PHYSICAL REVIEW B VOLUME 41, NUMBER 13 1 MAY 1990

Finite-size scaling of the Ising model in four dimensions

Pik-Yin Lai* and K. K. Mon
Department of Physics and Astronomy and Center for Simulational Physics, University of Georgia, Athens, Georgia 30602
(Received 8 December 1989)

The four-dimensional Ising model is studied to probe the possibility of observing in Monte Carlo simulations the logarithmic corrections to the mean-field theory near criticality. The finite-size-scaling behavior for the correlation length is proposed. The scaling forms of the finite-size renormalized coupling, susceptibility, and fourth field derivative at the renormalized tree-level approximation are derived. These results are used to analyze simulation data of the simple hypercubic lattices of sizes $4 \leq L \leq 14$, near criticality. Our simulation results are in agreement with the presence of logarithmic corrections and recent field-theoretical calculations of the specific heat. The approach to the nonscattering theory is observed and is consistent with the predicted logarithmic finite-size dependence.

I. INTRODUCTION

The field-theoretical renormalization-group (RG) approach[1] has been rather successful in the studies of critical phenomena below the upper critical dimension $d_>$. For $d > d_>$, mean-field theory is valid. At the upper critical dimension, renormalization group (RG) predicts logarithmic corrections[2] to the mean-field behavior. For the important Ising universality class, the upper critical dimension is four. These predicted logarithmic corrections to the power law singularities have been used to account[3] for the apparent deviation of the critical exponents from their mean-field values in earlier series expansion results[4] of the four-dimensional Ising model. These numerical studies are important because the exact solution of the Ising model in four dimensions is not known. For the same reason, Monte Carlo simulations have also been used to study the critical properties of the four-dimensional Ising model.[5–8] These Monte Carlo results indicated some evidence of the marginal behavior and the logarithmic corrections to the mean-field theory. However, there was no systematic finite-size-scaling analysis of the model and it is interesting to observe these effects by direct Monte Carlo simulations of finite systems. Although the four-dimensional Ising model may not be applicable to real magnetic systems, the possibility of logarithmic corrections at the upper critical dimension is expected to hold in general. When the upper critical dimension is three (like the case of uniaxial dipolar ferromagnet[9]) these logarithmic corrections may actually be observed experimentally. Furthermore, results of the four-dimensional Ising model is of some interest to the lattice gauge theory.[10,11]

The Ising model is in the same universality class[1] as the ϕ^4 field theory. The upper critical dimension for the ϕ^4 field theory is four, above which the theory becomes trivial (Gaussian) with the renormalized coupling vanishing in the critical regime and the critical exponents take their mean-field values. The renormalized coupling g_R can be expressed as

$$g_R = \frac{\chi^{(4)}}{\xi^d \chi^2} , \tag{1.1}$$

where ξ is the correlation length, and $\chi^{(4)}$ and χ are the fourth- and second-order cumulants of the order parameter distribution, respectively. One of the quantities that describes the approach to free field limit is the Fisher's exponent ω^* defined by[12]

$$\nu\omega^* \equiv d\nu + \gamma - 2\Delta , \tag{1.2}$$

where ν, γ are the critical exponents for the correlation length, susceptibility and Δ is the gap exponent. It is known that $\omega^* \geq 0$ holds in general[13] and is a measure of hyperscaling violations.[12] The renormalized coupling g_R near the critical point is governed by

$$g_R \sim t^{\nu\omega^*} , \tag{1.3}$$

where $t = T/T_c - 1$ and T_c is the critical temperature. At $d=2$, exact solution[14] shows $\omega^* = 0$ and hence hyperscaling holds and the model is a nontrivial interacting theory. For $d=3$, earlier works indicated possible hyperscaling violations,[15,16] but recent series expansion[17] and Monte Carlo simulation[18] results suggest ω^* is or close to zero. For $d=5$, Monte Carlo calculations confirmed mean-field predictions of large hyperscaling violations.[19] At $d=4$, the upper critical dimension, the common belief, suggested by renormalization-group calculations,[2] is that the logarithmic correction to Eq. (1.3) leads to a trivial (Gaussian) theory in the critical regime. In fact it was proven[20] rigorously that for the ϕ^4 theory in four dimensions, (i) there is at most a logarithmic deviation from the mean-field behavior in the critical regime, and (ii) $g_R \to 0$ as $t \to 0$, the renormalized coupling vanishes, and the limit is a free field. These predictions should be observable in a computer simulation and constitute in part the motivations for this work.

In the critical region, the correlation length diverges and finite-size effects[21–23] in computer simulation results can be significant. At the upper critical dimension, RG analysis by Rudnick *et al.*[24] for finite system predicts logarithmic size dependence corrections to the mean-field finite-size scaling predictions. They suggested[24] that a logarithmic finite-size dependence for the specific heat might be observable in simulations of the four-dimensional Ising system. In this paper, we report finite-size Monte Carlo studies of the Ising model in four dimensions. The results for the renormalized coupling, correlation length, susceptibility, fourth field derivative, and specific heat are analyzed following the finite-size scaling predictions. Our results are consistent with the presence of the theoretically predicted logarithmic corrections. This paper is organized as follows. In Sec. II, finite-size scaling predictions, incorporating the possibility of logarithmic corrections, are proposed for the correlation length. The finite-size scaling behavior of the renormalized coupling, susceptibility, fourth field derivative, and specific heat are obtained at the renormalized mean-field level. In Sec. III, we present simulation results for the four-dimensional Ising model to test the finite-size scaling predictions. The paper concludes with some remarks in Sec. IV.

II. THEORY

For the ϕ^4 theory, the susceptibility, fourth field derivative, and correlation length have already been shown[2] to behave as

$$\chi \sim t^{-\gamma}|\ln|t||^{1/3} , \tag{2.1}$$

$$\chi^{(4)} \sim t^{-\gamma-2\Delta}|\ln|t||^{1/3} , \tag{2.2}$$

$$\xi(t) \sim t^{-\nu}|\ln|t||^{1/6} , \tag{2.3}$$

near criticality with all the exponents taking their mean-field values ($\nu=\frac{1}{2},\gamma=1,\Delta=\frac{3}{2}$). Thus in four dimensions, RG predicts a Gaussian field theory near the critical point but the mean-field behavior has logarithmic corrections.

For numerical simulations, we have adopted the definition of correlation length for a finite hypercubic lattice with periodic boundary conditions used by Binder *et al.*,[19]

$$2d\xi_L^2 = \frac{\sum_{i,j} (r_i - r_j)^2(\langle s_i s_j \rangle - c_L)}{\sum_{i,j} (\langle s_i s_j \rangle - c_L)} , \tag{2.4}$$

where r_i is the position of the spin s_i at site i, $c_L=(1/L^d)\sum_i \langle s_i s_{i'} \rangle$, and i' is the site at $r_{i'}=r_i+\frac{1}{2}(1,1,1,1)L$. From phenomenological finite-size scaling,[21,22] we obtained the scaling functions of the correlation length,[23]

$$\xi_L(t)=LZ(tL^{1/\nu}|\ln|t||^{-1/3}) , \tag{2.5}$$

where $Z(x) \sim x^{-\nu}$ for $x \gg 1$. In the above, for simplicity the possibility of a dangerous irrelevant variable entering the scaling function of the correlation length is excluded; we will discuss this assumption further and consider simulation test below. For the free energy derived quantities like the finite-size renormalized coupling, susceptibility and specific heat, etc., it has been noticed that naive finite-size scaling breakdown[23,25,22] for $d \geq 4$ due to the presence of a second length scale which diverges slower than the correlation length.

To study the renormalized coupling, we considered the finite-size renormalized coupling g_L introduced by Binder[7]

$$g_L = \left[\frac{\chi_L^{(4)}}{L^d \chi_L^2} \right] , \tag{2.6}$$

where χ_L is the susceptibility and $\chi_L^{(4)}$ is the fourth field derivative; the subscripts L denote the corresponding finite-size quantities. For $d=4$ at the renormalized mean-field approximation,[24] $g_L(t)$ has the following finite-size scaling form (see Appendix):

$$g_L(t)=G(tL^{y_T^*}(\ln L)^{1/6}) . \tag{2.7}$$

Although for $d=4$,

$$y_T^* = \frac{1}{\nu} = 2 , \tag{2.8}$$

g_L does not scale as L/ξ. This will be discussed below. From its bulk value, $G(x)$ behaves as (see Appendix)

$$G(x) \sim x^{-d/y_T^*} \quad \text{for } x \gg 1 . \tag{2.9}$$

In order to observe the approach to the free field limit ($g_R \to 0$) as predicted in the theories, we use the parametrization,[16] $\xi_L(t)=cL$ for $t>0$ where c is some fixed constant. Under this parametrization, the renormalized coupling g_R can be studied by considering the large L limit of the finite-size renormalized coupling,

$$g_R \propto g_L \tag{2.10}$$

in the large L limit. Using Eqs. (2.5), (2.7), and (2.9), the finite-size scaling prediction is

$$g_L = G\left[Z^{-1}(c) L^{y_T^* - 1/\nu} (\ln L)^{1/6} \left| \frac{1}{\nu}\ln L - \ln Z^{-1}(c) + O(\ln\ln L) \right|^{1/3} \right] \tag{2.11}$$

$$\sim L^{-\omega^*} |\ln L|^{-d/2y_T^*} \tag{2.12}$$

for the large L limit. If the RG results are correct, then one has $y_T^*=2$, $\omega^*=0$, and g_L would approach zero as $(\ln L)^{-1}$ in $d=4$. The renormalized coupling g_R vanishes in the critical region in four dimensions and can be attributed to the presence of a second length scale which diverges as $t^{-1/y_T^*}|\ln t|^{-1/(6y_T^*)}$ in the bulk limit (see Appendix). This is in addition to the diverging correlation length, $\xi \sim t^{-\nu}|\ln t|^{\nu/3}$. This is analogous to the case[19,22,23] for $d>4$ where $\xi \sim t^{-1/2}$, but the second length scales as $t^{-2/d}$ diverging slower than the correlation length.

For $d=4$, due to the presence of the logarithms, the finite-size free-energy derived quantities like susceptibility and specific heat cannot be put into a scaling form of one scaling variable. However, if one is interested in the leading logarithmic corrections in $d=4$, a renormalized mean-field calculation[24] is sufficient and the scaling forms can be obtained at this level of approximation. The scaling behavior predicted at this tree-level approximation is exact[23] for large L. The finite-size behavior of the specific heat has been calculated (up to one-loop level) by Rudnick *et al.*[24] Using a similar technique, the finite-size scaling behavior of the susceptibility, fourth field derivative, and specific heat at the renormalized mean-field level[24] are derived in the Appendix and the results are listed below. These results will be tested in the next section.

$$\chi_L(t)=\chi_L(0)\psi(x) \quad \text{with } \chi_L(0)\sim L^2\sqrt{\ln L} \ , \tag{2.13}$$

$$\chi_L^{(4)}(t)=\chi_L^{(4)}(0)\psi^{(4)}(x) \quad \text{with } \chi_L^{(4)}(0)\sim L^8 \ln L \ , \tag{2.14}$$

$$C_L(t)=C_L(0)\Phi(x) \quad \text{with } C_L(0)\sim(\ln L)^{1/3} \ , \tag{2.15}$$

where $x=tL^2(\ln L)^{1/6}$. It is straightforward to show that these finite-size quantities peak at $t_{\max}$ with $t_{\max}^{-1} \sim L^2(\ln L)^{1/6}$, which agrees with the field-theoretical calculations of the specific heat by Rudnick *et al.*[24] It should be noted that an explicit RG calculation at four dimensions for the N-vector model in the large N limit also predicted[25] the susceptibility at criticality scales at $L^2\sqrt{\ln L}$ as in (2.13).

We now return to the possibility that the correlation function may have a dangerous irrelevant variable as discussed by Binder *et al.*[19] This would modify Eq. (2.5) to be

$$\xi_L(t)=L^{1+q_1y_1}Z(tL^{y_T^{**}}|\ln|t||^{-1/3} \ , \tag{2.16}$$

where $q_1y_1\le 0$ and $\nu=(1+q_1y_1)/y_T^{**}$. In the case of the three-dimensional Ising model,[18] q_1y_1 has been estimated to be very small, -0.0036 ± 0.006. We have also studied q_1y_1 here by evaluating $\xi_L(0)$ for different system sizes. The result will be given below.

III. MONTE CARLO RESULTS AND DISCUSSIONS

The system we simulate is the four-dimensional hypercubic Ising lattice with periodic boundary conditions at zero external field. Our lattice size ranges up to $L\le 14$. We use the standard Metropolis heat bath to update the spins. Typical Monte Carlo steps per spin is about 10^5. We have run sufficiently long to ensure that equilibrium is achieved, and confirmed the previous observation[6] that the relaxation time is rather short. Different runs (typically five) were used to estimate the statistical uncertainties. The critical coupling we used is taken from the high temperature series result by Gaunt *et al.*,[3] $T_c=6.6817 \pm 0.0015$.

We now present our Monte Carlo results and compare them with the finite-size-scaling theory predictions. Figure 1 is a plot of $\xi_L(0)$ versus L suggesting $q_1y_1=0$. See Eq. (2.16). A least-squares fit to the data gives

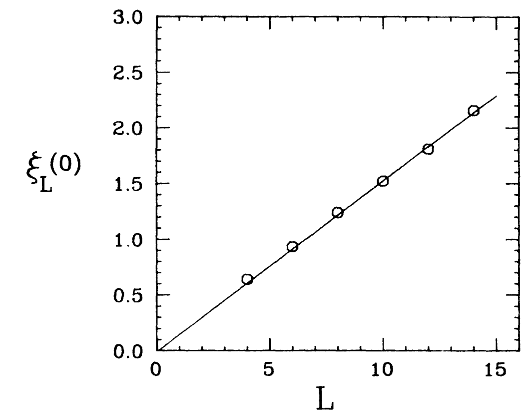

FIG. 1. $\xi_L(0)$ vs L. The solid line is the best fit straight line. The errors are smaller than the symbols.

$q_1y_1=-0.016\pm0.024$ and $Z(0)=0.154\pm0.002$. Figure 2 is a scaling plot of the finite-size correlation length versus the scaling variable $tL^2|\ln|t||^{-1/3}$. The data scale well for $L\ge 6$ and suggest the validity of (2.5) and the possibility that the RG exponents are correct.

For the renormalized coupling, field-theoretical calculation[23] for $d\ge 4$ at the tree-level approximation predicted the universal quantity $\langle s^4\rangle/\langle s^2\rangle^2=3+g_L(0)$ to be $\Gamma^4(\frac{1}{4})/(8\pi)^2\simeq 2.188$. Our simulation results for sizes up to $L=14$ give an estimate of ~ 1.92. To observe the approach to nonscattering theory at the upper critical dimension, we used the parametrization $\xi_L(t)=cL$ as suggested[16] in Eq. (2.12). Since $L^{-r}\simeq(1+r\ln L)^{-1}$ for r small, it is difficult to distinguish numerically the contributions of the decay of g_L from ω^* or the logarithmic part unless a large range of values for L is used. In practice, we choose $c\simeq 0.152$ such that $Z^{-1}(c)\simeq 1$. This choice will best manifest the logarithmic correction numerically. Figure 3 shows that g_L decreases for increasing L. The RG results predict $g_L\sim(\ln L)^{-1}$, and Fig. 3 shows that a straight line can be fitted for g_L^{-1} versus $\ln L$. In fact, least-squares fit indicates $g_L\sim(\ln L)^x$ with

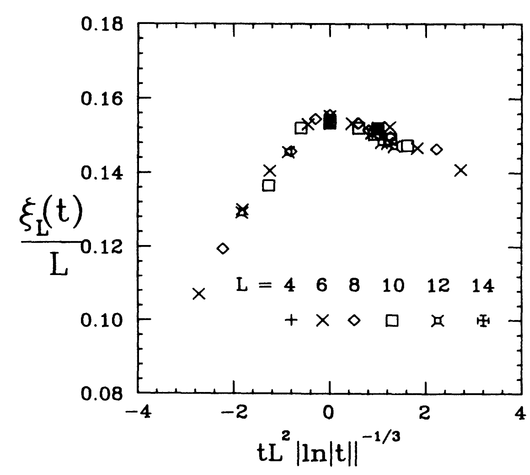

FIG. 2. Scaling plot of $\xi_L(t)/L$ vs $tL^2|\ln|t||^{-1/3}$.

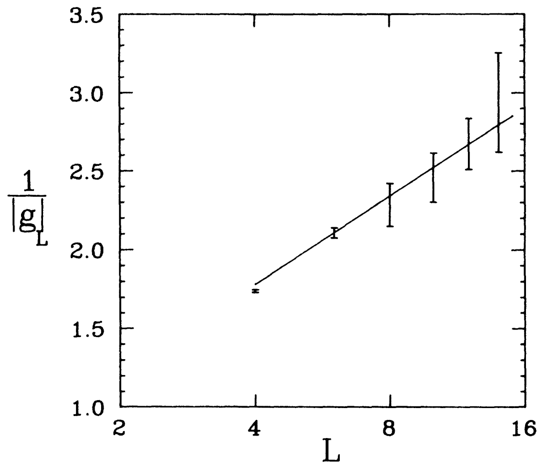

FIG. 3. $|1/g_L|$ vs $\ln L$ with $c=0.152$. The straight line is a linear least-squares fit for $L \geq 6$.

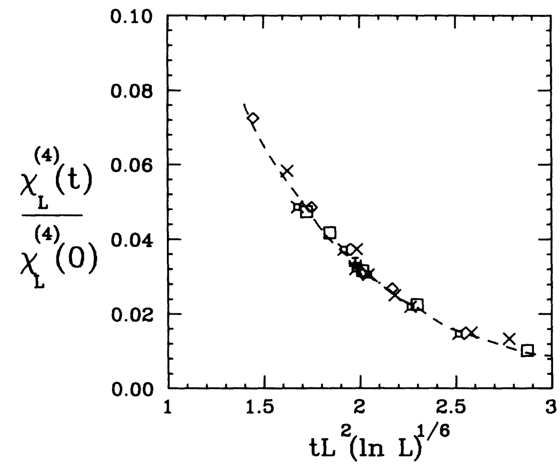

FIG. 5. Scaling plot of $\psi^{(4)}(x)$ with $x=tL^2(\ln L)^{1/6}$. The symbols for different values of L are the same as in Fig. 2. The dashed curve is a guide to the eye.

$x \simeq -1.1 \pm 0.3$ in agreement with the RG results. However, we stress that the range of $\ln L$ is rather small and our result is not a definitive test. For example, we also fitted g_L as some power of L and obtained $L^{-0.36\pm0.06}$. The approach to the trivial field theory in four dimensions has already been demonstrated in the series expansion work of the continuous spin Ising model[26] in which the renormalized coupling decreases as ξ increases in the spin-$\frac{1}{2}$ limit. Also, there were Monte Carlo results[27] of ϕ^4 theory in four-dimensional lattices indicating a noninteracting continuum limit.

We have computed the exponent y_T^* by using the relation

$$y_T^* = \frac{\ln[\dot{g}_{L_1}(0)/\dot{g}_{L_2}(0)]}{\ln(L_1/L_2)} , \tag{3.1}$$

where $\dot{g}$ denotes the temperature derivative of g. $\dot{g}_L(0)$ can be sampled directly in the simulation and the data were analyzed to obtain $y_T^*=2.03\pm0.05$ using data with $L \geq 6$. This is consistent with the RG result of 2.0.

In Figs. 4–7, the scaling functions ψ, $\psi^{(4)}$, and Φ for the susceptibility, fourth field derivative, and specific heat, respectively, with the scaling variable $x=tL^2(\ln L)^{1/6}$ are displayed. The data scale rather well for $L \geq 6$, consistent with Eqs. (2.13)–(2.15). Due to the unknown nonuniversal parameters involved in the field-theoretical results of the specific heat in Ref. 24, our data cannot be compared with the theory quantitatively. However, they do show qualitative agreement. To demonstrate the effects of the presence of the logarithmic corrections to mean-field theory, we show in Fig. 6 a scaling plot of $\psi^{(4)}(x)$ for the scaling variable $x=tL^2$, without the logarithmic corrections. The data apparently show systematic clustering into two groups and do not collapse onto a single curve as in Fig. 5. To see the logarithmic dependence in a more quantitative way, we plot-

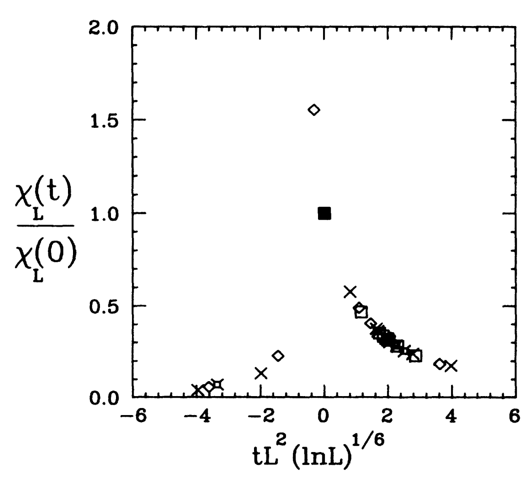

FIG. 4. Scaling plot of $\psi(x)$ with $x=tL^2(\ln L)^{1/6}$. The symbols for different values of L are the same as in Fig. 2.

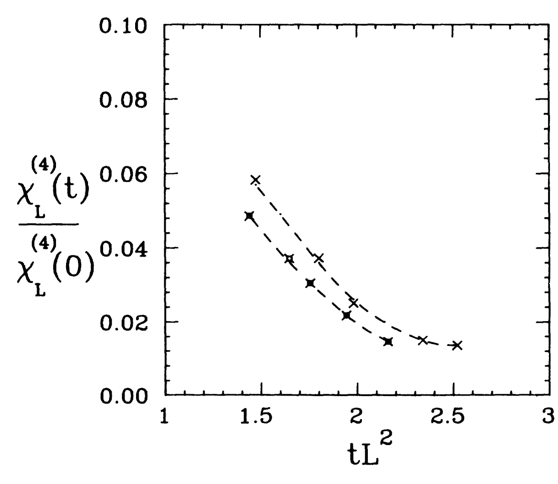

FIG. 6. Scaling plot of $\psi^{(4)}(x)$ with $x=tL^2$. Only $L=6$ and $L=12$ are shown for clarity. The errors are about the size of the symbols. The dashed curves are guides to the eye.

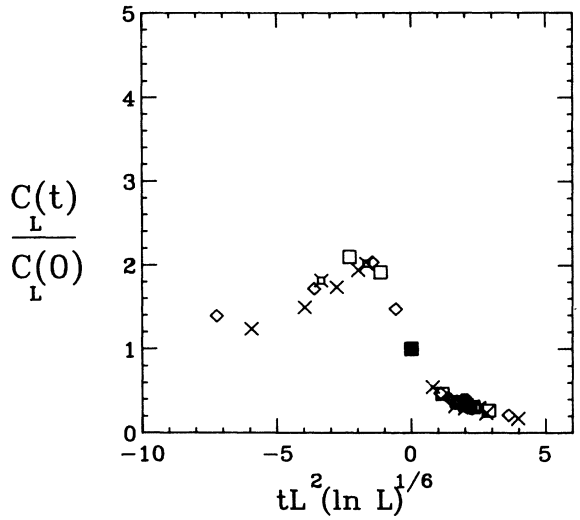

FIG. 7. Scaling plot of $\Phi(x)$ with $x = tL^2(\ln L)^{1/6}$. The symbols for different values of L are the same as in Fig. 2.

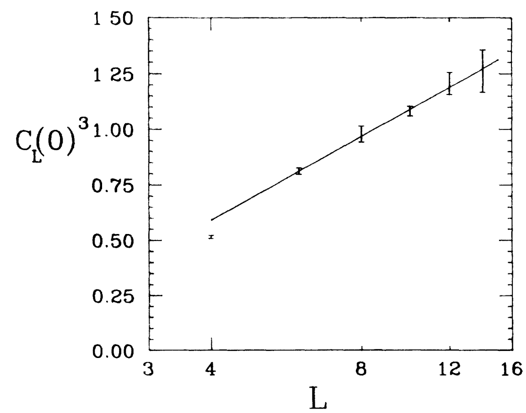

FIG. 9. $C_L(0)^3$ vs $\ln L$. The straight line is a linear least-squares fit.

ted $\chi_L(0)^2/L^4$, $\chi^{(4)}/L^8$, and $C_L(0)^3$ versus $\ln L$ in Figs. 8 and 9; they all show a linear dependence in $\ln L$ as suggested in Eqs. (2.13)–(2.15). We also fitted $\chi_L(0)/L^2$, $\chi^{(4)}/L^8$, and $C_L(0)$ as some power of $\ln L$ and obtained $(\ln L)^{0.45\pm0.08}$, $(\ln L)^{0.80\pm0.25}$, and $(\ln L)^{0.37\pm0.09}$, respectively, which are consistent with values $\frac{1}{2}$, 1 and $\frac{1}{3}$. It should be noted that if correlation length is the only length scale, then naive phenomenological finite-size analysis predicts that $\chi_L(0)/L^2$ is independent of L, which does not agree with our simulation results. Finally, we note that in the absence of logarithmic corrections, the results in Figs. 8 and 9 would be *independent* of L. This is not consistent with our data.

IV. CONCLUSION

We have obtained finite-size scaling behavior for the correlation length at four dimensions, allowing for possible logarithmic corrections. The finite-size scaling predictions, at the renormalized mean-field level, for the renormalized coupling, susceptibility, fourth field derivative, and specific heat are proposed. From our simulations of the four-dimensional Ising model, corrections to mean-field theory are exhibited. Although the range of lattice sizes considered is limited, the results scale very well and appear to be in the scaling limit. The simulation data are consistent with the theoretical predictions of logarithmic corrections and the approach to the trivial field theory is observed numerically.

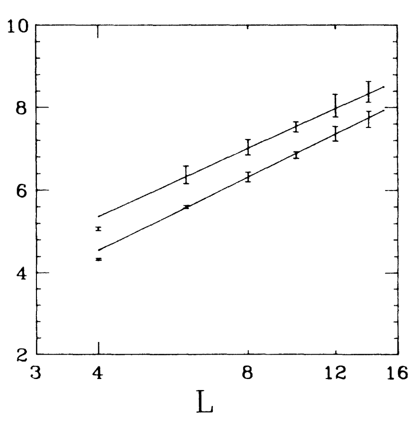

FIG. 8. Upper line $\chi_L^{(4)}(0)/L^8$, lower line $\chi_L(0)^2/L^4$. The straight lines are linear least-squares fits.

ACKNOWLEDGMENTS

We are indebted to Professor V. Privman, Professor R. Swendsen, and Dr. H. Guo for helpful communications and Professor D. Jasnow for useful comments and suggestions on the manuscript. This work was supported by the Petroleum Research Funds of the American Chemical Society and the Advanced Computational Methods Center of the University of Georgia. We thank them.

APPENDIX

In this Appendix we derive, using finite-size scaling theory[23,24] at the tree-level approximation, the finite-size scaling behavior of the susceptibility, fourth field derivative, and specific heat at four dimensions. The higher order loops[23] contribute corrections that go as L^{-2} and hence the tree approximation that leads to the scaling relations are exact for large L. We will use the same notations as in Ref. 24. For ϕ^4 model, one has[1,24]

$$\frac{du(\rho)}{d\ln\rho} = W(u(\rho)) , \tag{A1}$$

$$\frac{\rho^2 dt(\rho)}{d\ln\rho} = \gamma_{\phi^2}(u(\rho))\rho^2 t(\rho) , \tag{A2}$$

$$\frac{dm(\rho)}{d\ln\rho} = -\frac{\eta(u(\rho))m(\rho)}{2} , \tag{A3}$$

with, in $d=4$, $W(u)=\frac{3}{2}S_d u^2$, $\eta(u)=\frac{1}{24}u^2$, and $\gamma_{\phi^2}(u)=\frac{1}{2}s_d u$, where $S_d^{-1}=2^{d-1}\pi^{d/2}\Gamma(d/2)$. Integrating (A1) and (A2), one obtains

$$u(\rho)=\frac{u}{1-\frac{3}{2}S_d u\ \ln\rho}\ , \tag{A4}$$

$$\rho^2 t(\rho)=t\left[\frac{u(\rho)}{u}\right]^{1/3}\ , \tag{A5}$$

where u, t and m are the coupling, temperature, and magnetization at $\rho=1$, respectively. Thus $u(\rho)\rightarrow 0$ as $\rho\rightarrow 0$, approaching the trivial nonscattering field theory. ρ must be chosen to ρ^* for detailed calculations. ρ^* is determined by the relation[24]

$$\rho^{*2}t(\rho^*)+\tfrac{1}{2}u(\rho^*)\langle m^2(\rho^*)\rangle+\frac{1}{L^2}=(\kappa\rho^*)^2\ . \tag{A6}$$

At the renormalized mean-field level, the average $\langle\ \rangle$ is with respect to the mean-field Hamiltonian and the free energy depends only on the variable[23,24] z, where

$$z\equiv L^2\rho^{*2}t(\rho^*)/[u(\rho^*)]^{1/2}\ . \tag{A7}$$

Using Eqs. (A4) and (A5),

$$z=\frac{tL^2}{u^{1/3}}(1-\tfrac{3}{2}S_d u\ \ln\rho^*)^{1/6}\ ; \tag{A8}$$

m has the scaling form[23,24]

$$\langle m^{2j}(\rho^*)\rangle=[u(\rho^*)L^4]^{-j/2}D_j(z)\ ,$$

where

$$D_j(x)=\frac{\int_0^\infty d\phi\,\phi^{2j}\exp(-x\phi^2/2+\phi^4/4!)}{\int_0^\infty d\phi\,\exp(-x\phi^2/2+\phi^4/4!)}\ . \tag{A9}$$

In the finite-size regime (as in our simulations),

$$\rho^*\simeq(L\kappa)^{-1}\ , \tag{A10}$$

while in the bulk limit ρ^* is determined from $\rho^{*2}t(\rho^*)+\frac{1}{2}u(\rho^*)\langle m^2(\rho^*)\rangle=(\kappa\rho^*)^2$ and thus

$$\ln\rho^*\simeq\ln\frac{t}{\kappa^2}+O(\ln\ln t)\ . \tag{A11}$$

At this point, it is clear that the free-energy derived quantities have a characteristic length scale that goes as $t^{-1/2}(1-\frac{3}{2}S_d u\ \ln\rho^*)^{-1/12}$, which is different from the correlation length and leads to the breakdown of naive finite-size scaling at $d=4$. In the bulk limit, this free-energy length scale goes as $t^{-1/2}|\ln|t||^{-1/12}$, which diverges slower than the correlation length and thus $g_R\rightarrow 0$ in the critical region.

The finite-size susceptibility is given by

$$\chi_L(t)=L^4\langle m^2(\rho^*)\rangle \sim\frac{L^2}{[u(\rho^*)]^{1/2}}D_1(z)\ . \tag{A12}$$

Thus in the scaling regime $L\kappa\gg 1$, (A12) becomes

$$\chi_L(t)\sim L^2\sqrt{\ln L\kappa}D_1(tL^2(\ln L\kappa)^{1/6})\ , \tag{A13}$$

and Eq. (2.13) follows for $L\gg\kappa$. In the same way, $\chi_L^{(4)}(t)$ can be shown to have the finite-size scaling form at the renormalized mean-field level,

$$\chi_L^{(4)}(t)\sim L^8(\ln L\kappa)[D_2(tL^2(\ln L\kappa)^{1/6}) -3D_1^2(tL^2(\ln L\kappa)^{1/6})]\ . \tag{A14}$$

From their bulk limits in Eqs. (2.1) and (2.2),

$$\begin{aligned}&D_1(x)\sim x^{-2}\ ,\\&D_2(x)-3D_1^2(x)\sim x^{-4}\quad\text{for } x\gg 1\ .\end{aligned} \tag{A15}$$

It is easy to see that at the renormalized mean-field level, the finite-size renormalized coupling has the scaling form

$$g_L(t)=\frac{D_2(z)}{D_1^2(z)}-3\equiv G(z) \tag{A16}$$

and

$$G(x)\sim x^{-2}\quad\text{for } x\gg 1\ . \tag{A17}$$

For the specific heat, it is straightforward to get

$$C_L(t)\sim(\ln L\kappa)^{-1/3}\Phi(tL^2(\ln L\kappa)^{1/6})\ , \tag{A18}$$

where Φ is the second derivative of the free energy with respect to the scaling variable at the renormalized mean-field level. The specific heat up to the one-loop level has been calculated in Ref. 24.

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*Present address: Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, PA 15260.

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