, $H_p$, $2H_p$, and $4H_p$; (b) $H_m=4H_p$, $p_K=10$, and $b/a=1$ (smallest), 2, and 100 (largest); and (c) $b/a=2$, $H_m=4H_p$, and $p_K=0$ (flat), 1, 10, 100, and $\infty$ (sharpest).

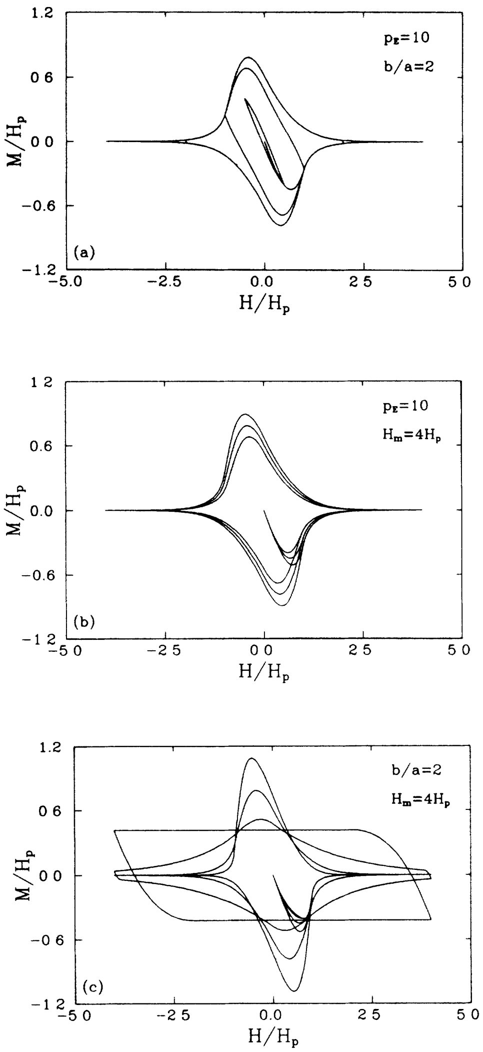


FIG. 3. Computed $M(H)$ curves, scaled by $H_p$, for the exponential model. The calculation conditions are (a) $b/a=2$, $p_E=10$, and $H_m=0.5H_p$, $H_p$, $2H_p$, and $4H_p$; (b) $H_m=4H_p$, $p_E=10$, and $b/a=1$ (smallest), 2, and 100 (largest); and (c) $b/a=2$, $H_m=4H_p$, and $p_E=0$ (flat), 1, 10, and 100 (sharpest).

tween $J_c$ and the width of the loop

$$\Delta M(H)=a(1-a/3b)J_c(H)\,, \tag{8}$$

which is generally valid when both the ascending and descending branches of the loop are in full-penetrated states and $|H|$ is reasonably larger than $H_p$. From Eq. (8) we see that when $b/a=1$, 2, and $\infty$, then $\Delta M(H)/aJ_c(H)=\frac{2}{3}$, $\frac{5}{6}$, and 1, so that the loops for different $b/a$ shown in the figures are equally distant in a large $H$ interval.

Figures 2(c) and 3(c) show the $p_K$- and the $p_E$-dependence of the high-$H_m$ loops. The curves of $p_K=p_E=0$ correspond to Bean's model. With increasing $p_K$ or $p_E$, the middle part of the loops becomes wider and the sides are compressed. In the figures we have not shown the loop for the exponential model that corresponds to $p_E\rightarrow\infty$, which is out of scale and shaped like a parallelogram with a maximum $M/H_p=2$.

In the case of high-$T_c$ superconductors, since the intergranular $J_c$ is often originated from Josephson junctions, which gives a $1/H$ behavior of $J_c$ at high $H$, Kim's model is expected to be the most satisfactory one. For this model, Fourier analysis of the loops gives that the maximum $\chi_m''$ ranges from 0.21 to 0.40 when $b/a=1$ and from 0.24 to 0.45 when $b/a\rightarrow\infty$. These $\chi_m''$ values are consistent with the ac susceptibility experimental data for many superconductors. The low-$p_K$ values of the maximum $\chi_m''$ can be experimentally found in some thin samples. There are two reasons for that: (1) when $a$ is small, the $p_K$ is also small, and (2) $J_c$ decreases with $H$ slower than $1/H$ at lower fields. For some samples, the experimental maximum $\chi_m''$ sometimes is even larger than the high-$p_K$ limit. In this case, we need the exponential model, which can give a maximum of $\chi_m''$ of 1.27 when $p_E\rightarrow\infty$. Further theoretical and experimental work will be published elsewhere.

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