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Quantized transmission of a saddle-point constriction

M. Büttiker
IBM Thomas J. Watson Research Center, Yorktown Heights, New York 10598
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Conductance quantization in the absence and in the presence of a magnetic field is discussed for a split-gate constriction with a local potential at the bottleneck forming a saddle. Simple criteria for the occurrence of conductance steps and the accuracy of quantization are given in terms of the curvatures of the saddle.

This paper presents a contribution to the ongoing theoretical discussion of the quantized conductance steps discovered by van Wees *et al.*[1] and Wharam *et al.*[2] in split-gate constrictions of a two-dimensional electron gas.[3] The literature[4-16] treats this problem by considering a hard-wall potential. In some papers the width of the conduction channel is also assumed to change abruptly, but in other papers[13-16] it is assumed to be a smooth function. Since constrictions in these experiments are electrostatically induced with a pair of split gates, the potential is a smooth function (without hard walls or sharp corners). The bottleneck of the constriction, therefore, forms a saddle. Thus the transmission and reflection at a saddle is a necessary part of every calculation that attempts to make contact with these experiments. In addition, the quantum-mechanical solution of this scattering problem is simple[17,18] and permits physical insight. A complete discussion of constriction conductances requires the consideration of carrier transmission from one equilibrium electron reservoir to another. However, if the transmission is *globally adiabatic*,[5,13-16] the calculation of the conductance due to the *local* scattering at the saddle alone is accurate up to exponentially small corrections.[16]

Near the bottleneck of the constriction the electrostatic potential can be expanded, and in terms of appropriate coordinates x and y is given by

$$V(x,y)=V_0-\tfrac{1}{2}m\omega_x^2x^2+\tfrac{1}{2}m\omega_y^2y^2. \quad (1)$$

Here, V_0 is the electrostatic potential at the saddle, and the curvatures of the potential are expressed in terms of the frequencies ω_x and ω_y. Now we neglect higher-order terms in x and y and discuss the conductance steps due to scattering at this quadratic potential. Let us first study the case of zero magnetic field. The total energy is given by the potential Eq. (1) supplemented by a kinetic energy $p^2/2m$. The Hamiltonian is separable into a transverse wave function associated with energies $\hbar\omega_y(n+\frac{1}{2})$, $n=0, 1, 2, 3, \ldots$, and a wave function for motion along x in an effective potential $V_0+\hbar\omega_y(n+\frac{1}{2})-1/2m\omega_x^2x^2$. This effective potential can be viewed as the band bottom of the nth quantum channel (subband) in the region of the saddle point.[13] In the absence of quantum tunneling the channels with threshold energy

$$E_n=V_0+\hbar\omega_y(n+\tfrac{1}{2}) \quad (2)$$

below the Fermi energy are open, and the channels with threshold energy E_n above the Fermi energy are closed. Quantum mechanically transmission and reflection at the saddle allows for channels which are neither completely open nor completely closed, but which permit transmission with a probability T_{mn}. Here, the index n refers to the incident channel, and the index m refers to the out-going channel. The transmission probabilities for this simple case are calculated in Ref. 17 and can be expressed with the help of the variable

$$\varepsilon_n=2[E-\hbar\omega_y(n+\tfrac{1}{2})-V_0]/\hbar\omega_x\,, \quad (3)$$

in the simple form,

$$T_{mn}=\delta_{mn}\frac{1}{1+e^{-\pi\varepsilon_n}}\,. \quad (4)$$

Only transmission probabilities for which the incident channel and the out-going channel are the same are nonzero. Because of the quadratic form of the potential there is no channel mixing. The transmission probabilities T_{nn} for $n=0, 1, 2, 3, 4, \ldots$, for the case $\omega_y/\omega_x=3$ are shown in Fig. 1 as a function of $(E-V_0)/\hbar\omega_x$. For $\varepsilon_n \ll 0$ the transmission probability is exponentially small,

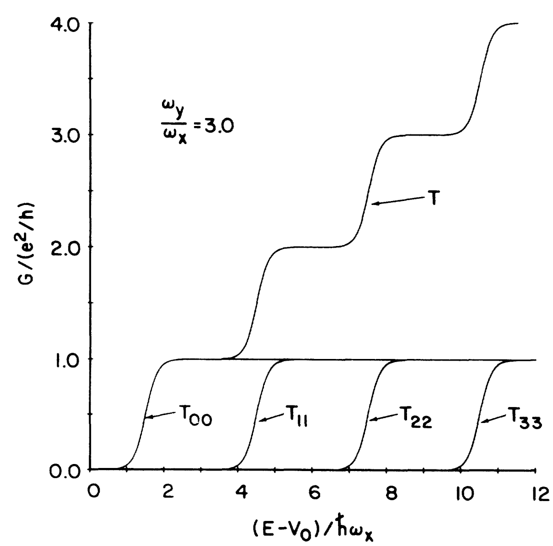

FIG. 1. Single-channel transmission probabilities T_{nn} and the total transmission probability (conductance) $T=\sum_n T_{nn}$ as a function of $(E-V_0)/\hbar\omega_x$ for a ratio of $\omega_y/\omega_x=3$. The opening of successive quantum channels over narrow energy intervals leads to the quantization of the conductance.

$T_{nn} \approx \exp(\pi\varepsilon_n)$. For $\varepsilon_n \gg 0$ the transmission probability is close to one, $T_{nn} \approx 1-\exp(-\pi\varepsilon_n)$. The transition from zero transmission probability to a transmission probability close to one occurs near $\varepsilon_n = 0$, i.e., in the neighborhood of the classical threshold energy E_n given by Eq. (2). The size of the energy interval needed for the transition is determined by $\hbar\omega_x$. The conductance, in the case that we deal with transmission from one equilibrium electron reservoir to another equilibrium reservoir, is determined by the sum of all transmission probabilities, $T = \sum_{mn} T_{mn}$, and is given by[3,4,19,20]

$$G = \frac{e^2}{h} T . \tag{5}$$

The total transmission probability T (or the two-terminal conductance) shows, therefore, a series of well-developed steps if the transition region for the opening of a quantum channel is small compared to the channel separation. Since the width of the transition region is $\hbar\omega_x$ and the separation of quantum channels at the saddle is determined by $\hbar\omega_y$, well-pronounced steps occur if

$$\omega_y \gtrsim \omega_x . \tag{6}$$

Figure 2 shows a series of conductance traces with ratios of ω_y/ω_x increased in increments of 0.25 in the interval from 0 to 5. It is seen that already for ratios which are moderately larger than 1 the conductance shows considerable structure, and if the ratio approaches the maximum value shown the plateaus are entirely flat. The quality of the quantization can best be discussed by considering the energy derivative of the total transmission probability (the conductance). From Eq. (3) we find

$$\frac{dT}{dE} = \sum_{n=0}^{n=\infty} \frac{\pi}{2\hbar\omega_x} \frac{1}{\cosh^2(\pi\varepsilon_n/2)} , \tag{7}$$

which for $\omega_y \gg \omega_x$ is minimal at the energies,

$$E_n \cong V_0 + \hbar\omega_y(n+1) , \tag{8}$$

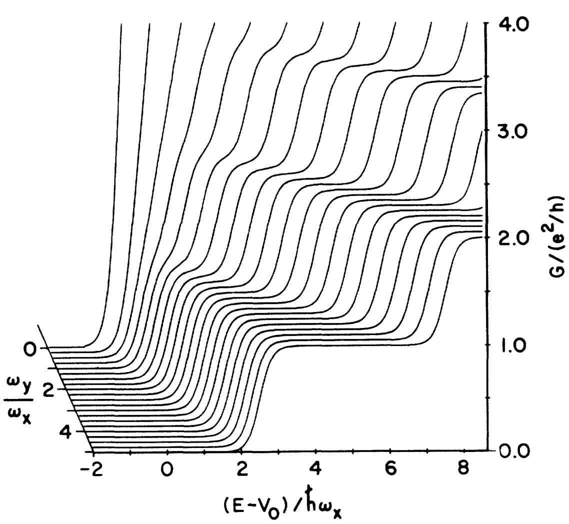

FIG. 2. Constriction conductance as a function of $(E-V_0)/\hbar\omega_x$ for differing saddle potentials characterized by a ratio ω_y/ω_x for ratios in an interval from 0 to 5 in increments of 0.25.

and is maximal at the energies given by Eq. (2), i.e., at the classical threshold of the quantum channel. The nearly flat portions of the total transmission probability are centered around the energies determined by Eq. (8). Evaluated at the center of the plateau, Eq. (7) yields a slope,

$$\frac{dT}{dE} = (4\pi/\hbar\omega_x) e^{-2\pi\omega_y/\omega_x} , \tag{9}$$

accurate up to corrections of the order $(e^{-2\pi\omega_y/\omega_x})^2$. Thus for large ratios ω_y/ω_x the slope at the center of the plateaus is exponentially small. The maximum slope at the center of a conductance step, i.e., for an energy $E = E_n$, is $\pi/2\hbar\omega_x$. For $\omega_y \gg \omega_x$ the quantized value $G = (e^2/h)n$ is approached up to exponentially small corrections of the order of $e^{-3\pi\omega_y/\omega_x}$.

Next we investigate the constriction resistance in a magnetic field B (perpendicular to the x and y plane). Cyclotron motion with frequency $\omega_c = |eB/mc|$ gives rise to a new energy scale which affects the transmission behavior. For the saddle-point potential considered here the single-channel transmission probability has been calculated by Fertig and Halperin.[18] With the help of the definition $\Omega^2 = \omega_c^2 + \omega_y^2 - \omega_x^2$ the energies that govern transmission and reflection at the saddle in a magnetic field are[18]

$$E_1 = \frac{1}{2} \frac{\hbar}{\sqrt{2}} [(\Omega^4 + 4\omega_x^2\omega_y^2)^{1/2} - \Omega^2]^{1/2} \tag{10}$$

and

$$E_2 = \frac{\hbar}{\sqrt{2}} [(\Omega^4 + 4\omega_x^2\omega_y^2)^{1/2} + \Omega^2]^{1/2} . \tag{11}$$

The transmission probability is a function of

$$\varepsilon_n = \frac{E - E_2(n+\frac{1}{2}) - V_0}{E_1} , \tag{12}$$

and is, as in the absence of a field, given by

$$T_{mn} = \delta_{mn} \frac{1}{1+\exp(-\pi\varepsilon_n)} . \tag{13}$$

Therefore, in the presence of a field E_2 takes the place of $\hbar\omega_y$ and E_1 plays the role $\hbar\omega_x/2$ plays at zero field. Again, there is no channel mixing. In the limit of zero applied field these formulas reduce to the results presented above. At high fields, when ω_c exceeds ω_x and ω_y, Eqs. (10) and (11) simplify considerably. The relevant energies are[18]

$$E_1 \approx \hbar\omega_x\omega_y/2\omega_c \approx (m/2)\omega_x\omega_y l_B^2 , \tag{14}$$

where we have used the magnetic length $l_B^2 = \hbar c/|eB|$ and

$$E_2 \approx \hbar\omega_c . \tag{15}$$

In the limit where Eqs. (14) and (15) are applicable, a simple interpretation is again possible. In a high magnetic field carriers execute rapid cyclotron motion around a guiding center with energy $E_G = E - \hbar\omega_c(n+\frac{1}{2})$, which follows the equipotential contours of the potential Eq. (1). In the absence of tunneling the trajectories of the states in a high magnetic field are determined by $E_G = V(x,y)$. For $E_G < V_0$ this describes a trajectory which is repelled by the saddle. The closest approach of such a trajectory to the saddle point is determined by $E_G = V(x_n, 0)$. For $E_G > V_0$ the trajectory passes through the saddle. The closest approach of such a trajectory to the saddle point is determined by $E_G = V(0, y_n)$. As has been pointed out in Ref. 18, the transmission probabilities in terms of x_n or y_n are in the high-magnetic-field limit

$$T_{mn} = \delta_{mn} \frac{1}{1+\exp[\pi(\omega_x/\omega_y)(x_n/l_B)^2]} , \qquad (16)$$

if $\varepsilon_n < 0$, and are

$$T_{mn} = \delta_{mn} \frac{1}{1+\exp[-\pi(\omega_y/\omega_x)(y_n/l_B)^2]} , \qquad (17)$$

if $\varepsilon_n > 0$. Since $(x_{n+1}^2 - x_n^2)/l_B^2 = 2\omega_c^2/\omega_x^2 \gg 1$, it is at most one of the transmission probabilities which is between zero and one (up to exponentially small corrections). All the high-magnetic-field states, except possibly one, are either completely reflected at the saddle or completely transmitted.[1] The transition from complete reflection to complete transmission occurs over a very narrow energy interval $\approx (m/2)\omega_x\omega_y l_B^2 \ll \hbar\omega_c$. At high enough magnetic fields, conductance steps always occur regardless of the ratio of ω_x and ω_y. The slope at the center of the plateaus in high magnetic fields is determined by

$$\frac{dT}{dE} \approx (2\pi/E_1)e^{-\pi E_2/E_1} \approx \left[\frac{4\pi\omega_c}{\sqrt{\omega_x\omega_y}}\right] e^{-2\pi\omega_c^2/\omega_x\omega_y} . \qquad (18)$$

Therefore, at high fields the plateaus are flat with high accuracy.

The transition from the zero-magnetic-field case to the high-magnetic-field case is determined by the general equations (10) and (11). The conductance shows a marked structure, if $E_2 > E_1$. This implies $\Omega^2 > 4\omega_x\omega_y$ or

$$\omega_c + \omega_y \geq \omega_x . \qquad (19)$$

Clearly, there are saddles which for $\omega_c = 0$ exhibit little structure, but which for moderate fields show very sharp conductance steps. Such an example is shown in Fig. 3. For $\omega_y = \omega_x$ the zero-field conductance shows only an indication of a steplike structure. Already, for $\omega_c = \omega_x$ the plateaus are very flat and for even higher fields the steplike structure is well described by Eqs. (14)-(18).

The local-scattering problem treated here is also a solution of the global-transmission problem, if every carrier transmitted at the saddle leaves the constriction in the forward direction and if every carrier reflected at the saddle leaves the constriction in the backward direction.[5] This can be ensured if the saddle potential is smoothly connected to hornlike regions which widen in an adiabatic fashion.[5] The adiabatic widening of the constriction also guarantees that channels reaching the bottleneck are completely filled.[13-16] The fact that nonequilibrium populations of differing channels produce deviations from quantization has been emphasized in Ref. 21 in connection with the quantum Hall effect and has been highlighted in a number of experiments.[22] Similarly, in experiments on constrictions[1,2,23] such nonequilibrium effects probably account for major deviations from quantization. In the

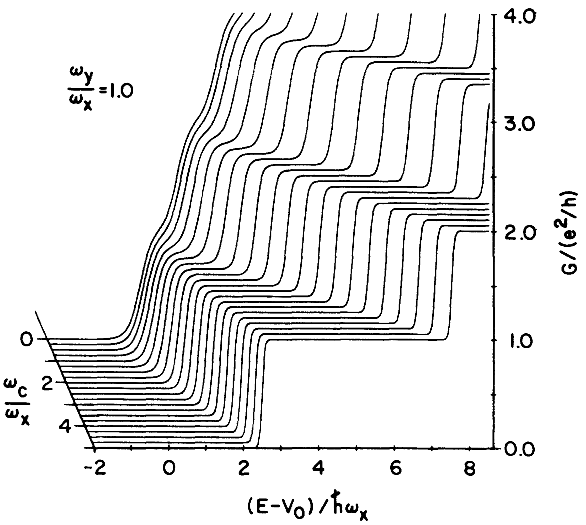

FIG. 3. Constriction conductance in a magnetic field as a function of $(E - V_0)/\hbar\omega_x$. The strength of the magnetic field is determined by the cyclotron frequency ω_c. As an example a saddle with $\omega_y/\omega_x = 1$ is treated which shows poor quantization at zero field. ω_c/ω_x is increased in increments of 0.25 from 0 to 5.

model discussed above the conductance steps are approximately equally spaced as functions of gate voltage if the saddle-point energy V_0 depends linearly on the gate voltage and if the frequencies ω_x and ω_y do not (strongly) depend on the gate voltage. The saddle-point potential discussed here is likely to be of relevance if only a few channels are transmitted. As the point contact widens and the number of transmitted channels increases, the potential can be expected to be increasingly flat in the center in the transverse direction, $\omega_y \rightarrow 0$. In this limit, for small applied gate voltages the model calculations[13-16] that treat the constriction in a hard-wall approximation might become more suitable. The important question of the precise shape of the potential in such a constriction has received little attention so far. Reference 24 estimates the magnitudes of impurity potentials. Reference 25 studies the self-consistent potential in the presence of wall imperfections and demonstrates that screening produces a smooth potential. A self-consistent calculation[26] of the potential near the tip of a tunneling microscope, viewed as a constriction on an atomic scale, finds that the self-consistent potential has the form of a saddle. An important aspect[27] we have neglected is that due to the electrostatic fields, the motion perpendicular to the two-dimensional electron gas is affected and gives rise to scattering.

In the simple model discussed above the scattering at the saddle preserves the quantum number of the channel. This might generate the impression that the absence of channel mixing is also a necessary condition for the quantization. This is not the case, as seen from a number of works,[7,8] and from the following simple argument.[19] Since the conductance is only determined by the total transmission probability T and since[20] $T = \mathrm{Tr}(t^\dagger t)$ where t stands for the matrix of transmission amplitudes, the conductance is left invariant if t is multiplied by arbitrary unitary matrices U_1 and U_2. If t is a diagonal matrix that exhibits quantization, then $U_1^{-1}tU_2$ is a transmission matrix that is mixing channels but gives rise to the same conductance as t. Finally, a simple estimate of the accuracy of the local saddle-point approximation discussed here can be obtained by comparing this approximation with a problem which is exactly solvable on a global scale (for a more thorough discussion see Ref. 16). The potential

$$V(x,y)=(V_0/\cosh^2\alpha x)+\frac{1}{2}m\omega_y^2y^2$$

with $\alpha^2=\frac{1}{2}m\omega_x^2/V_0$ describes transmission from reservoir to reservoir if $V_0 \gg \hbar\omega_y$. But since $\omega_x \approx \omega_y$, the parameter α is very small and the transmission probability of the hyperbolic cosine potential is given by Eq. (4) to within an accuracy which is far better than the exponentially small terms retained in Eq. (9) to estimate the slope of the plateaus. The discussion of the saddle-point transmission presented in this paper is thus valuable not only because it is simple but might also contain, from a practical point of view, many of the essential features needed for comparison with experiments.

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