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Magnon damping in the two-dimensional quantum Heisenberg antiferromagnet at short wavelengths

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Dyson-Maleev boson formalism is used to calculate magnon damping $\Gamma_{\mathbf{k}}$ in the two-dimensional quantum Heisenberg antiferromagnet at low temperatures T and for wavelengths *short* compared to the thermal de Broglie wavelength. From the evaluation of the second-order self-energy it is found that $\Gamma_{\mathbf{k}} \propto T^3 Z(|\mathbf{v}_{\mathbf{k}}|)$ as $T \rightarrow 0$, where $\mathbf{v}_{\mathbf{k}}$ is the gradient of dispersion relation of free magnons. For $\mathbf{k}$ close to the boundary of the Brillouin zone, where $|\mathbf{v}_{\mathbf{k}}|$ is small, the function Z has the expansion $Z(|\mathbf{v}_{\mathbf{k}}|)=1+O(|\mathbf{v}_{\mathbf{k}}|^2)$. For general $|\mathbf{v}_{\mathbf{k}}|$, we have calculated Z numerically. Although there is no long-range order at any $T \neq 0$, the staggered correlation length $\xi(T)$ is exponentially large as $T \rightarrow 0$. It is shown explicitly in the present work that, at low temperatures, magnons with momentum $\mathbf{k}$ in the regime $|\mathbf{k}|a \gg (T/E_\pi)^{1/3}$ (where a is the lattice spacing and E_π is the energy of zone-boundary magnons) are well-defined quasiparticles. We present evidence to support the argument that this remains true as long as $|\mathbf{k}|\xi \gg 1$.

I. INTRODUCTION

It is now widely believed that the magnetic properties of undoped La_2CuO_4 can be modeled by a spin $S=\frac{1}{2}$ nearest-neighbor quantum Heisenberg antiferromagnet (QAFM) on a square lattice with an exchange coupling J of the order of 1500 K.[1-5] The Hamiltonian of the QAFM is given by

$$\mathcal{H}=J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j \ , \tag{1.1}$$

where the $\mathbf{S}_i$ are dimensionless spin S operators at lattice site i. A hypercubic lattice of N sites has DN nearest-neighbor bonds in D dimensions; the sum in Eq. (1.1) is over all distinct bonds. In the present paper we are interested in the case $D=2$.

Recently, the low-temperature behavior of the QAFM has been discussed by Chakravarty and co-workers.[6,7] The results appear to be in quantitative agreement with the neutron-scattering experiments[2,5] in La_2CuO_4. In Ref. 6 it was argued that there are well-defined magnon excitations for large wave vectors $\mathbf{k}$, i.e., for $k\xi \gg 1$, where ξ is the two-dimensional staggered correlation length. In this paper we give a partial microscopic justification of this physically plausible argument. Moreover, the short-wavelength spin dynamics of QAFM has been probed in two-magnon Raman-scattering experiments.[8] Because the two-magnon Raman cross section is dominated by magnetic excitations with momenta close to the boundary of the Brillouin zone (BZ), it is important to understand magnon damping at short wavelengths.

Chakravarty, Halperin, and Nelson[6] used the mapping of the QAFM onto the quantum nonlinear σ model to study the *long-wavelength* properties of QAFM. From their renormalization-group analysis, and a similar theory recently developed by Chakravarty and the present author for the quantum Heisenberg *ferromagnet*,[9] it is evident that in addition to the lattice spacing a and the exponentially large order parameter correlation length ξ, the thermal de Broglie wavelength λ_{th} is a third characteristic length scale of the system. For a QAFM, λ_{th} can be written in terms of the spin-wave velocity c as

$$\frac{\lambda_{\text{th}}}{2\pi}=\frac{\hbar c}{T} \ , \tag{1.2}$$

where we have set the Boltzmann constant equal to unity. Since at low temperatures $a \ll \lambda_{\text{th}} \ll \xi$, the thermal de Broglie wavelength separates a short-wavelength quantum regime, $k \gg \lambda_{\text{th}}^{-1}$, from the long-wavelength regime $k \ll \lambda_{\text{th}}^{-1}$, where ξ is the only relevant length scale. Note that for $T \rightarrow 0$ the region $k \gg \lambda_{\text{th}}^{-1}$ comprises almost the entire BZ and is certainly of experimental interest.

Magnon damping in the regime $\xi^{-1} \ll k \ll \lambda_{\text{th}}^{-1}$ has been studied by a combination of spin-wave hydrodynamics, renormalization-group methods, and the dynamic scaling hypothesis.[6,7] Here we consider *short-wavelength* magnons. For momenta[10] in the range $k^3 \gg \lambda_{\text{th}}^{-1}$ we present a controlled microscopic calculation of the magnon damping $\Gamma(\mathbf{k},\hbar\omega)$ on resonance, i.e., at frequency $\hbar\omega=E_{\mathbf{k}}$, where $E_{\mathbf{k}}$ is the quasiparticle energy. We find that in this regime $\Gamma(\mathbf{k},E_{\mathbf{k}})$ is given by

$$\frac{\hbar\Gamma(\mathbf{k},E_{\mathbf{k}})}{E_\pi}=\left[\frac{4\pi}{3}\right]S^{-2}\left[\frac{T}{E_\pi}\right]^3 Z(|\mathbf{v}_{\mathbf{k}}|) \times\left[1+O\left[\frac{T}{E_\pi},(2S)^{-1}\right]\right] \ , \tag{1.3}$$

where E_π is the energy of the zone-boundary magnons, $\mathbf{v}_{\mathbf{k}}$ is the gradient of the free magnon dispersion relation, and the function Z is unity for magnons with momentum precisely at the zone boundary, where $\mathbf{v}_{\mathbf{k}}=0$. For general $|\mathbf{v}_{\mathbf{k}}|$ a numerical evaluation of Z is given (see Fig. 1). In

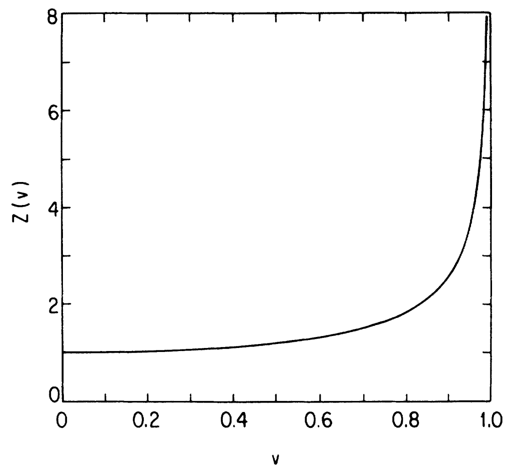

FIG. 1. Graph of the function $Z(v)$ defined in Eq. (2.27), obtained by numerical integration.

Sec. III we shall discuss Eq. (1.3) and carefully examine the range in parameter space for which it is valid.

In order to apply standard many-body techniques for interacting Bosons, we shall use the Dyson-Maleev transformation[11] to recast the spin operators in terms of Boson operators. A detailed and careful discussion of the method can be found in a paper by Harris, Kumar, Halperin, and Hohenberg,[12] henceforth referred to as HKHH, who have obtained low-temperature expressions for the damping of magnons in a QAFM in *three dimensions* and in the limit of long wavelengths. HKHH found that in this case the quasiparticle picture is certainly valid, and that the simple "golden-rule" expression (see the following) yields an accurate estimate for the magnon lifetime, which is not drastically changed if vertex corrections or damping of intermediate states are taken into account as well. However, in the present paper we treat the two-dimensional problem in which there is *no long-range order* at $T\neq 0$. Thus, our problem is very different from the one treated by HKHH.

The golden-rule expression for the magnon damping is most conveniently obtained from the discontinuity of the second-order self-energy (Ref. 13) $\Sigma^{(2)}(\mathbf{k},z)$ across the real frequency axis

$$\hbar\Gamma_{\mathbf{k}}\equiv\hbar\Gamma(\mathbf{k},E_{\mathbf{k}})=2\,\mathrm{Im}\left[\lim_{\eta\to 0}\Sigma^{(2)}(\mathbf{k},E_{\mathbf{k}}-i\eta)\right] . \tag{1.4}$$

$\Sigma^{(2)}$ is calculated within second-order perturbation theory and hence contains two powers of the interaction part of the Hamiltonian. However, the justification of the perturbative treatment of the damping of short-wavelength magnons is subtle because it relies on the cancellation of unphysical singularities in the Dyson-Maleev (DM) vertices. They become singular if the momentum of one of the outgoing magnons approaches zero (see Appendix A). Nevertheless, due to an exact cancellation of these singular terms in Eq. (1.4), we obtain in the two-dimensional QAFM a finite result for $\Gamma_{\mathbf{k}}$. In order to make this cancellation of the singular terms manifest, the various terms appearing in the expressions of the interaction vertices have to be carefully grouped together. In Appendix A we introduce a convenient parametrization of the DM vertices, which will be extremely useful for the evaluation of Eq. (1.4).

We shall also ignore kinematic interactions between spin waves,[11] although it is clear that such an approximation in two dimensions is on much less firm grounds than it is for $D=3$. Nonetheless, because the population of short-wavelength magnons is not large for low temperatures, the neglect of kinematic interactions cannot introduce serious errors. Clearly, for $T\neq 0$ and $D=2$, our perturbative treatment cannot be applied to long wavelengths, where elementary spin-wave excitations cannot be defined in a naive sense. Note, however, that a nonperturbative renormalization-group treatment seems to hold.[9] The reason is discussed in detail in Ref. 9, and is quite complex. We would like to emphasize that the justification of the neglect of the kinematic interaction in this case is quite different from that given in the classic paper by Dyson[11] for $D=3$, where the system has long-range order below the Néel temperature T_N.

In Sec. II we shall show that the evaluation of Eq. (1.4) does not lead to divergencies. Physically, it is clear that at low temperatures short-wavelength magnons should be well-defined quasiparticles, because the staggered correlation length ξ is exponentially large, although there is no true long-range order[14] at any $T\neq 0$. As argued in Ref. 6, magnons with momentum $k\gg\xi^{-1}$ can propagate in a locally ordered region of a size much larger than their wavelength. Thus it seems very plausible that at low temperatures the value for $\Gamma_{\mathbf{k}}$ obtained from Eq. (1.4) gives a reliable estimate of the damping of short-wavelength magnons.

II. EVALUATION OF THE GOLDEN-RULE EXPRESSION FOR THE MAGNON DAMPING AT SHORT WAVELENGTHS

The golden-rule expression for the magnon damping in two dimensions, as defined in Eq. (1.4) is[12]

$$\frac{\hbar\Gamma_{\mathbf{k}}}{E_\pi}=\left[\frac{1}{4\pi^3}\right]\left[\frac{1}{S\alpha(T,S)}\right]^2[1+n(\epsilon_{\mathbf{k}}/\tau)]^{-1}\times\int_{\mathrm{BZ}}d^2p\int_{\mathrm{BZ}}d^2q\,\delta(\epsilon_{\mathbf{k}}+\epsilon_{\mathbf{p}}-\epsilon_{\mathbf{k}+\mathbf{q}}-\epsilon_{\mathbf{p}-\mathbf{q}})\times B(\mathbf{q},\mathbf{p},\mathbf{k},\tau)W(\mathbf{q},\mathbf{p},\mathbf{k}) , \tag{2.1}$$

where $\alpha(T,S)$ is the Hartree-Fock renormalization factor for the quasiparticle energies (see below), and

$$E_\pi\equiv 2DJS\alpha(T,S) \tag{2.2}$$

is the energy of zone-boundary magnons. $\epsilon_{\mathbf{k}}$ is the dimensionless dispersion relation for free magnons defined in Eq. (A8). Although it must be remembered that we are interested in the case $D=2$, we shall continue to write D in equations that are valid for arbitrary D. The dimensionless temperature τ is defined to be

$$\tau\equiv\frac{T}{E_\pi} , \tag{2.3}$$

and $n(x)$ is the Bose-Einstein occupation factor,

$$n(x) \equiv \frac{1}{\exp(x)-1} \ . \tag{2.4}$$

The integrations in Eq. (2.1) are over the magnetic BZ (Ref. 10)

$$\{|p^1+p^2| \le \pi\} \cap \{|p^1-p^2| \le \pi\}$$

(similarly for **q**). The function B is defined by

$$B(\mathbf{q},\mathbf{p},\mathbf{k},\tau) \equiv n(\epsilon_{\mathbf{p}}/\tau)[1+n(\epsilon_{\mathbf{k}+\mathbf{q}}/\tau)][1+n(\epsilon_{\mathbf{p}-\mathbf{q}}/\tau)] \ , \tag{2.5}$$

and the function $W(\mathbf{q},\mathbf{p},\mathbf{k})$ is given in terms of the DM vertices, defined in Appendix A, by

$$\begin{aligned} W(\mathbf{q},\mathbf{p},\mathbf{k}) \equiv & V^{(1)}(\mathbf{q},\mathbf{p}-\mathbf{q},\mathbf{k}+\mathbf{q})V^{(1)}(-\mathbf{q},\mathbf{p},\mathbf{k}) \\ & + V^{(4)}(-\mathbf{q},\mathbf{k}+\mathbf{q},-\mathbf{p})V^{(4)}(\mathbf{q},\mathbf{k},-\mathbf{p}+\mathbf{q}) \\ & + V^{(4)}(\mathbf{k}-\mathbf{p}+\mathbf{q},\mathbf{p}-\mathbf{q},-\mathbf{p}) \\ & \times V^{(4)}(-\mathbf{k}+\mathbf{p}-\mathbf{q},\mathbf{k},-\mathbf{k}-\mathbf{q}) \ . \end{aligned} \tag{2.6}$$

In deriving Eq. (2.1), we have substituted for the quasiparticle energies $E_{\mathbf{k}}$ in Eq. (1.4) the Hartree-Fock expression[15]

$$E_{\mathbf{k}} = 2DJS\epsilon_{\mathbf{k}} + \Sigma^{(1)}(\mathbf{k}) \equiv 2DJS\alpha(T,S)\epsilon_{\mathbf{k}} \ . \tag{2.7}$$

Here $\Sigma^{(1)}(\mathbf{k})$ is the first-order self-energy. The Hartree-Fock renormalization factor $\alpha(T,S)$ does not depend on **k** and has to satisfy a self-consistency equation, which for finite S has a solution only if $T \le T_S \equiv 2DJSR_S$. The numerical value of R_S is of order unity for small S and approaches infinity as $S \to \infty$, while $\alpha(T,S)$ is, even for small S, of order unity for all temperatures where it is defined. Although the temperature and spin dependence of $\alpha(T,S)$ is irrelevant for the asymptotic behavior of $\Gamma_{\mathbf{k}}$, we shall keep this factor in the evaluation of Eq. (1.4) in order to show how the Hartree-Fock corrections to the propagators modify our final result.

We shall now evaluate Eq. (2.1) at low temperatures. Note that our dimensionless temperature τ can be written in terms of the thermal de Broglie wavelength as

$$\tau = \frac{2\pi}{\sqrt{D}} \lambda_{\mathrm{th}}^{-1} \ , \tag{2.8}$$

where we have used the free magnon expression for the spin-wave velocity $c = 2\sqrt{D}JS/\hbar$. We are interested in magnons with momentum **k** in the regime $\lambda_{\mathrm{th}}^{-1} \ll k$. However, for technical reasons discussed in detail in Sec. III, we need the stronger condition

$$\tau \ll k^3 \ , \tag{2.9}$$

in order to be able to evaluate Eq. (1.4) in a controlled way. Although at low temperatures the area of the BZ defined by Eq. (2.9) includes magnons with wavelengths much larger than the lattice spacing, we shall in the present paper refer to it as the *short-wavelength regime*, because the damping of all magnons with momentum in this regime is given by same asymptotic form as $\tau \to 0$, namely, Eq. (1.3). Note that for $k = O(1)$ Eq. (2.9) is equivalent to $\tau \ll 1$, but for small k it is more restrictive. These assumptions imply that $\epsilon_{\mathbf{k}}/\tau \gg 1$, and consequently $n(\epsilon_{\mathbf{k}}/\tau)$ in Eq. (2.1) is exponentially small and can be dropped. However, the factor $n(\epsilon_{\mathbf{p}}/\tau)$ in Eq. (2.5) cuts off the **p** integration in Eq. (2.1) at $\epsilon_{\mathbf{p}} \approx \tau \ll 1$. Hence, to a very good approximation, we can use the small-momentum expansion for $\epsilon_{\mathbf{p}}$, which is given by

$$\epsilon_{\mathbf{p}} = \frac{p}{\sqrt{D}} + O(p^3) \ . \tag{2.10}$$

Let us now consider the scattering surface, i.e., the set of points in the **q** plane satisfying

$$\epsilon_{\mathbf{k}} + \epsilon_{\mathbf{p}} - \epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{p}-\mathbf{q}} = 0 \ . \tag{2.11}$$

For each given **p** and **k**, this equation defines a set of curves $\mathbf{q}(t) \equiv \mathbf{q}(t;\mathbf{p},\mathbf{k})$ in the **q** plane. We use a real variable t to parametrize these curves. Note that if $\mathbf{q}(t;\mathbf{p},\mathbf{k})$ solves Eq. (2.11), then so does $\mathbf{q}'(t;\mathbf{p},\mathbf{k}) \equiv \mathbf{p}-\mathbf{k}-\mathbf{q}(t;\mathbf{p},\mathbf{k})$. We have argued that in this problem $p \lesssim \tau \ll 1$, while **k** is assumed to satisfy Eq. (2.9). In this case, as pointed out by HKHH, the scattering surface consists of two disjoint pieces: a curve $\mathbf{q}(t)$ with $q \ll k$, and a curve $\mathbf{q}'(t)$ with $q' = O(k)$. However, from the symmetry of the integrand in Eq. (2.1) it follows that the disjoint curves give equal contributions to the integral. [The identity $W(\mathbf{q},\mathbf{p},\mathbf{k}) = W(\mathbf{p}-\mathbf{k}-\mathbf{q},\mathbf{p},\mathbf{k})$ follows from the symmetry relations listed in Appendix A, Eqs. (A31) and (A32).] We therefore need to consider only the piece of the scattering surface with $q \ll k$ and multiply the final result by a factor of 2. Because $\epsilon_{\mathbf{k}+\mathbf{q}}/\tau \gg 1$ for this piece, we can set $1+n(\epsilon_{\mathbf{k}+\mathbf{q}}/\tau) \approx 1$ in Eq. (2.5), neglecting exponentially small corrections of order $\exp(-\epsilon_{\mathbf{k}+\mathbf{q}}/\tau)$. Note that for the other piece of the scattering surface the factor $n(\epsilon_{\mathbf{p}-\mathbf{q}}/\tau)$ is exponentially small, while $n(\epsilon_{\mathbf{k}+\mathbf{q}}/\tau)$ has to be kept. We could have equally well chosen the piece of the scattering surface with $q = O(k)$.

We shall now derive an explicit expression for the piece of the scattering surface with $p,q \ll k$. In this case we use in Eq. (2.11) the small-momentum limit of $\epsilon_{\mathbf{p}-\mathbf{q}}$ [Eq. (2.10)]. Furthermore, because $q \ll k$, we expand

$$\epsilon_{\mathbf{k}+\mathbf{q}} = \epsilon_{\mathbf{k}} + \frac{1}{\sqrt{D}} \mathbf{v}_{\mathbf{k}} \cdot \mathbf{q} + O(q^2) \ , \tag{2.12}$$

where

$$\mathbf{v}_{\mathbf{k}} \equiv \sqrt{D}\, \nabla \epsilon_{\mathbf{k}} \ . \tag{2.13}$$

As shown in Sec. III, this approximation can only be justified if the **p** integration in Eq. (2.1) is cut off at $p \lesssim k^3$, a condition that is satisfied in the short-wavelength region defined via Eq. (2.9). The equation for the scattering surface then becomes

$$p - |\mathbf{p}-\mathbf{q}| - \mathbf{q}\cdot\mathbf{v}_{\mathbf{k}} = 0 \ . \tag{2.14}$$

For $D = 2$, this is an equation for an ellipse in the **q** plane and it is straightforward to obtain the following parametric representation,

$$\mathbf{q}(t) = p[\hat{\mathbf{p}} + \mathbf{r}(t)], \quad t \in [0,2\pi] \ , \tag{2.15}$$

with

$$\mathbf{r}(t)=\left[\frac{1-v(\mathbf{k})\cos\varphi}{1-v^2(\mathbf{k})}\right]\times([\cos t-v(\mathbf{k})]\hat{\mathbf{v}}+\{[1-v^2(\mathbf{k})]^{1/2}\sin t\}\hat{\mathbf{v}}_\perp)\ , \tag{2.16}$$

and where $v(\mathbf{k})$, the length of the vector $\mathbf{v_k}$ defined in Eq. (2.13), is given by

$$v(\mathbf{k})\equiv|\mathbf{v_k}|=\frac{\gamma_\mathbf{k}}{\epsilon_\mathbf{k}}\left[\frac{1}{D}\sum_{\mu=1}^{D}(\sin k^\mu)^2\right]^{1/2}\ . \tag{2.17}$$

Here $\hat{\mathbf{v}}\equiv\mathbf{v_k}/v$, $\hat{\mathbf{p}}\equiv\mathbf{p}/p$, $\hat{\mathbf{v}}_\perp$ is a unit vector perpendicular to $\hat{\mathbf{v}}$, and $\cos\varphi\equiv\hat{\mathbf{v}}\cdot\hat{\mathbf{p}}$ is the cosine of the angle between $\mathbf{v_k}$ and $\mathbf{p}$. $\gamma_\mathbf{k}$ is given in Eq. (A9). We have normalized $\mathbf{v_k}$ such that $\mathbf{v_k}\to\hat{\mathbf{k}}$ for small k, and hence $0\le v(\mathbf{k})\le 1$. It is easy to see that $v(\mathbf{k})$ is the eccentricity of the ellipse given in Eqs. (2.15) and (2.16). Note also that for $\mathbf{k}$ precisely at the zone boundary $v(\mathbf{k})=0$, and the scattering surface is a circle with radius p around the point $\mathbf{p}$.

Having obtained a convenient parametric representation of the scattering surface $\mathbf{q}(t;\mathbf{p},\mathbf{k})$, we can eliminate the δ function and write the $\mathbf{q}$ integration as an integral over the parameter t (we set $D=2$ from now on)

$$\frac{\hbar\Gamma_\mathbf{k}}{E_\pi}=\left[\frac{1}{2\pi^3}\right]\left[\frac{1}{S\alpha(T,S)}\right]^2\int_0^{2\pi}d\varphi\int_0^{2\pi}dt\int_0^\infty p\,dp\,\tilde{B}(t,\varphi,p,\mathbf{k},\tau)\tilde{W}(t,\varphi,p/\sqrt{2},\mathbf{k})\ , \tag{2.18}$$

where

$$\tilde{B}(t,\varphi,p,\mathbf{k},\tau)\equiv\left[\frac{|d\mathbf{q}(t)/dt|}{|\nabla_\mathbf{q}(\epsilon_{\mathbf{k}+\mathbf{q}}+\epsilon_{\mathbf{p}-\mathbf{q}})|_{\mathbf{q}=\mathbf{q}(t)}}\right]n(\epsilon_\mathbf{p}/\tau)\times[1+n(\epsilon_{p\mathbf{r}(t)}/\tau)]\ , \tag{2.19}$$

and

$$\tilde{W}(t,\varphi,p/\sqrt{2},\mathbf{k})\equiv W(\mathbf{q}(t;\mathbf{p},\mathbf{k}),\mathbf{p},\mathbf{k})\ . \tag{2.20}$$

Here $\mathbf{p}=p[\cos\varphi,\sin\varphi]$ and the p^1 axis is in the direction of $\mathbf{v_k}$. Because the Bose factor $n(\epsilon_\mathrm{p}/\tau)$ cuts off the p integration at $p\approx\tau$, we have extended the upper limit for the p integration in Eq. (2.18) to infinity without changing the value of the integral. From Eqs. (2.15) and (2.16) it is easy to show that, for $p\ll 1$,

$$\frac{|d\mathbf{q}(t)/dt|}{|\nabla_\mathbf{q}(\epsilon_{\mathbf{k}+\mathbf{q}}+\epsilon_{\mathbf{p}-\mathbf{q}})|_{\mathbf{q}=\mathbf{q}(t)}}=\sqrt{2}\left[\frac{1}{1-v^2(\mathbf{k})}\right]^{3/2}p[1-v(\mathbf{k})\cos\varphi]\times[1-v(\mathbf{k})\cos t][1+O(p)]\ . \tag{2.21}$$

Clearly the function $\tilde{B}(t,\varphi,p,\mathbf{k},\tau)$ diverges for $p\to 0$ as p^{-1}. In the integral, Eq. (2.18), this singularity is cancelled by a factor p from the Jacobian. Moreover, we show in Appendix B that the function $\tilde{W}(t,\varphi,p/\sqrt{2},\mathbf{k})$ has a finite limit as $p\to 0$ [see Eq. (2.25)], and hence *no infrared divergencies are encountered.* Note also that the use of the long-wavelength expansion for $\epsilon_{p\mathbf{r}(t)}$, Eq. (2.10), is justified, because $|p\mathbf{r}(t)|$ is a small quantity in the regime $\tau\ll k^3$ we are considering here. To prove this, we calculate the length of the vector $\mathbf{r}$. From Eq. (2.16) we obtain

$$|\mathbf{r}(t)|\equiv r(t,\varphi;v(\mathbf{k}))=\frac{[1-v(\mathbf{k})\cos t][1-v(\mathbf{k})\cos\varphi]}{1-v^2(\mathbf{k})}\ . \tag{2.22}$$

Obviously for zone-boundary magnons, for which $v(\mathbf{k})$ is small, $|\mathbf{r}(t)|$ is of order unity, and hence $|p\mathbf{r}|\ll 1$ (keeping in mind that $p\lesssim\tau\ll 1$). Also in the case $k\ll 1$, where we may use the small momentum limit for $v(\mathbf{k})$ [see Eq. (2.28)], we find that $|p\mathbf{r}|\approx p/k^2\lesssim\tau/k^2$ is a small quantity, because according to Eq. (2.9) we have $\tau/k^2\ll k\lesssim 1$.

Writing $p=\sqrt{2}\tau x$ we obtain

$$\frac{\hbar\Gamma_\mathbf{k}}{E_\pi}=\left[\frac{8}{\pi}\right]\left[\frac{1}{S\alpha(T,S)}\right]^2\tau^3\tilde{Z}(\mathbf{k},\tau)[1+O(\tau)]\ , \tag{2.23}$$

where the function $\tilde{Z}$ is defined by

$$\tilde{Z}(\mathbf{k},\tau)=\left[\frac{1}{2\pi}\right]^2\left[\frac{1}{1-v^2(\mathbf{k})}\right]^{3/2}\int_0^{2\pi}d\varphi\int_0^{2\pi}dt\int_0^\infty dx[1-v(\mathbf{k})\cos t][1-v(\mathbf{k})\cos\varphi]\tilde{W}(t,\varphi,\tau x,\mathbf{k})\times\left[\frac{x}{\exp(x)-1}\right]\left[\frac{x}{1-\exp[-r(t,\varphi;v(\mathbf{k}))x]}\right]\ . \tag{2.24}$$

Note that the x integration is cut off at $x=O(1)$. Because $\tau\ll 1$, we may expand $\tilde{W}(t,\varphi,\tau x,\mathbf{k})$ for small τx. As shown in Appendix B, the lowest-order term in this expansion is independent of τx and is given by

$$\tilde{W}(t,\varphi,\tau x,\mathbf{k})=\tfrac{1}{2}[1-v^2(\mathbf{k})]\frac{[1-v(\mathbf{k})\cos\varphi]}{[1-v(\mathbf{k})\cos t]}+O(\tau x)\ . \tag{2.25}$$

The final result for the damping of magnons in the regime $\tau^{1/3}\ll k$, which also includes zone-boundary magnons, is

$$\frac{\hbar\Gamma_{\mathbf{k}}}{E_\pi}=\left[\frac{4\pi}{3}\right]\left[\frac{1}{S\alpha(T,S)}\right]^2\tau^3 Z(v(\mathbf{k}))[1+O(\tau)] , \tag{2.26}$$

where $Z(v)$ is defined by

$$Z(v)\equiv\left[\frac{1}{1-v^2}\right]^{1/2}\left[\frac{1}{2\pi}\right]^2\left[\frac{3}{\pi^2}\right]\int_0^{2\pi}d\varphi\int_0^{2\pi}dt\int_0^\infty dx[1-v\cos\varphi]^2\left[\frac{x}{\exp(x)-1}\right]\left[\frac{x}{1-\exp[-r(t,\varphi;v)x]}\right] . \tag{2.27}$$

To the leading order in τ and $(2S)^{-1}$, $\alpha(T,S)$ can be set to unity. This is consistent, because α has an expression of the form

$$\alpha(T,S)=1+O(T/E_\pi,(2S)^{-1})$$

[see Eq. (A12) for $\alpha(0,S)$], and the correction in Eq. (1.3) depends also on the higher-order contributions to the self-energy.

The integration in Eq. (2.27) becomes trivial for $v=0$ (corresponding to magnons with momenta precisely at the boundary of the BZ), because in this case $r(t,\varphi;0)=1$, independent of t and φ. We have normalized Z such that $Z(v=0)=1$. Although we have not succeeded in performing the integration in Eq. (2.27) analytically for general v, it is easy to show that for small v, i.e., for $\mathbf{k}$ close to the zone boundary, $Z(v)$ increases quadratically:

$$Z(v(\mathbf{k}))=1+O(v^2(\mathbf{k})) .$$

For $v\to1$ it diverges as $(1-v)^{-1/2}$. Note that for small k the function $v(\mathbf{k})$ has an expansion

$$v(\mathbf{k})=1-g_D(\hat{\mathbf{k}})k^2+O(k^4) , \tag{2.28}$$

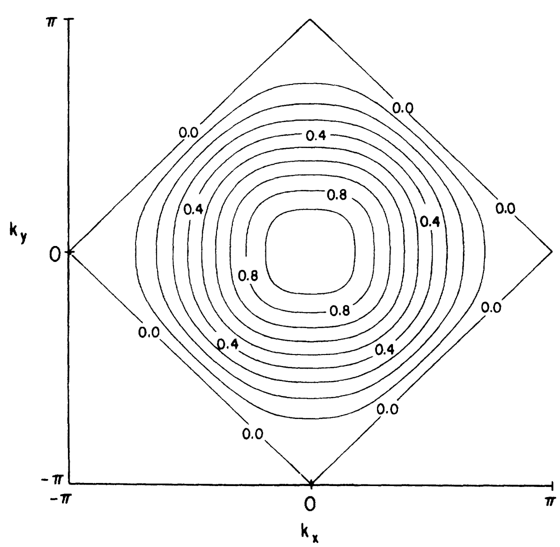

FIG. 2. Curves of constant magnon damping in the Brillouin zone of the two-dimensional QAFM. The curves are defined via the equation $|\mathbf{v}_{\mathbf{k}}|\equiv v=\text{const}$, where the vector $\mathbf{v}_{\mathbf{k}}$ is the gradient of the magnon dispersion relation [see Eq. (2.13)]. Note that $0\le|\mathbf{v}_{\mathbf{k}}|\le1$. Neighboring contours correspond to an increase of 0.1 in v.

where $g_D(\hat{\mathbf{k}})$ depends only on the direction of $\mathbf{k}$ and is strictly positive. In $D=2$ it follows that for $k\to0$

$$Z(v(\mathbf{k}))=\frac{9\zeta(3)}{\sqrt{2}\pi^2}[g_2(\hat{\mathbf{k}})]^{-1/2}k^{-1}[1+O(k)] , \tag{2.29}$$

where $\zeta(3)\simeq1.202$. A numerical evaluation of $Z(v)$ is shown in Fig. 1. According to Eq. (2.26) the momentum dependence of $\Gamma_{\mathbf{k}}$ enters only through $|\mathbf{v}_{\mathbf{k}}|$; i.e., surfaces of constant damping are curves on which the gradient of the magnon dispersion relation has constant length. These curves, defined by $|\mathbf{v}_{\mathbf{k}}|\equiv v=\text{const}$, are shown in Fig. 2. Because $Z(v)$ deviates significantly from unity for $v\gtrsim0.8$, it is clear that at low temperatures the damping of magnons in a large region of the BZ is not very different from the damping of magnons at the zone boundary.

III. DISCUSSION

In this section we shall carefully examine the range of validity of Eq. (2.26). In particular, we shall show that the analysis presented in Sec. III is only consistent for momenta k in the short-wavelength region $k\gg\tau^{1/3}$ [cf. Eq. (2.9)]. Furthermore, we shall use the golden-rule expression, Eq. (2.1), to *estimate* the damping of magnons at longer wavelengths.

Three independent parameters in Eq. (2.26) are T, S, and the momentum $\mathbf{k}$. We have assumed that $k^3\gg\tau$; i.e., the magnons that we consider have energies much larger than the thermal energy, or, equivalently, their wavelength is much shorter than the thermal de Broglie wavelength. Furthermore, we have implicitly assumed that the perturbation expansion for the self-energy can be truncated at the second order and have ignored the kinematic interactions between spin waves. As argued in Sec. I, for $k\xi\gg1$ the perturbation expansion in powers of $(2S)^{-1}$ should be well defined.

HKHH have shown that in three dimensions, and in the limit of long wavelengths vertex corrections can be safely ignored. For $D=2$, it is clearly not justified to neglect vertex corrections at long wavelengths. This has been explicitly demonstrated by Kosevich and Chubukov,[16] who have found that divergent vertex corrections signal the breakdown of perturbation theory, if the wavelength of the magnons becomes comparable to the order-parameter correlation length, i.e., if $k\xi=O(1)$. In the present paper, however, we study short-wavelength magnons. We have convinced ourselves by direct calculation of the lowest-order diagrams contributing to the renormalization of the bare vertices, that in the regime $\tau\ll k^3$ the "dressed" vertices modify our final result, Eq. (2.26), by nonsingular corrections order $(2S)^{-1}$ and

$(2S)^{-1}\tau^2/k$. These corrections do not contribute to the asymptotic behavior we are considering here. It is clear that these approximations amount to a possible multiplicative correction factor of the form

$$Y(S)\equiv 1+y_1(2S)^{-1}+y_2(2S)^{-2}+\cdots \ .$$

This will modify the coefficient of the T^3 law for $\Gamma_{\mathbf{k}}$.

Let us now discuss the range of validity of Eq. (2.26) in momentum space. In deriving this equation, we have made the basic assumption that Eq. (2.14) is an accurate approximation for the exact scattering surface, which is defined via Eq. (2.11). At low temperatures this assumption is certainly satisfied for zone-boundary magnons, for which k is of order unity. We now show that the *lower limit* for k, where Eq. (2.14) ceases to correctly describe the true scattering surface, is given by $p^{1/3}$. This criterion has been given by HKHH for $D=3$. That it is also true for $D=2$ can be easily seen from Eqs. (2.15) and (2.16): If we insert the long-wavelength limit for $v(\mathbf{k})$ [cf. Eq. (2.28)], we see that the characteristic value of $|\mathbf{q}(t)|$ is $q\approx p/k^2$. As our approximate equation for the scattering surface was derived by expanding $\epsilon_{\mathbf{k}+\mathbf{q}}$ (assuming $q\ll k$), we need, for consistency, the condition $p/k^2\lesssim k$. Remembering that the $\mathbf{p}$ integration in Eq. (2.1) is cut off at $p\approx\tau$, we conclude that Eq. (2.26) is valid, *even for small k,* as long as $\tau^{1/3}\ll k$. This is equivalent to Eq. (2.9), and implies $\tau\ll 1$. Note that for $\tau^{1/3}\ll k\ll 1$ we find, from Eqs. (2.26) and (2.29), that

$$\frac{\hbar\Gamma_{\mathbf{k}}}{E\pi}=\left[\frac{6\sqrt{2}\zeta(3)}{\pi}\right]S^{-2}\tau^3[g_2(\hat{\mathbf{k}})]^{-1/2}k^{-1} \ . \tag{3.1}$$

We emphasize that this expression is only valid for $\tau\ll k^3$, and hence the damping in this regime is certainly small compared to the energy.

As mentioned in Sec. I, we expect that the magnons with large momenta compared to the inverse correlation length are well-defined quasiparticles. Because ξ^{-1} is exponentially small at low temperatures, there is a large range in the momentum space,

$$\xi^{-1}\ll k\lesssim\tau^{1/3} \ , \tag{3.2}$$

where our approximation for the scattering surface, Eq. (2.14), is not valid. Nonetheless, perturbation theory may be applicable, although in the regime $k\ll\lambda_{\text{th}}^{-1}$ vertex corrections will become important, such that it may not be justified to truncate the perturbation expansion at the second order. Unfortunately, evaluation of Eq. (2.1) is not so straightforward in the regime defined by Eq. (3.2), as the scattering surface cannot be described by a single analytic expression, valid for all $\mathbf{p}$ contributing to the integral. In three dimensions, discussed by HKHH, the situation is different. Because of the presence of an extra factor of p in the Jacobian, the integration in Eq. (2.1) is dominated by thermal momenta $p\approx\tau$. Thus, it is sufficient to estimate the integral using a form of the scattering surface that is valid for $p\approx\tau$. By contrast, in $D=2$ the integrand is *not* dominated by a single range of momenta p, and all momenta up to $p\approx\tau$ contribute to the integral. The short-wavelength magnons are the simplest to consider. Because of the inequality $p\lesssim\tau\ll k^3$, Eq. (2.14) is an accurate description of the scattering surface for all $\mathbf{p}$ where the integrand is nonvanishing.

Although the technical difficulties mentioned have prevented us from evaluating Eq. (2.1) in the long-wavelength regimes with the same accuracy as we have done for $k\gg\tau^{1/3}$, we can give a rough estimate. If we neglect all angular dependencies and count the powers of p and k in the integrand, we find that in the range $\xi^{-1}\ll k\ll\tau^3$

$$\Gamma_{\mathbf{k}}\sim\tau^2k \ . \tag{3.3}$$

This estimate agrees with the one given by Kosevich and Chubukov,[16] who claim that it is actually valid for $\xi^{-1}\ll k\ll\tau$.

IV. CONCLUSIONS

In this paper we have shown that at low temperatures conventional spin-wave techniques, when applied carefully, can be used to calculate in the two-dimensional QAFM the damping of magnons for all momenta that are large compared to the inverse correlation length. Thus, even in $D=2$ magnons are found to be well-defined quasiparticles; i.e., their damping is small compared to their energy. Let us summarize the various expressions for the magnon damping in the low-temperature regimes we have considered in the present paper:

$$\Gamma_{\mathbf{k}}\propto\begin{cases}\tau^2k & \text{if } \xi^{-1}\ll k\ll\tau^3\\ \tau^3[g_2(\hat{\mathbf{k}})]^{-1/2}k^{-1} & \text{if } \tau^{1/3}\ll k\ll 1\\ \tau^3Z(|v_{\mathbf{k}}|) & \text{if } \tau^{1/3}\ll k\approx 1 \ .\end{cases} \tag{4.1}$$

Although we have no definite predictions for the behavior in the regime $\tau^3\lesssim k\lesssim\tau^{1/3}$, it is clear that it must be an interpolation between the behavior in the neighboring regimes. Thus the damping is expected to be small.

Note added in proof. After completion of this work I received a copy of work prior to publication by S. Tyc and B. I. Halperin where they use DM formalism to calculate magnon damping for $\xi^{-1}\ll k\ll 1$. In the regime $\tau^{1/3}\ll k\ll 1$, their result agrees with Eq. (3.1). Furthermore, they show that for $\xi^{-1}\ll k\ll\tau^3$ the correct asymptotic behavior is given by $\Gamma_{\mathbf{k}}\propto k\tau^2\ln\tau^{-1}$.

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APPENDIX A: THE DYSON-MALEEV VERTICES

In this appendix we shall introduce a convenient parametrization of the two-magnon vertices on a bipartite lattice in D dimensions. We shall also give explicit expressions of the vertices in the various long-wavelength limits. Although such expressions can be found in Appendix A of HKHH, the underlying symmetries are not transparent, and there seem to be some inconsistencies in their long-wavelength limits.[17] The DM transformation is

$$\mathcal{S}_i^+ = \sqrt{2S}\,[1-(2S)^{-1}a_i^\dagger a_i]a_i \;,$$
$$\mathcal{S}_j^+ = \sqrt{2S}\,b_j^\dagger[1-(2S)^{-1}b_j^\dagger b_j] \;,$$
$$\mathcal{S}_i^- = \sqrt{2S}\,a_i^\dagger, \quad \mathcal{S}_j^- = \sqrt{2S}\,b_j \;,$$
$$\mathcal{S}_i^z = S - a_i^\dagger a_i, \quad \mathcal{S}_j^z = -(S - b_j^\dagger b_j) \;, \tag{A1}$$

where the Boson operators a_i (b_j) refer to sublattice A (B). Next we introduce Fourier transformed operators $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$ via

$$a_i \equiv \frac{1}{\sqrt{N/2}} \sum_{\mathbf{k}} \exp(i\mathbf{k}\cdot\mathbf{r}_i)a_{\mathbf{k}} \;, \tag{A2}$$

$$b_j \equiv \frac{1}{\sqrt{N/2}} \sum_{\mathbf{k}} \exp(-i\mathbf{k}\cdot\mathbf{r}_j)b_{\mathbf{k}} \;, \tag{A3}$$

where the sums are over the $N/2$ points of the first BZ of the QAFM, and $\mathbf{r}_i$ is the location of lattice site i. Note that HKHH have introduced the Fourier transformation on sublattice B with an opposite sign in the exponential. Finally, the part of the Hamiltonian that is quadratic in the Boson operators is diagonalized via a canonical transformation to the magnon quasiparticle operators $\alpha_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$. The most general form of this transformation is given by

$$a_{\mathbf{k}} \equiv u_{\mathbf{k}}^*(\alpha_{\mathbf{k}} - x_{\mathbf{k}}^*\beta_{\mathbf{k}}^\dagger) \;, \tag{A4}$$

$$b_{\mathbf{k}}^\dagger \equiv u_{\mathbf{k}}(-x_{\mathbf{k}}\alpha_{\mathbf{k}} + \beta_{\mathbf{k}}^\dagger) \;, \tag{A5}$$

where

$$u_{\mathbf{k}} \equiv \left[\frac{1+\epsilon_{\mathbf{k}}}{2\epsilon_{\mathbf{k}}}\right]^{1/2} \exp(i\phi_{\mathbf{k}}) \;, \tag{A6}$$

$$x_{\mathbf{k}} \equiv \left[\frac{1-\epsilon_{\mathbf{k}}}{1+\epsilon_{\mathbf{k}}}\right]^{1/2} \exp(-2i\phi_{\mathbf{k}}) \tag{A7}$$

with

$$\epsilon_{\mathbf{k}} \equiv (1-\gamma_{\mathbf{k}}^2)^{1/2} \;, \tag{A8}$$

and[10]

$$\gamma_{\mathbf{k}} \equiv \frac{1}{D}\sum_{\mu=1}^{D} \cos k^\mu \;. \tag{A9}$$

The phases $\phi_{\mathbf{k}}$ are arbitrary. From Eqs. (A4) and (A5) it is obvious that a factor $\exp(-i\phi_{\mathbf{k}})$ is associated with each quasiparticle operator $\alpha_{\mathbf{k}}$ or $\beta_{\mathbf{k}}$. These phases do, however, cancel if one calculates physical observables, as it should be, and we shall choose all phase factors to be unity from now on.

For convenience we normal order the Hamiltonian and write the final result in the form

$$\mathcal{H}_{\rm DM} = E_{\rm DM}^{(0)} + :\mathcal{H}_{\rm DM}^{(2)}: + :\mathcal{H}_{\rm DM}^{(4)}: \;. \tag{A10}$$

Here $E_{\rm DM}^{(0)}$ is the ground-state energy and is given by

$$E_{\rm DM}^{(0)} = -NDJ[S\alpha(0,S)]^2 \;, \tag{A11}$$

where

$$\alpha(0,S) = 1 + \frac{C}{2S} \tag{A12}$$

is the Hartree-Fock renormalization factor at $T=0$, introduced in Eq. (2.7). The numerical constant C is defined by

$$C \equiv 1 - \frac{1}{(N/2)}\sum_{\mathbf{k}} \epsilon_{\mathbf{k}} \tag{A13}$$

and equals 0.158 in two dimensions.[18] The Hamiltonian can be written in a very compact form if we express it in terms of the two-component operators

$$\Psi_{\mathbf{k}}^\dagger \equiv (\alpha_{\mathbf{k}}^\dagger, \beta_{\mathbf{k}}) \;. \tag{A14}$$

The free part of the Hamiltonian is then

$$:\mathcal{H}_{\rm DM}^{(2)}: = E_\pi(0)\sum_{\mathbf{k}} \epsilon_{\mathbf{k}} :\Psi_{\mathbf{k}}^\dagger \Psi_{\mathbf{k}}: \;, \tag{A15}$$

where $E_\pi(0) = 2DJS\alpha(0,S)$ [see Eq. (2.2)].

The expression for $:\mathcal{H}_{\rm DM}^{(4)}:$ is a linear combination of ten terms quartic in the quasiparticle operators. Each of these terms can be characterized by an ordered pair of two of the four matrices σ^+, σ^-, p^+, and p^-, defined by

$$\sigma^+ \equiv \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \sigma^- \equiv \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$
$$p^+ \equiv \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad p^- \equiv \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{A16}$$

If we use the abbreviation

$$\{p^-;\sigma^+\}_{(\mathbf{q},1,2)} \equiv :(\Psi_{1+\mathbf{q}}^\dagger p^- \Psi_1)(\Psi_{2-\mathbf{q}}^\dagger \sigma^+ \Psi_2): = \alpha_{2-\mathbf{q}}^\dagger \beta_2^\dagger \beta_1^\dagger \beta_{1+\mathbf{q}} \;, \tag{A17}$$

(the expressions involving other matrices are defined analogously) the normal ordered interaction part of the DM Hamiltonian can be written as

$$\begin{aligned} :\mathcal{H}_{\rm DM}^{(4)}: = \frac{DJ}{N}\sum_{\mathbf{q},1,2} [&[\{p^+;p^+\}+\{p^-;p^-\}]V^{(1)} \\ &-2[\{p^+;\sigma^-\}+\{p^-;\sigma^+\}]V^{(2)} + 2[\{\sigma^+;p^+\}+\{\sigma^-;p^-\}]V^{(3)} \\ &-2[\{p^+;p^-\}+\{p^-;p^+\}]V^{(4)} + [\{\sigma^+;\sigma^+\}+\{\sigma^-;\sigma^-\}]V^{(5)}] \;, \end{aligned} \tag{A18}$$

where $V^{(j)} \equiv V^{(j)}(\mathbf{q},1,2)$, $j=1,\ldots,5$, are the DM vertices. For notational simplicity we have written 1 for $\mathbf{k}_1$ and 2 for $\mathbf{k}_2$, and have omitted the arguments $(\mathbf{q},1,2)$ in the sum. Note that our vertices $V^{(j)}$ correspond to the expressions $\Phi^{(j)}$ given by HKHH for $J=1,2,3,4$, while our $V^{(5)}$ corresponds to $\Phi^{(7)}$ of HKHH.

The symmetries of the DM vertices become very transparent by writing them as follows:

$$V^{(1)}(\mathbf{q},1,2)=A[x_{1+\mathbf{q}}F_1+x_{2-\mathbf{q}}F_2]\ , \tag{A19}$$

$$V^{(2)}(\mathbf{q},1,2)=A[x_{1+\mathbf{q}}x_{2-\mathbf{q}}F_1+F_2]\ , \tag{A20}$$

$$V^{(3)}(\mathbf{q},1,2)=A[x_{2-\mathbf{q}}F_3+x_{1+\mathbf{q}}F_4]\ , \tag{A21}$$

$$V^{(4)}(\mathbf{q},1,2)=A[x_{1+\mathbf{q}}x_{2-\mathbf{q}}F_3+F_4]\ , \tag{A22}$$

$$V^{(5)}(\mathbf{q},1,2)=A[x_{2-\mathbf{q}}F_1+x_{1+\mathbf{q}}F_2]\ , \tag{A23}$$

where $F_j=F_j(\mathbf{q},1,2)$, $(j=1,2,3,4)$, and $A=A(\mathbf{q},1,2)$ are functions of the momentum transfer $\mathbf{q}$ as well the incoming momenta $\mathbf{k}_1$ and $\mathbf{k}_2$, which we shall define and discuss now. The function A is given by

$$A(\mathbf{q},1,2)\equiv\frac{u_{1+\mathbf{q}}u_{2-\mathbf{q}}}{u_1u_2}\ . \tag{A24}$$

To define the functions F_j, we first introduce the two auxiliary functions

$$f(\mathbf{q},1,2)\equiv u_1^2[\gamma_{\mathbf{q}+1-2}-x_1\gamma_{\mathbf{q}-2}]\ , \tag{A25}$$

$$g(\mathbf{q},1,2)\equiv u_1^2[x_1\gamma_{\mathbf{q}+1-2}-\gamma_{\mathbf{q}-2}]\ . \tag{A26}$$

The F_j are then given by

$$F_1(\mathbf{q},1,2)\equiv u_2^2[f(\mathbf{q},1,0)-x_2f(\mathbf{q},1,2)]\ , \tag{A27}$$

$$F_2(\mathbf{q},1,2)\equiv u_2^2[x_2g(\mathbf{q},1,0)-g(\mathbf{q},1,2)]\ , \tag{A28}$$

$$F_3(\mathbf{q},1,2)\equiv u_2^2[f(\mathbf{q},1,2)-x_2f(\mathbf{q},1,0)]\ , \tag{A29}$$

$$F_4(\mathbf{q},1,2)\equiv u_2^2[x_2g(\mathbf{q},1,2)-g(\mathbf{q},1,0)]\ . \tag{A30}$$

Note that if we interchange the two incoming momenta and reverse the sign of the momentum transfer, the F_j transform into each other according to

$$\begin{bmatrix}F_1\\F_2\\F_3\\F_4\end{bmatrix}_{(\mathbf{q},1,2)}=\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}F_1\\F_2\\F_3\\F_4\end{bmatrix}_{(-\mathbf{q},2,1)}\ . \tag{A31}$$

It follows that

$$V^{(j)}(\mathbf{q},1,2)=V^{(j)}(-\mathbf{q},2,1)\quad\text{for } j=1,4,5\ , \tag{A32}$$

which implies, in particular, that under the summation sign in Eq. (A18) we may replace $\{p^-;p^+\}V^{(4)}$ by $\{p^+;p^-\}V^{(4)}$. The vertices $V^{(2)}$ and $V^{(3)}$ do *not* have the symmetry (A32) and hence the order of the matrices in the curly braces *cannot* be interchanged.

The advantage of our parametrization becomes evident if one considers the limiting behavior of these functions as the momenta approach zero. For small k we have

$$x_{\mathbf{k}}=1-\frac{k}{\sqrt{D}}+\frac{k^2}{2D}+O(k^3)\ , \tag{A33}$$

$$\gamma_{\mathbf{k}}=1-\frac{k^2}{2D}+O(k^4)\ , \tag{A34}$$

$$u_{\mathbf{k}}=\left[\frac{\sqrt{D}}{2}\right]^{1/2}\frac{1}{\sqrt{k}}+O(\sqrt{k})\ . \tag{A35}$$

Hence, although the factor $u_{\mathbf{k}}^2$ is singular as $k\rightarrow 0$, the difference in the square brackets in Eqs. (A25)–(A30) exactly cancel these singularities and the long-wavelength limits of the functions f, g, and F_j are finite. The singular behavior of the DM vertices is entirely contained in the function A, which exhibits square-root singularities as $\mathbf{q}$ approaches $-\mathbf{k}_1$ or $\mathbf{k}_2$. Note, however, that A satisfies

$$A(\mathbf{q},\mathbf{k}_1,\mathbf{k}_2)A(-\mathbf{q},\mathbf{k}_1+\mathbf{q},\mathbf{k}_2-\mathbf{q})=1\ . \tag{A36}$$

It turns out that in the expression for the second-order self-energy the momentum dependence of the vertices is such that always two factors of A appear in the above combination; thus the singular terms in the DM vertices precisely cancel.

The various long-wavelength limits of the DM vertices, and hence the behavior of the function $W(\mathbf{q},\mathbf{k}_1,\mathbf{k}_2)$ defined in Eq. (2.6), can be obtained from the corresponding limits of the functions F_j. We now give a list of these limits, which might be useful also for other calculations done with the Dyson-Maleev formalism. We denote by $\hat{\mathbf{k}}$ the unit vector in the direction of $\mathbf{k}$, and define a vector $\mathbf{G}_{\mathbf{k}}$ with components

$$G_{\mathbf{k}}^{\mu}\equiv -D\partial^{\mu}\gamma_{\mathbf{k}}=\sin k^{\mu} \tag{A37}$$

and a matrix $\mathcal{M}_{\mathbf{k}}$ with elements

$$\mathcal{M}_{\mathbf{k}}^{\mu\nu}\equiv -D\partial^{\mu}\partial^{\nu}\gamma_{\mathbf{k}}=\delta^{\mu\nu}\cos k^{\mu}\ . \tag{A38}$$

The normalization is such that $\mathbf{G}_{\mathbf{k}}\rightarrow\mathbf{k}$ and $\mathcal{M}_{\mathbf{k}}$ becomes a D-dimensional unit matrix for $\mathbf{k}\rightarrow 0$. First, we give the limits if one of the incoming momenta approaches zero:

$$F_1(\mathbf{q},1,2)\overset{k_1\rightarrow 0}{\sim}\frac{u_2^2}{2}\left[(\gamma_{\mathbf{q}}-x_2\gamma_{\mathbf{q}-2})-\frac{\hat{\mathbf{k}}_1}{\sqrt{D}}\cdot(\mathbf{G}_{\mathbf{q}}-x_2\mathbf{G}_{\mathbf{q}-2})\right]+O(k_1) \tag{A39}$$

$$\overset{k_2\rightarrow 0}{\sim}\frac{u_1^2}{2}\left[(\gamma_{\mathbf{q}+1}-x_1\gamma_{\mathbf{q}})-\frac{\hat{\mathbf{k}}_2}{\sqrt{D}}\cdot(\mathbf{G}_{\mathbf{q}+1}-x_1\mathbf{G}_{\mathbf{q}})\right]+O(k_2)\ , \tag{A40}$$

$$F_2(\mathbf{q},1,2) \overset{k_1\to 0}{\sim} \frac{u_2^2}{2}\left[(\gamma_{\mathbf{q}-2}-x_2\gamma_{\mathbf{q}})+\frac{\hat{\mathbf{k}}_1}{\sqrt{D}}\cdot(\mathbf{G}_{\mathbf{q}-2}-x_2\mathbf{G}_{\mathbf{q}})\right]+O(k_1) \tag{A41}$$

$$F_3(\mathbf{q},1,2) \overset{k_1\to 0}{\sim} \frac{u_2^2}{2}\left[(\gamma_{\mathbf{q}-2}-x_2\gamma_{\mathbf{q}})-\frac{\hat{\mathbf{k}}_1}{\sqrt{D}}\cdot(\mathbf{G}_{\mathbf{q}-2}-x_2\mathbf{G}_{\mathbf{q}})\right]+O(k_1) \tag{A43}$$

$$\overset{k_2\to 0}{\sim} \frac{u_1^2}{2}\left[(\gamma_{\mathbf{q}+1}-x_1\gamma_{\mathbf{q}})+\frac{\hat{\mathbf{k}}_2}{\sqrt{D}}\cdot(\mathbf{G}_{\mathbf{q}+1}-x_1\mathbf{G}_{\mathbf{q}})\right]+O(k_2)\ , \tag{A44}$$

$$F_4(\mathbf{q},1,2) \overset{k_1\to 0}{\sim} \frac{u_2^2}{2}\left[(\gamma_{\mathbf{q}}-x_2\gamma_{\mathbf{q}-2})+\frac{\hat{\mathbf{k}}_1}{\sqrt{D}}\cdot(\mathbf{G}_{\mathbf{q}}-x_2\mathbf{G}_{\mathbf{q}-2})\right]+O(k_1) \tag{A45}$$

$$\overset{k_2\to 0}{\sim} \frac{u_1^2}{2}\left[(\gamma_{\mathbf{q}}-x_1\gamma_{\mathbf{q}+1})-\frac{\hat{\mathbf{k}}_2}{\sqrt{D}}\cdot(\mathbf{G}_{\mathbf{q}}-x_1\mathbf{G}_{\mathbf{q}+1})\right]+O(k_2)\ . \tag{A46}$$

If both incoming momenta approach zero, we have

$$F_1(\mathbf{q},1,2) \overset{k_1,k_2\to 0}{\sim} \frac{1}{4}\left[\gamma_{\mathbf{q}}-\hat{\mathbf{k}}_1\mathcal{M}_{\mathbf{q}}\hat{\mathbf{k}}_2-\frac{\hat{\mathbf{k}}_1+\hat{\mathbf{k}}_2}{\sqrt{D}}\cdot\mathbf{G}_{\mathbf{q}}\right]+O(k_1,k_2)\ , \tag{A47}$$

$$F_2(\mathbf{q},1,2) \overset{k_1,k_2\to 0}{\sim} \frac{1}{4}\left[\gamma_{\mathbf{q}}-\hat{\mathbf{k}}_1\mathcal{M}_{\mathbf{q}}\hat{\mathbf{k}}_2+\frac{\hat{\mathbf{k}}_1+\hat{\mathbf{k}}_2}{\sqrt{D}}\cdot\mathbf{G}_{\mathbf{q}}\right]+O(k_1,k_2)\ , \tag{A48}$$

$$F_3(\mathbf{q},1,2) \overset{k_1,k_2\to 0}{\sim} \frac{1}{4}\left[\gamma_{\mathbf{q}}+\hat{\mathbf{k}}_1\mathcal{M}_{\mathbf{q}}\hat{\mathbf{k}}_2-\frac{\hat{\mathbf{k}}_1-\hat{\mathbf{k}}_2}{\sqrt{D}}\cdot\mathbf{G}_{\mathbf{q}}\right]+O(k_1,k_2)\ , \tag{A49}$$

$$F_4(\mathbf{q},1,2) \overset{k_1,k_2\to 0}{\sim} \frac{1}{4}\left[\gamma_{\mathbf{q}}+\hat{\mathbf{k}}_1\mathcal{M}_{\mathbf{q}}\hat{\mathbf{k}}_2+\frac{\hat{\mathbf{k}}_1-\hat{\mathbf{k}}_2}{\sqrt{D}}\cdot\mathbf{G}_{\mathbf{q}}\right]+O(k_1,k_2)\ . \tag{A50}$$

Note that the symmetries, Eq. (A31), are now obvious, since $\gamma_{-\mathbf{q}}=\gamma_{\mathbf{q}}$, $\mathcal{M}_{-\mathbf{q}}=\mathcal{M}_{\mathbf{q}}$ and $\mathbf{G}_{-\mathbf{q}}=-\mathbf{G}_{\mathbf{q}}$.

Because F_j $(j=1,2,3,4)$ contains only terms that are nonsingular in $\mathbf{q}$, one can simply set $\mathbf{q}=0$ to obtain the leading behavior for small-momentum transfer. Obviously F_j, and hence also the DM vertices, are then independent of the momentum transfer $\mathbf{q}$. Using the fact that the vector $\mathbf{G}_{\mathbf{q}}$ is related to the velocity $\mathbf{v}_{\mathbf{k}}$ defined in Eq. (2.13) via

$$\mathbf{v}_{\mathbf{k}}=\frac{2}{\sqrt{D}}u_{\mathbf{k}}^2x_{\mathbf{k}}G_{\mathbf{k}}\ , \tag{A51}$$

we obtain from Eqs. (A39)–(A46)

$$F_1(\mathbf{q},1,2) \overset{k_1 q\to 0}{\sim} \frac{1}{4}[2u_2^2(1-x_2\gamma_2)-\hat{\mathbf{k}}_1\cdot\mathbf{v}_2]+O(k_1,q) \tag{A52}$$

$$\overset{k_2,q\to 0}{\sim} \frac{1}{4}\left[2u_1^2(\gamma_1-x_1)-\hat{\mathbf{k}}_2\cdot\frac{\mathbf{v}_1}{x_1}\right]+O(k_2,q)\ , \tag{A53}$$

$$F_2(\mathbf{q},1,2) \overset{k_1,q\to 0}{\sim} \frac{1}{4}\left[2u_2^2(\gamma_2-x_2)-\hat{\mathbf{k}}_1\cdot\frac{\mathbf{v}_2}{x_2}\right]+O(k_1,q) \tag{A54}$$

$$\overset{k_2,q\to 0}{\sim} \frac{1}{4}[2u_1^2(1-x_1\gamma_1)-\hat{\mathbf{k}}_2\cdot\mathbf{v}_1]+O(k_2,q)\ , \tag{A55}$$

$$F_3(\mathbf{q},1,2) \overset{k_1,q\to 0}{\sim} \frac{1}{4}\left[2u_2^2(\gamma_2-x_2)+\hat{\mathbf{k}}_1\cdot\frac{\mathbf{v}_2}{x_2}\right]+O(k_1,q) \tag{A56}$$

$$\overset{k_2,q\to 0}{\sim} \frac{1}{4}\left[2u_1^2(\gamma_1-x_1)+\hat{\mathbf{k}}_2\cdot\frac{\mathbf{v}_1}{x_1}\right]+O(k_2,q)\ , \tag{A57}$$

$$F_4(\mathbf{q},1,2) \overset{k_1,q\to 0}{\sim} \frac{1}{4}[2u_2^2(1-x_2\gamma_2)+\hat{\mathbf{k}}_1\cdot\mathbf{v}_2]+O(k_1,q) \quad (A58)$$

$$\overset{k_2,q\to 0}{\sim} \frac{1}{4}[2u_1^2(1-x_1\gamma_1)+\hat{\mathbf{k}}_2\cdot\mathbf{v}_1]+O(k_2,q)\ . \quad (A59)$$

The limits that all momenta are small are most easily obtained from Eqs. (A47)–(A50):

$$F_1(\mathbf{q},1,2) \overset{k_1,k_2,q\to 0}{\sim} \tfrac{1}{4}(1-\hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}}_2)+O(k_1,k_2,q)\ , \quad (A60)$$

$$F_2(\mathbf{q},1,2) \overset{k_1,k_2,q\to 0}{\sim} \tfrac{1}{4}(1-\hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}}_2)+O(k_1,k_2,q)\ , \quad (A61)$$

$$F_3(\mathbf{q},1,2) \overset{k_1,k_2,q\to 0}{\sim} \tfrac{1}{4}(1+\hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}}_2)+O(k_1,k_2,q)\ , \quad (A62)$$

$$F_4(\mathbf{q},1,2) \overset{k_1,k_2,q\to 0}{\sim} \tfrac{1}{4}(1+\hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}}_2)+O(k_1,k_2,q)\ . \quad (A63)$$

APPENDIX B: DERIVATION OF EQ. (2.25)

In this appendix we shall calculate the coefficient $\tilde{W}_0(t,\varphi,\mathbf{k})$ in the expansion

$$\tilde{W}(t,\varphi,\tau x,\mathbf{k})=\sum_{n=0}^{\infty}\tilde{W}_n(t,\varphi,\mathbf{k})(\tau x)^n\ , \quad (B1)$$

where $\tilde{W}(t,\varphi,\tau x,\mathbf{k})$ is defined via Eqs. (2.20) and (2.6). First, note that the momentum dependence of the DM vertices in the expression for $W(\mathbf{q},\mathbf{p},\mathbf{k})$ is such that all singular terms, which are exclusively contained in the function $A(\mathbf{q},1,2)$ defined in Eq. (A24), cancel according to Eq. (A36). Because the remaining terms in the DM vertices do not contain any singularities, $\tilde{W}(t,\varphi,z,\mathbf{k})$ is analytic in z, which implies the existence of a convergent expansion of the form given previously. We now calculate that the leading term $\tilde{W}_0$ is this expansion. Substituting to Eq. (2.15) into Eq. (2.6) yields

$$\begin{aligned}\tilde{W}(t,\varphi,p/\sqrt{2},\mathbf{k})&=W(p(\hat{\mathbf{p}}+\mathbf{r}),p\hat{\mathbf{p}},\mathbf{k})\\&=\tilde{V}^{(1)}(p(\hat{\mathbf{p}}+\mathbf{r}),-p\mathbf{r},\mathbf{k}+p(\hat{\mathbf{p}}+\mathbf{r}))\tilde{V}^{(1)}(-p(\hat{\mathbf{p}}+\mathbf{r}),p\hat{\mathbf{p}},\mathbf{k})\\&\quad+\tilde{V}^{(4)}(-p(\hat{\mathbf{p}}+\mathbf{r}),\mathbf{k},-p\hat{\mathbf{p}})\tilde{V}^{(4)}(p(\hat{\mathbf{p}}+\mathbf{r}),\mathbf{k},p\mathbf{r})\\&\quad+\tilde{V}^{(4)}(\mathbf{k}+p\mathbf{r},-p\mathbf{r},-p\hat{\mathbf{p}})\tilde{V}^{(4)}(-\mathbf{k}-p\mathbf{r},\mathbf{k},-\mathbf{k}-p(\hat{\mathbf{p}}+\mathbf{r}))\ ,\end{aligned} \quad (B2)$$

where we have defined

$$\tilde{V}^{(j)}(\mathbf{q},\mathbf{k}_1,\mathbf{k}_2)\equiv\frac{V^{(j)}(\mathbf{q},\mathbf{k}_1,\mathbf{k}_2)}{A(\mathbf{q},\mathbf{k}_1,\mathbf{k}_2)}\ , \quad (B3)$$

and made use of Eq. (A36). Note that the modified vertices $\tilde{V}^{(j)}$ do not contain any singular terms. It is clearly a consequence of our elliptic approximation to the scattering surface that in the arguments of the $\tilde{V}^{(j)}$ the momentum p appears only up to linear order. Obviously the coefficients $\tilde{W}_n$ for $n\geq 2$ have to be calculated using more accurate descriptions of the scattering surface. Here we are, however, only interested in $\tilde{W}_0$, and since in the short-wavelength regime the magnon momentum k is large compared to p and $|p\mathbf{r}|$, we may set the terms $p\hat{\mathbf{p}}$ and $p\mathbf{r}$ in Eq. (B2) as zero when they appear in the sum with $\mathbf{k}$.

We proceed by showing that

$$\tilde{V}^{(j)}(-\mathbf{k},\mathbf{k},-\mathbf{k})=0 \quad \text{for } j=3,4\ , \quad (B4)$$

and that therefore the last term in Eq. (B2) does not contribute to $\tilde{W}_0$. To proof this, we note that Eqs. (A29)–(A30) imply that

$$F_j(-\mathbf{k},\mathbf{k},-\mathbf{k})=u_{\mathbf{k}}^4[\gamma_{\mathbf{k}}(1+x_{\mathbf{k}}^2)-2x_{\mathbf{k}}]=0 \quad \text{for } j=3,4\ , \quad (B5)$$

by construction of the Bogoliubov transformation [cf. Eqs. (A6)–(A8)]. Since $\tilde{V}^{(3)}$ and $\tilde{V}^{(4)}$ are linear combinations of the functions F_3 and F_4, Eq. (B4) follows trivially. The analyticity of the $\tilde{V}^{(j)}$ implies then that the last term in Eq. (B2) vanishes at least as fast as p as $p\to 0$ and hence cannot contribute to $\tilde{W}_0$.

Next we observe that the momentum dependence in the remaining two terms of Eq. (B2) is such that for each vertex the momentum transfer and one of the incoming momenta are small. The corresponding limiting forms of the vertices are easily derived from Eqs. (A52)–(A59). Using

$$u_{\mathbf{k}}^2(1-x_{\mathbf{k}}^2)=1\ , \quad (B6)$$

we obtain

$$\tilde{V}^{(1)}(\mathbf{q},1,2) \overset{k_1,q\to 0}{\sim} \tfrac{1}{2}(1-\mathbf{v}_2\cdot\hat{\mathbf{k}}_1)+O(k_1,q) \quad \text{(B7)}$$

$$\overset{k_2,q\to 0}{\sim} \tfrac{1}{2}(1-\mathbf{v}_1\cdot\hat{\mathbf{k}}_2)+O(k_2,q)\ , \quad \text{(B8)}$$

$$\tilde{V}^{(4)}(\mathbf{q},1,2) \overset{k_1 q\to 0}{\sim} \tfrac{1}{2}(1+\mathbf{v}_2\cdot\hat{\mathbf{k}}_1)+O(k_1,q) \quad \text{(B9)}$$

$$\overset{k_2,q\to 0}{\sim} \tfrac{1}{2}(1+\mathbf{v}_1\cdot\hat{\mathbf{k}}_2)+O(k_2,q)\ . \quad \text{(B10)}$$

Substituting Eqs. (B7) and (B10) for $\tilde{V}^{(1)}$ and $\tilde{V}^{(4)}$ in Eq. (B2) then yields

$$\begin{aligned}\tilde{W}_0(t,\varphi,\mathbf{k}) &= \tfrac{1}{4}\{[1-\mathbf{v}_{\mathbf{k}}\cdot(-\hat{\mathbf{r}})][1-\mathbf{v}_{\mathbf{k}}\cdot\hat{\mathbf{p}}] \\ &\quad +[1+\mathbf{v}_{\mathbf{k}}\cdot(-\hat{\mathbf{p}})](1+\mathbf{v}_{\mathbf{k}}\cdot\hat{\mathbf{r}})\} \\ &= \tfrac{1}{2}(1-\mathbf{v}_{\mathbf{k}}\cdot\hat{\mathbf{p}})(1+\mathbf{v}_{\mathbf{k}}\cdot\hat{\mathbf{r}})\ . \end{aligned} \quad \text{(B11)}$$

Note that the contributions of the terms containing the vertices $V^{(1)}$ and $V^{(4)}$ are identical. We bring this equation into the form of Eq. (2.25) by substituting the definition

$$1-\mathbf{v}_{\mathbf{k}}\cdot\hat{\mathbf{p}} = 1-v(\mathbf{k})\cos\varphi\ , \quad \text{(B12)}$$

and the identity

$$1+\mathbf{v}_{\mathbf{k}}\cdot\hat{\mathbf{r}} = \frac{1-v^2(\mathbf{k})}{1-v(\mathbf{k})\cos t}\ , \quad \text{(B13)}$$

which is easily derived from Eq. (2.16).

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[13]On a bipartite lattice it is useful to introduce two types of magnon quasiparticle operators, $\alpha_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$ (see Appendix A). Consequently the self-energy is a 2×2 matrix with diagonal elements $\Sigma_{\alpha\alpha}$ and $\Sigma_{\beta\beta}$, and off diagonal elements $\Sigma_{\alpha\beta}$ and $\Sigma_{\beta\alpha}$. Unlike the Hartree-Fock expression $\Sigma^{(1)}$, the second-order self-energy $\Sigma^{(2)}$ is not diagonal. However, to obtain the magnon damping to leading order in $T/(2DJS)$ and S^{-1}, it is sufficient to calculate the diagonal element $\Sigma^{(2)}_{\alpha\alpha}$. (See Refs. 12 and 16.) In Eq. (1.4) we have omitted the matrix indices, and it is understood that $\Sigma^{(2)}_{\alpha\alpha}$ is meant.

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[17]To give an example, consider the long-wavelength limits of the vertex $\Phi^{(4)}(\mathbf{q},\mathbf{k}_1,\mathbf{k}_2)$ given by HKHH: According to their Eq. (A17c), we have $\Phi^{(4)}(\mathbf{q},\mathbf{k}_1,\mathbf{k}_2) \overset{k_1,k_2\to 0}{\sim} 0$, while Eq. (A21b) gives $\Phi^{(4)}(\mathbf{q},\mathbf{k}_1,\mathbf{k}_2) \overset{k_1,k_2,q\to 0}{\sim} 2\epsilon_{\mathbf{k}_1}\epsilon_{\mathbf{k}_2}(1+\hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}}_2)$. Obviously the latter expression cannot be obtained from the former one by taking the limit $q\to 0$ because *different orders in the small momenta have been kept.* This is rather confusing and not very useful for our calculations.

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