Scattering from a magnetic strip: Analytic description of transmission and conductance

Y. Avishai
Department of Physics, Ben-Gurion University, Beer Sheva, Israel

Y. B. Band
Allied Signal Inc., Corporate Technology Center, Morristown, New Jersey 07960
and Department of Chemistry, Ben-Gurion University, Beer Sheva, Israel
(Received 14 November 1989)

The transmission and reflection amplitudes for charged particle scattering from a magnetic field in an infinite strip $\{0 \le x \le X, -\infty < y < \infty\}$ are obtained analytically, and the conductance per unit transverse length g is given in closed form. It is found that the dimensionless quantity hg/e^2k scales with the variable $R = X/kL^2$ (where k^2 is the electron energy and L is the magnetic length) and vanishes without residual tunneling if the strip width exceeds the cyclotron diameter ($R > 2$). In the region $(0 < R < 2)$, g decreases monotonically with R without Aharonov-Bohm oscillations.

Recently, there has been great interest in experimental and theoretical investigations of the ballistic conductance and magnetoconductance of small systems in which electron motion is geometrically confined.[1] In two terminal systems, confinement leads to the phenomenon of quantized conductance,[1,2] which persists also in the presence of magnetic fields.[2,3] In the context of multiterminal conductance measurements,[4] it has been suggested that the effect of confinement gives rise to the quantized Hall effect due to conduction via edge states.[5] But what happens when the motion of the electrons is not confined and no edge states exist, e.g., in an infinitely long strip containing a magnetic field? What is the dependence of the magnetoconductance on strip width and field strength? Here we present the analytic solution for the reflection and transmission coefficients and the conductance per unit transverse length for the case of a constant magnetic field confined to an infinitely long strip. We find that the conductance per unit length decreases with increasing magnetic field and with strip width, as is classically expected, but scales with Fermi energy, magnetic field, and strip width via the dimensionless ratio, strip width divided by the cyclotron radius. It vanishes identically for strip widths larger than the cyclotron diameter. The quantum-mechanical conductance is smaller than the classical conductance when the conductance is finite.

Consider the quantum-mechanical motion in the X-Y plane of a charged particle with (effective) mass m, charge e, and Fermi energy $E = \hbar^2 k^2/2m$. A constant magnetic field $B\hat{\mathbf{z}}$, perpendicular to the plane is active in the vertical strip $0 \le x \le X$. We choose the Landau gauge for the vector potential $\mathbf{A}$, i.e., $A_y = 0$ $(x \le 0)$, $A_y = Bx$ $(0 \le x \le X)$, $A_y = BX$ $(X \le x)$. The Schrödinger equation in the three domains takes the form

$$\left[-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right]\psi(x,y) = k^2\psi(x,y) \quad (x \le 0)\ , \tag{1a}$$

$$\left[-\frac{\partial^2}{\partial x^2} + \left[i\frac{\partial}{\partial y} + \frac{x}{L^2}\right]^2\right]\psi(x,y) = k^2\psi(x,y) \quad (0 \le x \le X)\ , \tag{1b}$$

$$\left[-\frac{\partial^2}{\partial x^2} + \left[i\frac{\partial}{\partial y} + \frac{X}{L^2}\right]^2\right]\psi(x,y) = k^2\psi(x,y) \quad (X \le x)\ , \tag{1c}$$

where $L = \sqrt{\hbar c/|eB|}$ is the magnetic length. We impose plane-wave boundary conditions in the x direction in the regions $x \le 0$ and $x \ge X$. The wave function within the strip is determined from Eq. (1b) by imposing continuity of the wave function and its derivatives with respect to x at the boundaries $x = 0, X$. In the y direction we impose periodic boundary conditions with period $Y = Y_p = 2\pi(L^2/X)p$, where the integer p controls the length of the system in the transverse direction. We intend to study an infinite strip and eventually let $p \to \infty$, and our results will be independent of the precise sequence Y_p. The reason for the restriction $Y = Y_p$ will become clear shortly.

A complete and orthonormal set of functions for the transverse motion in the regions $x \le 0$ and $x \ge X$ is given by $\phi_m(y) = e^{iq_m y}/\sqrt{Y}$, $q_m = 2\pi m/Y$ $(m = 0, \pm 1, \pm 2, \ldots)$. The energies of the transverse and longitudinal motions are q_m^2 and $k_m^2 = k^2 - q_m^2$, respectively. There are $2N+1$ channels for $k_m^2 > 0$ $(m = -N, -N+1, \ldots, N)$. All other channels are evanescent but are nevertheless treated exactly.

An important point in the construction of the solution is that, while the energy of the wave function in the region $0 \le x \le X$ cannot be separated, the wave function can nevertheless be expressed as the product $\psi(x,y) = \phi_m(y) f_m(x)$. The functions $f_m(x)$ satisfy a second-order differential equation with no finite singular points,

$$\left[-\frac{\partial^2}{\partial x^2} + \left[\frac{x}{L^2} - q_m \right]^2 \right] f_m(x) = k^2 f_m(x) . \quad (2)$$

Since we need the solution of Eq. (1b) only in the finite domain $0 \le x \le X$, we need the two linearly independent solutions of Eq. (2) to match wave function and derivative at 0 and X. These are the parabolic cylinder (Weber) functions,[6] $f_m^{(1)}(x) = y_1(\nu, z(x,q_m))$, $f_m^{(2)}(x) = y_2(\nu, z(x,q_m))$, where $\nu = -[(kL)^2 - 1]/2$, and $z(x,q_m) = \sqrt{2}(x/L - q_m L)$.

The solution of Eqs. (1a)–(1c) corresponding to boundary conditions of an initial channel n incident from the left, reflected waves to the left and transmitted waves to the right is as follows:

$$\psi_n(x,y) = \begin{cases} \phi_n(y) e^{ik_n x} + \sum_{m=-\infty}^{\infty} \phi_m(y) e^{-ik_m x} R_{mn} & (x \le 0) , \quad (3a) \\ \sum_{m=-\infty}^{\infty} \phi_m(y) [f_m^{(1)}(x) a_{mn} + f_m^{(2)}(x) b_{mn}] & (0 \le x \le X) , \quad (3b) \\ e^{iXy/L^2} \sum_{m=-\infty}^{\infty} \phi_m(y) e^{ik_m(x-X)} T_{mn} & (X \le x) . \quad (3c) \end{cases}$$

The coefficients a_{mn} and b_{mn} as well as the reflection and transmission matrices R_{mn} and T_{mn} are determined by the matching conditions at $x=0$ and X. Due to our special choice of the transverse dimension $Y = Y_p$ the gauge transformation does not spoil the boundary conditions and the wave function $\psi_n(x,y)$ in Eq. (3c) is indeed periodic in y, since $X/L^2 = (2\pi/Y)p$ and therefore

$$\exp(iXy/L^2)\phi_m(y) = \phi_{m+p}(y) .$$

We can now use the orthonormality and completeness of $\{\phi_m(y)\}$ to obtain matching equations at $x=0,X$. After some algebra we find the (flux normalized) transmission matrix,

$$t_{mn} = \left[\frac{k_m}{k_n} \right]^{1/2} T_{mn} = J \left[\frac{k_m}{k_n} \right]^{1/2} \frac{2ik_{m-p}}{D(q_{m-p})} \delta_{m-p,n} , \quad (4)$$

where $J = W\{f_m^{(2)}(x), f_m^{(1)}(x)\}$ is the Wronskian [$J = (\sqrt{2}/L)$ and D is the determinant],

$$D(q_m) = \det \begin{bmatrix} f_m^{(1)\prime}(x_1) + ik_m f_m^{(1)}(x_1) & f_m^{(2)\prime}(x_1) + ik_m f_m^{(2)}(x_1) \\ f_m^{(1)\prime}(x_2) - ik_{m-p} f_m^{(1)}(x_2) & f_m^{(2)\prime}(x_2) - ik_{m-p} f_m^{(2)}(x_2) \end{bmatrix} . \quad (5)$$

We need to compute the sum over channels to evaluate the conductance G,[7]

$$G = \frac{e^2}{h} \sum_{mn} |t_{mn}|^2 = 4|J|^2 \frac{e^2}{h} \sum_{n=-N}^{N-p} \frac{k_{n+p} k_n}{|D(q_{n+p})|^2} . \quad (6)$$

The last equality results from the occurrence of $(\delta_{m-p,n})^2 = \delta_{m-p,n}$ in the double sum over channels m and n. The sum over channels in Eq. (6) is limited to those with real momenta. The significance of this restriction will be discussed shortly.

We now divide G by the transverse length Y to obtain the conductance per unit width $g = G/Y$, and then go to the limit $Y \to \infty$ by letting $p \to \infty$ in the relation $Y = Y_p = 2\pi(L^2/X)p$. In this limit we use

$$\sum_n F(q_n)/Y \to (1/2\pi) \int F(q) dq ,$$

$$k_n \to k(q) = (k^2 - q^2)^{1/2} ,$$

$$k_{n+p} \to k(q + X/L^2) .$$

The transmission amplitudes of Eq. (4) now become

$$t(q',q) = J \left[\frac{k(q')}{k(q)} \right]^{1/2} \frac{2ik(q' - X/L^2)}{D(q')} \times \delta \left[q' - \left[q + \frac{X}{L^2} \right] \right] \quad (7)$$

Here, the quantity X/L^2 is the momentum transferred in the course of passage through the strip, and the δ function represents momentum balance in the transverse direction. Hence, the present problem is not strictly single channel, since the transmission matrix is not diagonal. However, each initial channel is transmitted into only one definite final channel. We finally obtain

$$g = \frac{e^2}{h} \frac{2}{\pi} |J|^2 \int_{-k}^{k - X/L^2} \frac{k(q + X/L^2) k(q)}{|D(q + X/L^2)|^2} dq . \quad (8)$$

This exact result, which gives the conductance of an infinite magnetic strip per unit transverse length in closed form, is independent of the manner in which Y tends to infinity. Note that in order to evaluate g from Eq. (8) we do not need to compute $|t(q',q)|^2$, thus we avoid the pathological Dirac δ^2 factor. Had we started with an infinite strip at the onset, we would have had to evaluate an integral over $dq\,dq'$, which includes a term $[\delta(q'-X/L^2-q)]^2$. The integral over dq' then gives G as $\delta(0)$ times the right-hand side of Eq. (8). The (somewhat artificial) limiting procedure adopted shows that the infinite term $\delta(0)$ occurs because the width of the system is infinite (which implies an infinite conductance) and that removing $\delta(0)$ is equivalent to calculating g rather than G. In other words, to avoid the integral over a square of a Dirac δ function we started with a discrete sum over channels that contains a square of a Kronecker δ function.

The physics behind the integration limits in Eq. (8) is transparent. The initial momentum $k(q)$ is real if the transverse momentum $|q|$ is less than the magnitude k of the total momentum. The final momentum $k(q+X/L^2)$ is real if the transverse momentum of the transmitted particle, $|q+Y/L^2|$, is less than k. These two conditions can be fulfilled only if $X<2\rho$, where ρ is the cyclotron radius, $\rho=kL^2$. Thus, once a complete circular orbit is accommodated within the strip, the transmission vanishes without residual tunneling due to momentum-energy conservation. Note that for particles with a horizontal trajectory, transmission vanishes for $X\geq\rho$. However, the conductance is calculated by summing over all channels, i.e., all transverse momenta, and therefore oblique trajectories can be transmitted provided $X<2\rho$. This is in concurrence with the reservoir picture,[8] since electrons are ejected from the reservoirs in all possible directions.

Direct numerical evaluation of the parabolic cylinder functions $y_1(\nu,z)$ and $y_2(\nu,z)$ is rather difficult if required for large ν and z (necessary for realistic values of k, L, and X). Fortunately, there exists a scaling law for the dimensionless quantity hg/e^2k. Clearly, this quantity must be expressible in terms of the only two independent dimensionless quantities in the problem, namely, kL and $R=X/kL^2=X/\rho$. However, hg/e^2k depends only on R. This remarkable scaling property is not anticipated. The scaling behavior simplifies calculation of the conductance, since parameters such that both ν and z are small can be chosen, and yet all the allowed values of $R\in[0,2]$ are spanned.

We have shown that $(h/e^2)(\pi g/k)=0$ for $R\geq 2$. Let us now consider the limit $R\to 0$. Here, the interaction with the magnetic strip switches off, and the sum of $|t_{mn}|^2$ given in Eq. (6) (divided by the transverse length Y) merely counts the number of physical channels per unit length. This can be understood by noting from (4) that summing $|t_{mn}|^2$ over m gives the ratio of outgoing to incoming flux for channel n. Hence the double sum in (6) yields the sum of such ratios for all physical channels, which in the continuum limit equals k/π. Therefore, $h\pi g/e^2k\to 1$ as $R\to 0$.

We can learn more about scaling by studying the classical conductance. Consider electron reservoirs to the right and left of the strip, with equal particle density N and Fermi velocity v. If a small potential difference, ΔV, is applied across the strip, a net current of particles with energies in the range $e\Delta V$ above the (quasi-) Fermi energy E_F flows from left to right. Particles from the right cannot contribute to the current at zero temperature because they would arrive with energies below the Fermi energy and the Pauli principle excludes this possibility. Particles from the left with energy in the pertinent energy range cross to the right provided their incident angle θ (measured from the y axis), is less than a critical angle, $\theta_L=\arccos(R-1)$. These particles result in a flux in the x direction of particles per unit energy emerging at the right end of the strip from a transverse length Y, given by $(dN/dE)Yv\sin\theta'$. Here $\theta'(\theta)$ is the angle between the final velocity and the y axis. The function relating θ' to θ must be determined from the classical equations of motion (see below). Summing over all incident angles (isotropy implies an angular probability distribution of $d\theta/2\pi$) we obtain the net current to the right from a portion Y of the strip in the linear response approximation,

$$I=Ye\Delta V\frac{dN}{dE}e\frac{1}{2\pi}\int_0^{\theta_L}v\sin\theta'(\theta)d\theta\ , \tag{9}$$

where, $\theta_L=\arccos(R-1)$. The integral yields the average of the normal component of the final velocity v_f.

To find the dependence of $\sin\theta'$ on θ we solve the equations of motion subject to the initial conditions $x(0)=0$, $v_x(0)=v_{x0}$, $v_y(0)=v_{y0}$, with $v_{x0}^2+v_{y0}^2=v^2$. Here θ is given by $\cos\theta=v_{y0}/v$. The solution for $x(t)$ is $x(t)=(v_{x0}\sin\omega t-v_{y0}\cos\omega t+v_{y0})/\omega$, where $\omega=eB/(mc)$. Setting $x(t)=X$, solving for $t=t(X)$, and substituting into the expression for $dx(t)/dt$, we obtain the final velocity in the x direction, from which $\sin\theta'$ is obtained in terms for θ:

$$dx(t)/dt|_{t=t(X)}=v\sin\theta'=v[1-(R-\cos\theta)^2]^{1/2}\ . \tag{10}$$

Thus, the classical conductance per unit transverse length, $g_c=I/(Y\Delta V)$, is given by

$$g_c=\frac{dN}{dE}ve^2\frac{1}{2\pi}\int_0^{\theta_L}[1-(R-\cos\theta)^2]^{1/2}d\theta\ . \tag{11}$$

In the "purely" classical theory there is no natural dimensionless quantity with units of conductance, and therefore g_c does not depend only on R. We can, however, "semiquantize" g_c in two stages: first we impose the de Broglie relation $mv=\hbar k$, and second we invoke the Pauli principle (without spin degeneracy) on the two-dimensional electron gas so that the density N is given by $N=k^2/4\pi$, and, therefore, $(dN/dE)v=k/h$. Substitution into Eq. (11) yields the semiquantized conductance per unit transverse length $g_{\rm sq}$,

$$g_{\rm sq}=\frac{ke^2}{2\pi h}\int_0^{\arccos(R-1)}\sqrt{1-(R-\cos\theta)^2d\theta}\ . \tag{12}$$

This expression coincides with the quantum-mechanical result for $R=0$ and $R>2$ (for $R=0$, $g_{\rm sq}=ke^2/\pi h$ and for $R\geq 2$, $g_{\rm sq}=0$).

In Fig. 1 we plot $\pi hg/ke^2$ as a function of R, as determined quantum mechanically and classically. The quantum conductance is always less than the classical result when the conductance is finite because of the finite probability of reflection off the boundary of the magnetic field in the quantum case. The quantum-mechanical conductance does not possess an oscillatory structure that might arise due to the Aharonov-Bohm effect.[9] Such effects

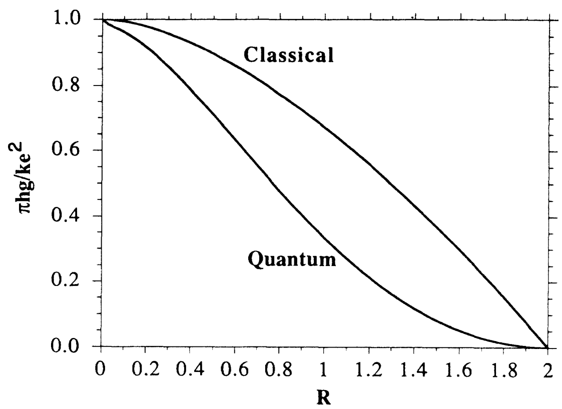

FIG. 1. The dimensionless quantity $\pi hg/ke^2$ as a function of the scaling variable $R = X/kL^2$ computed from Eq. (13). The electrical conductance per unit transverse length for arbitrary values of X, k, and L is equal to e^2/hg and can be readily deduced from this graph.

have been observed experimentally and studied theoretically in impure systems,[10] but recently they have also been studied experimentally[11] and theoretically[12] in pure systems where propagation is ballistic. In all these cases there is an areal scale (the mean free path squared for impure systems, system size for the quantum dot experiment), which determines the period of oscillation by the requirement that the flux through this area change by one unit. In the present case, however, there is no areal scale and oscillations do not appear.

From a theoretical point of view, the present results are remarkable because they predict that the conductance decays to zero without any residual tunneling. Practically, the presence of edges in a strip of finite transverse length would modify the results obtained here, since an edge current will flow along the boundary of the strip. Thus, even for $R > 2$, the conductance will be quantized, $G = ne^2/h$, where n is the number of edge states. Note, however, that the conductance per unit length will vanish as $Y \to \infty$, and that for finite Y the edge-state conductance can be subtracted from the experimentally determined conductance to determine the residual conductance.

Magnetoconductance of a two-dimensional magnetic strip can be studied in experiments on heterostructure interfaces. To estimate ranges of R and g, which can be studied in present day systems, let us assume a strip of width of 1 μ and a magnetic field of 1 T. Two-dimensional electron gas densities of $\approx 8 \times 10^{16}$ electrons/m^2 can be obtained and correspond to a Fermi momentum of the order of 0.1 Å^{-1}. This gives $R \approx 1.5$ and $g = 0.002\,55 e^2/\text{h Å} \approx 10$ mho/cm.

In conclusion, the quantum-mechanical transmission probability and the conductance vanish for barriers wider than the cyclotron diameter without any residual tunneling. The quantum conductance scales with energy, magnetic field and strip width via the dimensionless ratio $R = X/kL^2$. Aharonov-Bohm oscillations in the conductance are absent. The classical conductance of the strip scales with R only if semiquantized and is larger than the full quantum result when the conductance is finite. For tapered (smoothly rising) magnetic fields, the vanishing of the conductance past a certain barrier width remains valid, but scaling is of course not exactly valid.

ACKNOWLEDGMENTS

We wish to thank Y. Imry for suggesting that this problem could be reduced to one resembling a one-dimensional tunneling problem, M. Büttiker, B. Horovitz, and M. Azbel for useful discussions, and Z. Melamed for helping us with the numerical evaluation of the parabolic cylinder functions. One of us (Y.A.) would like to thank Allied Signal Inc. for the hospitality extended to him when this work was initiated. This work was partially supported by Grant No. 81169101 from the American-Israel Binational Science Foundation.

[1]D. A. Wharam, T. J. Thornton, R. Newbury, M. Pepper, H. Ajmed, J. E. F. Frost, D. G. Hasko, D. C. Peacock, D. A. Ritchie, and G. A. C. Jones, J. Phys. C **21**, L209 (1988); B. J. van Wees, H. van Houten, C. W. J. Beenakker, J. G. Williamson, L. P. Kouwenhoven, D. van der Marel, and C. T. Fuxton, Phys. Rev. Lett. **60**, 848 (1988).

[2]D. E. Khmel'nitskii and R. E. Shekhter, Pis'ma Zh. Eksp. Teor. Fiz. **48**, 218 (1988) [JETP Lett. **48**, 238 (1988)]; E. G. Haanappel and D. van der Marel, Phys. Rev. B **39**, 5484 (1989); **39**, 7811 (1989); Y. Avishai and Y. B. Band, *ibid.* **40**, 19 640 (1989); L. I. Glazman, G. B. Lesovick, D. E. Khmelnitzkii, and R. I. Shekhter, Pis'ma Zh. Eksp. Teor. Fiz. **48**, 218 (1988) [JETP Lett. **48**, 238 (1988)]; L. I. Glazman and A. V. Khaetzkii, *ibid.* **48**, 546 (1988) [*ibid.* **48**, 591 (1988)]; Europhys. Lett. **9**, 263 (1989); L. Escapa and N. Garcia, J. Phys. C **1**, 2125 (1989); G. Kirczenow, Solid State Commun. **68**, 715 (1988); G. Kirczenow, Phys. Rev. B **39**, 10 452 (1989); A. Szafer and A. D. Stone, Phys. Rev. Lett. **62**, 300 (1989); R. Landauer, J. Phys. C (to be published).

[3]B. J. van Wees, L. P. Kouwenhoven, H. van Houten, C. W. J. Bennakker, J. E. Mooij, C. T. Foxton, and J. J. Harris, Phys. Rev. B **38**, 3625 (1988).

[4]L. J. van der Pauw, Philips Res. Rep. **13**, 1 (1958).

[5]M. Buttiker, Phys. Rev. B **38**, 9375 (1988); P. Streda, J. Kucera, and A. H. MacDonald, Phys. Rev. Lett. **59**, 1973 (1988); T. Ohtzuki and Y. Ono, Solid State Commun. **65**, 403 (1988); B. Kramer, Y. Ono, and T. Ohtzuki, Sur. Sci. **196**, 127 (1988).

[6]M. Abramovitz and I. Stegun, *Mathematical Functions* (N.B.S. Applied Mathematics Series, Washington, D.C., 1970), Chap. 19, Eqs. (19.2.1)–(19.2.6).

[7]D. S. Fisher and P. A. Lee, Phys. Rev. B **23**, 6851 (1981); P. A. Lee and D. S. Fisher, Phys. Rev. Lett. **47**, 882 (1981).

[8]R. Landauer, IBM J. Res. Dev. **1**, 223 (1957); Philos. Mag. **21**, 863 (1970); R. Landauer, IBM J. Res. Dev. **1**, 223 (1957); Philos. Mag. **21**, 863 (1970); M. Buttiker, Y. Imry, R. Landauer, and S. Pinhas, Phys. Rev. B **31**, 6207 (1985); M. Ya Azbel, J. Phys. C **14**, L225 (1981); Y. Imry, *Directions in Condensed Matter Physics*, edited by G. Grinstein and G. Mazenko (World Scientific, Singapore, 1986), pp. 101–163; Y. Geffen, Y. Imry, and M. Ya Azbel, Phys. Rev. Lett. **52**, 129 (1984); Surf. Sci. **142**, 203 (1984).

[9]D. Bohm and Y. Aharonov, Phys. Rev. **115**, 485 (1959); **123**,

1511 (1961).

[10]D. Yu. Sharvin and Yu. V. Sharvin, Pis'ma Zh. Eksp. Teor. Fiz. **34**, 285 (1981) [JETP Lett. **34**, 272 (1981)]; B. L. Alt'shuler, A. G. Aronov, and B. Z. Spivak, Pis'ma Zh. Eksp. Teor. Fiz. **33**, 101 (1981) [JETP Lett. **33**, 94 (1981)]; R. A. Webb, S. Washburn, C. P. Umbach, and R. B. Laibowitz, Phys. Rev. Lett. **54**, 2696 (1985); V. Chandrasekhar, M. J. Rooks, S. Wind, and D. E. Prober, *ibid.* **55**, 1610 (1985).

[11]B. J. van Wees, L. P. Kouwenhoven, C. J. P. Harmans, J. G. Williamson, C. E. Timmering, M. E. I. Broekaart, H. van Houten, C. T. Fuxton, and J. J. Harris (unpublished).

[12]U. Sivan, Y. Imry, and C. Hartzstein (unpublished); U. Sivan and Y. Imry, Phys. Rev. Lett. **61**, 1001 (1988); J. K. Jain, *ibid.* **60**, 2074 (1988); J. K. Jain and S. A. Kivelson, *ibid.* **60**, 1542 (1988); M. Buttiker (unpublished); Y. Avishai and Y. B. Band, Phys. Rev. B **40**, 3429 (1989).