Theory of the fractional quantum Hall effect

J. K. Jain

Department of Physics, State University of New York at Stony Brook, Stony Brook, New York 11794-3800

(Received 24 October 1989; revised manuscript received 13 December 1989)

A theoretical framework is presented which provides a unified description of the integer and the fractional quantum Hall effects. The main assertion is that new candidate incompressible states can be constructed by taking products of some known incompressible states, and all incompressible states can thus be generated starting from the states at integer filling factors. The crucial difference from previous theories is that the higher Landau levels play an essential role in identifying the correlations responsible for the fractional quantum Hall effect. The quasiparticle excitations of the fractional states can be understood simply in this approach by analogy to the quasiparticles of the integer states. Numerical results show that these trial states very accurately describe the transition from the $\frac{1}{3}$ state to the $\frac{2}{5}$ state for a four-electron system. It is further shown that the predictions of the theory are completely consistent with the phenomenology of the fractional quantum Hall effect; in particular, the predicted order of stability of the various fractions is in agreement with experiments. Even though the fractional quantum Hall effect is found to be possible at all rational filling factors in this approach, it is indicated why the odd-denominator fractions are in general more stable than the even-denominator ones.

I. INTRODUCTION

The experimental observation of the phenomenon termed "fractional quantum Hall effect[1,2] (FQHE)" has posed theorists with an extremely well-defined and fascinating problem. It is clear, especially from the work of Laughlin,[2-4] that a plateau in the Hall resistance at h/pe^2 requires incompressibility at filling factor $\nu=p$ in a disorder-free system. Any successful theory of the FQHE must then be able to answer the following questions. (i) What is the physics of incompressibility at noninteger filling factors? (ii) What are the wave functions describing the physically important correlations of the incompressible states? (iii) What are the excited states and their properties? (iv) Why does FQHE occur only for rational values of p ($\equiv P/Q$, where P and Q are integers)? (v) What is the order of stability of the various fractional states? Usually the fractions with smaller P and Q are the first ones to appear, but why is it that sometimes fractions involving larger values of P and Q are observed while those involving smaller values of P and Q are not? (For example $\frac{6}{13}$ has been observed whereas $\frac{5}{13}$ has not.) Why is $\frac{5}{2}$ the only observed even-denominator fraction so far? (vi) What is the role of spin? In this paper we present a theory that, we believe, answers all these questions.

The most widely accepted theory of the FQHE, which we will call the "standard" theory, consists of the Laughlin wave functions[4] for the fractions $1/m$, where m is an odd integer, and its hierarchical generalizations[5-7] to all other fractions with odd denominators. [Other approaches have also been proposed (see Ref. 8 for a Wigner-crystal-based theory, Ref. 9 for a cooperative-ring-exchange approach, Ref. 10 for a Ginzburg-Landau approach, and Refs. 11 and 12 for various microscopic trial wave functions).] While it answers some of the questions raised above to a certain extent, the author finds it unsatisfactory in the following respects.

First consider the hierarchy theory, which has been developed in order to explain FQHE at filling factors other than $1/m$. As the magnetic field is varied away from one of the "magic" $1/m$ filling factors, quasiparticles are created to accommodate the extra flux. These quasiparticles have fractional charge[4] and are believed to obey fractional statistics.[6,13] The basic idea of the hierarchy theory is that at certain filling factors the quasiparticles may themselves condense into a Laughlin-type correlated state to produce stiffness at these new filling factors. The quasiparticles of this new state can in turn again condense at some other filling factor to produce stability at the next level of hierarchy, and so on. The main conceptual difficulty with this picture arises due to its use of quasiparticles. It takes a large number of quasiparticles to form a Laughlin-type condensate; for example, the $\frac{2}{5}$ state is obtained from the $\frac{1}{3}$ state when there are half as many quasielectrons as electrons. It is not clear how meaningful the concept of quasiparticles is when there is one quasielectron for every two electrons and the distance between the quasiparticles is less than their size. In fact, by the time one gets to the experimentally observed $\frac{6}{13}$ state through the sequence $\frac{1}{3}\rightarrow\frac{2}{5}\rightarrow\frac{3}{7}\rightarrow\frac{4}{9}\rightarrow\frac{5}{11}\rightarrow\frac{6}{13}$, many more quasielectrons have been created than there are electrons, and it is difficult to intuitively appreciate how the $\frac{1}{3}$ state can be responsible for the stability of the $\frac{6}{13}$ state. Furthermore, it is not even clear to the author why it should be more useful to consider the quasiparticles rather than the electrons in such situations. A consideration of the electrons rather than quasiparticles is also desirable because the quasiparticles possess rather unusual properties like fractional statistics, and the states containing many quasiparticles are expected to be quite nontrivial when written in terms of quasiparticle coordinates. In short, even though the hierarchical theory provides a possible classification of the FQH states, it does not provide a detailed microscopic understanding of the phenomenon.

Secondly, in physics one is used to finding common themes underlying seemingly unrelated phenomena. In the case of the QHE, however, the situation is quite the opposite; while the experimental observation of all the fractions is essentially identical, there are several theories for their explanation. To begin with, there are two distinct frameworks for understanding the QHE at integer and fractional filling factors. While the QHE at integer filling factors[14] [the integer QHE (IQHE)] is explained neglecting electron-electron interactions,[3,15] the observation of QHE at noninteger filling factors (FQHE) is believed to arise from a condensation of the two-dimensional (2D) electrons into a "new collective state of matter"[4] as a result of repulsive interelectron interactions. Even within the FQHE there are several levels of understanding. Laughlin's theory explains the QHE at the so-called "fundamental" fractions $1/m$, where m is an odd integer. Other fractions with odd denominators are obtained in a hierarchical scheme[5-7] starting from the Laughlin fractions. After the observation of FQHE at $\frac{5}{2}$ (Ref. 16) possibility of FQHE at even-denominator fractions has attracted considerable amount of attention, and its origin is being suggested as some sort of electron pairing.[17-19] Since all the fractions occur under similar experimental conditions with similar experimental manifestation, in our opinion, and based on general esthetic grounds, this is far from a satisfactory state of affairs.

Another approach is to write trial wave functions as a function of the electron coordinates, as is the case with the Laughlin states, and understand their incompressibility from a microscopic point of view without appealing to the quasiparticles. Such wave functions have indeed been written to describe the incompressible states at filling factors other than $1/m$,[10,11] and some have been shown to have good overlaps with the true states for small number of electrons. However, no simple overall picture has yet emerged.

The purpose of this paper is to describe in detail a new class of extremely simple trial wave functions for the FQHE.[20,21] These result from a desire to incorporate the IQHE and FQHE into a single theoretical framework and reveal that there is a fundamental connection between the two. Since the IQHE is very well understood, this connection constitutes a powerful tool for understanding various aspects of the FQHE. For example, not only can the incompressible states at fractional filling factors be written in terms of IQH states, but the excitations of the FQH states can also be understood in terms of excitations of the IQH states. Predictions of the theory are completely consistent with experiments. Unlike previous theories, the FQHE is predicted to be in principle possible at *all* rational filling factors, but at the same time it is indicated why the odd-denominator fractions are experimentally so much more abundant.

The plan of the paper is as follows. Section II provides a brief review of the standard theory of the FQHE. Section III introduces the new trial wave functions for the incompressible states at fractional filling factors and their quasiparticle excitations. In Sec. IV the quasielectron state and the $\frac{2}{5}$ state are studied numerically and found to be extremely accurate representations of the true states. It is discussed why the higher Landau-level states play a crucial role in identifying the physics of the FQH states. Section V shows that the order of stability of the various states predicted by the theory is in excellent agreement with that observed experimentally. Section VI provides a generalization of these trial states. The role of spin is discussed in Sec. VII. The present approach is compared with the hierarchy approach in Sec. VIII. The paper is concluded in Sec. IX.

II. REVIEW

We start with a summary of the important results.

The fractions observed in the lowest Landau level (LL) to date are[1,22-26] 1, $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$, $\frac{2}{3}$, $\frac{2}{5}$, $\frac{2}{7}$, $\frac{2}{9}$, $\frac{3}{5}$, $\frac{3}{7}$, $\frac{3}{11}$, $\frac{4}{5}$, $\frac{4}{7}$, $\frac{4}{9}$, $\frac{4}{11}$, $\frac{4}{13}$, $\frac{5}{9}$, $\frac{5}{11}$, $\frac{6}{11}$, $\frac{6}{13}$, $\frac{7}{11}$, $\frac{7}{13}$, and $\frac{9}{13}$. Some of these fractions are observed only in ρ_{xx}. The purer the sample, the greater is the number of observed fractions, and there is more or less a definite order in which new fractions appear as the sample quality is improved.

The IQHE is a relatively well-understood phenomenon. In the presence of a transverse magnetic field LL's are formed, each containing a large number of states which are degenerate in the absence of disorder. The degeneracy of each LL is equal to eB/hc per unit area. Due to Pauli principle, each state can be occupied at most by one electron, and the number of filled LL's (i.e., the filling factor) is

$$\nu = (\text{density}) \times hc/eB \ . \tag{1}$$

In the presence of disorder the degeneracy of the states in a LL is lifted, the density of states is broadened, and a Landau band is produced with localized states in the tail and extended states at the center. Laughlin has shown,[3] with the help of a general and elegant gauge argument, that the Hall resistance is quantized to h/pe^2, where p is an integer, so long as the Fermi level lies in a mobility gap.

In a translationally invariant system the Hall resistance at any filling factor ν is given by $h/\nu e^2$. Based on Laughlin's gauge argument for the IQHE, one believes that in order to explain QHE at a fractional filling factor p in the physical system, it is sufficient to demonstrate that there is a gap in the excitation spectrum at this filling factor in an ideal impurity-free system. The impurities and inhomogeneties in the physical system create localized states, and one feels that so long as the Fermi level lies in the mobility gap, the Hall resistance should remain quantized at its "unperturbed" value of h/pe^2.

For an explanation of the QHE at fractional values of ν, Laughlin proposed[4] trial states describing highly correlated incompressible quantum fluids at filling factors $1/m$. There is essentially no doubt regarding the validity of the Laughlin states for two reasons: they have excellent overlaps with the exact wave functions for a small number of particles, and they have been shown to be the exact nondegenerate ground states for certain model electron-electron interactions.[5,27,28] There have been several attempts to understand what is responsible for the incompressibility of the Laughlin states. One property of the Laughlin wave functions, that all the zeros of the wave function are bound to the particles, has attracted a substantial amount of attention.[10,18]

As mentioned before, in the hierarchy theory the next level of stable states (the "daughter" states) is obtained at each step when the quasiparticles of the "parent" state condense into a Laughlin-type state. Iteration of this procedure predicts stability at all rational filling factors P/Q in the range $0<\nu<1$, where Q is an odd integer, and FQHE is expected for all these filling factors unless the state in question is unstable to some other type of state (e.g., the Wigner-crystal state). The hierarchical scheme is successful in predicting the order of stability of the experimentally observed fractions. Recently, an even-denominator fraction has been observed,[16] which on the face of it is inconsistent with the standard theory, but trial states have been constructed which may explain this fraction,[17,18] even though the relevance of these trial states has been questioned.[19]

Nonhierarchical trial wave functions have also been proposed for certain filling factors.[11,12,18] Some of these are found to have large overlaps with the true states for few-particle systems.[11,12]

III. NEW TRIAL WAVE FUNCTIONS

In this work we start with the appealing viewpoint that the distinction between QHE at different types of fractions is to some extent artificial and there must be a single theory that describes QHE at all fractions. Believing that such a theory exists, the strategy for uncovering it is actually quite obvious: one must take as prototype the best-understood parts, i.e., the IQHE and the Laughlin states, and then see if one can understand them in a unified framework, which then hopefully will also give FQHE at other fractions. The author has earlier proposed such a scheme[20,21] in which one can understand the FQHE and the IQHE in a single framework. The idea is to assert that the Laughlin states

$$\chi_{1/m}=\prod_{\substack{j,k\\(j<k)}}(z_j-z_k)^m\exp\left[-\frac{1}{4}\left[\frac{eB}{\hbar c}\right]\sum_i|z_i|^2\right], \tag{2}$$

where m is an odd integer and $z_j=x_j+iy_j$ denotes the position of the jth electron, are incompressible due to their similarity to χ_1, where χ_1 is the incompressible IQH state at filling factor 1:

$$\chi_1=\begin{vmatrix}1 & 1 & 1 & \cdots\\ z_1 & z_2 & z_3 & \cdots\\ z_1^2 & z_2^2 & z_3^2 & \cdots\\ \vdots & \vdots & \vdots & \ddots\end{vmatrix}\exp\left[-\frac{1}{4}\left[\frac{eB}{\hbar c}\right]\sum_i|z_i|^2\right]$$
$$=\prod_{\substack{j,k\\(j<k)}}(z_j-z_k)\exp\left[-\frac{1}{4}\left[\frac{eB}{\hbar c}\right]\sum_i|z_i|^2\right]. \tag{3}$$

This can be emphasized by writing

$$\chi_{1/m}=\prod_{\substack{j,k\\(j<k)}}(z_j-z_k)^{m-1}\chi_1, \tag{4}$$

where the factor multiplying χ_1 simply serves to add $m-1$ flux quanta (one flux quantum is $\phi_0=hc/e$) to each electron. [Note that the magnetic field, which is determined by the argument of the exponential, is constant; the amount of flux penetrating the sample is increased due to an increase in the size of the sample as a result of multiplication by the factor $\prod_{j<k}(z_j-z_k)^{m-1}$.] It was argued in Ref. 20 that this addition of $m-1$ flux quanta to each electron does not completely destroy the correlations in χ_1 responsible for its incompressibility. This is quite clear in the context of a mean-field approximation[29] in which one can think of adding $m-1$ flux quanta to each electron as attaching a flux tube to each electron carrying a flux $(m-1)\phi_0$. The correlations remain unaltered in this picture because these flux tubes are unobservable. Consequently, the resulting state can also be expected to be incompressible, which is clearly true for the Laughlin states. One can now generalize this notion and say that given *any* incompressible state χ_{p_1} (e.g., an IQH state, for which $p_1=$integer) a new candidate incompressible state can be constructed as

$$\chi_p=\prod_{\substack{j,k\\(j<k)}}(z_j-z_k)^{m-1}\chi_{p_1}, \tag{5}$$

where the filling factor is

$$p=\frac{p_1}{(m-1)p_1+1}. \tag{6}$$

This filling factor can be obtained either by counting the average number of flux quanta per electron, or by counting the total number of occupied states in a given LL. The average number of flux quanta per electron is equal to the inverse of the filling factor. In χ_{p_1} there are $1/p_1$ flux quanta per electron, so that χ_p has $(m-1)+1/p_1$ flux quanta per electron, which leads to the filling factor given in Eq. (6). In order to count the number of single-particle states occupied in a given LL, let us consider the disk geometry. Then the number of occupied states in the thermodynamic limit is given by the largest power of a coordinate z_j in the polynomial part of the wave function. In χ_{p_1} this power is N/p_1, which implies that in χ_p this power is $N/p_1+N(m-1)$, which also yields the same filling factor.

We find it convenient to write these state as products of incompressible states as

$$\chi_p=\chi_{p_1}\chi_1^{m-1}\equiv[p_1,1,1,\ldots], \tag{7}$$

where the m elements of $[p_1,1,\ldots]$ denote the filling factors of the m states in the product. The exponential part of a factor χ_ν in the product is defined to be

$$\exp\left[-\frac{1}{4}\left[\frac{e_\nu B}{\hbar c}\right]\sum_i|z_i|^2\right] \tag{8}$$

with the charge e_ν given by

$$e_\nu = ep/\nu \ . \tag{9}$$

One can check that this provides the correct exponential factor

$$\exp\left[-\frac{1}{4}\left[\frac{eB}{\hbar c}\right]\sum_i |z_i|^2\right] \tag{10}$$

for χ_p. We find this "division" of the electronic charge very natural for two reasons: with this choice each χ_ν in the product occupies the same area as χ_p, and, as we will see later, e_ν is closely related to the charges of the quasiparticle excitations. For example, for the Laughlin states $e_1 = e/m$ which is also the charge of the quasiparticles. In fact, at some level one can think of the Laughlin states as the states in which the charge-(e/m) quasiparticles occupy IQH states χ_1. This view has been taken in Ref. 21 to construct the FQH states discussed in this paper.

Often, when there is no ambiguity, we will omit the exponential factor for notational facility.

The states χ_p satisfy the usual requirements. They are translationally invariant because they are products of translationally invariant states. They are also eigenstates of angular momentum, which follows because χ_p is a homogeneous polynomial with all terms of the same degree, i.e., the replacement $z_j \to z_j e^{i\theta}$ amounts to multiplying the wave function by a phase factor $e^{iL\theta}$, where L is the total angular momentum.

A. The stable filling factors

One starts with integer values of p_1 to obtain some FQH states, which we term the "fundamental" FQH states. Once a state at p is thus obtained, this state as well as the closely related states[30] at $n \pm p$ can be used[31] to construct further states $[n \pm p, 1, 1, \ldots]$. All odd denominator fractions can be obtained in this manner. We prove this by constructing a state at P_0/Q_0 where Q_0 is odd. If Q_0 is 1, there is nothing to prove. If $Q_0 \neq 1$, the state at P_0/Q_0 can be obtained from a state at P_1/Q_0, where $2P_1 < Q_0$, using

$$\frac{P_0}{Q_0} = j \pm \frac{P_1}{Q_0} \ , \tag{11}$$

where j is an appropriately chosen positive integer or zero and $P_1 \le P_0$. P_1/Q_0 can in turn be obtained from P_1/Q_1 by writing a state of the type $[P_1/Q_1, 1, 1, \ldots]$ where

$$\frac{Q_1}{P_1} = \frac{Q_0}{P_1} - 2(m-1) \tag{12}$$

such that $m-1 > 1$. Thus, P_0/Q_0 can be obtained from P_1/Q_1 where $P_1 \le P_0$ and $Q_1 < Q_0$. This process can now be iterated until $Q_n = 1$; P_n then is the IQH state from which P_0/Q_0 can be obtained.

In this paper we will discuss only the fundamental FQH states (for which p_1 = integer), unless mentioned otherwise, since they have the simplest structure. The other states, which have a more complex structure, can be studied analogously.

We would like to point out that some of the states that belong to this scheme have also been considered elsewhere.[11,12] These are the states which are entirely in the lowest LL. The examples are, of course, the Laughlin states and their hole analogs, the state $[1-\frac{1}{3}, 1, 1]$ at $\nu = \frac{2}{7}$, etc. The new states here are the ones in which the higher LL states play a role.

B. Quasiparticle excitations

The real power of the present approach lies in the fact that it recognizes the underlying IQH structure of the FQH states and thereby enables an understanding of various aspects of the FQHE by analogy to the IQHE. As an example we demonstrate how one can understand the quasiparticle excitations of the FQH states in a trivial fashion.

The FQH states in the present scheme are product states where each state in the product is an incompressible state itself. It is natural to expect that a quasiparticle of the product state can be obtained by creating a quasiparticle in one of the states of the product. Let us consider the example of the Laughlin states first. According to the above prescription a quasihole is given by

$$\chi_{1/m}^{+} = \chi_1^{+} \chi_1^{m-1} \ , \tag{13}$$

where χ_1^{+} is the state χ_1 with a hole. The wave function for the state χ_1 with a hole at the origin is

$$\chi_1^{+} = \begin{vmatrix} z_1 & z_2 & z_3 & \cdots \\ z_1^2 & z_2^2 & z_3^2 & \cdots \\ z_1^3 & z_2^3 & z_3^3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix} \ , \tag{14}$$

which is equal to

$$\chi_1^{+} = \left[\prod_k z_k\right] \chi_1 \ , \tag{15}$$

so that the quasihole state is

$$\chi_{1/m}^{+} = \left[\prod_j z_j\right] \chi_{1/m} \ . \tag{16}$$

Remarkably, and encouragingly, this is precisely Laughlin's trial wave function for a quasihole at the origin.[4] Thus, recognition of the underlying IQH structure of the Laughlin states provides a simple interpretation for Laughlin's trial wave function for the quasihole. The state for a quasielectron at the origin can in the same spirit be written as

$$\chi_{1/3}^{-} = \begin{vmatrix} z_1^* & z_2^* & z_3^* & z_4^* & \cdots \\ 1 & 1 & 1 & 1 & \cdots \\ z_1 & z_2 & z_3 & z_4 & \cdots \\ z_1^2 & z_2^2 & z_3^2 & z_4^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{vmatrix} \chi_1^2 \ . \tag{17}$$

This wave function is different from various other wave functions proposed in the literature for the quasielectron.[4,32] We will later give results of numerical calculations on small systems to show that this trial state accurately describes the true quasielectron.

Clearly, there will be several kinds of quasiparticles in general. Let us take the example of the simplest new FQH state $\chi_1^2\chi_2$ at $\nu=\frac{2}{5}$. There are three kinds of quasiholes: one of the type $\chi_1^+\chi_1\chi_2$, and two of type $\chi_1^2\chi_2^+$ because a hole in χ_2 can be created in either of the two LL's. Similarly there are several types of quasielectrons. We claim that the lowest-energy quasihole is obtained by creating a hole in the state χ_2 and the lowest-energy quasielectron is obtained by adding an electron in the lowest unoccupied LL of this state. Intuitively one can understand why these quasiparticles have the lowest energy by saying that since χ_2 corresponds to the most filled LL's, its quasiparticles experience the smallest "effective" magnetic field. A more quantitative way of understanding this is to realize that (as shown below) these quasiparticles have the smallest charge, which implies that they have the lowest energy because the quasiparticle energy is proportional to the square of its charge.[4] Quasiparticles of other FQH states are completely analogous. States containing several quasiparticles can be constructed in a similar fashion. Since the total number of particles is constant in each of the factors χ_ν, adding or removing a particle at one point is necessarily accompanied by removal or addition, respectively, of a particle at another point in the sample, which is often conveniently chosen to be at the boundary.

The charge and statistics of the quasiparticles of the FQH states have aroused a considerable amount of interest because they constitute an example of particles that are fractionally charged[4] and are neither bosons nor fermions, i.e., obey fractional statistics.[6,13] We consider the excitation described by the wave function

$$\prod_i (z_i - z_0)\chi_p \ . \tag{18}$$

The Berry phase calculation of Arovas, Schrieffer, and Wilczek[13] can be carried out without modification for this excitation, and one can show that it has charge $-ep$ and obeys p statistics (i.e., interchange of two of these excitations produces a phase πp). It is clear that $\prod_i (z_i-z_0)\chi_{p_1}$ consists of p_1 holes at z_0, one in each of the p_1 LL's. Therefore the above excitation is equivalent to p_1 quasiholes of the type $\chi_{p_1}^+\chi_1^{m-1}$ or to one quasihole of the type $\chi_1^+\chi_1^{m-2}\chi_{p_1}$. From simple counting arguments we expect these quasiholes to have charge $-ep/p_1$ and $-ep$, respectively, and to obey p/p_1^2 and p statistics, respectively. It is clear now that the charge of the quasihole corresponding to a hole in the state χ_ν is simply e_ν as defined in Eq. (9). A similar analysis for quasielectrons is difficult because of their more complicated wave functions, but since a quasielectron-quasihole pair is a neutral boson, the quasielectron $\chi_{p_1}^-\chi_1^{m-1}$ must have charge ep/p_1 and obey $-p/p_1^2$ statistics. Thus, we have shown that the smallest charge, and hence the lowest-energy quasiparticles of the FQH state are images of the quasiparticles of the state in the product which has the largest filling factor. The lowest-energy quasiparticles of a fundamental state P/Q have charge e/Q and obey $1/PQ$ statistics, which is in agreement with the result of the hierarchy theory. We would like to emphasize that wave functions for the state containing more than one quasiparticle can be written without regard to their charge or statistics.

IV. NUMERICAL STUDY OF SMALL SYSTEMS

Study of small particle systems has played an important role in the field of the FQHE. First let us consider the trial states that are entirely in the lowest LL. Both the Laughlin states $\chi_{1/m}$ and their quasiholes $\chi_{1/m}^+$ have been shown to be quite accurate for few particle systems.[3] Another example is the state $[\frac{2}{3},1,1]$ at $\nu=\frac{2}{7}$ where the state at $\frac{2}{3}$ is the hole analog of the Laughlin state at $\frac{1}{3}$. This state has also been shown to have excellent overlap with the true state.[12]

A test of the present approach requires a study of the states in which the higher LL's play a role. Here we study[33] the simplest such states, namely the quasielectron state $\chi_{1/m}^-$ and the $\frac{2}{5}$ state $\chi_{2/5}=\chi_2\chi_1^2$. We consider the limit $\hbar\omega_c\to\infty$ so that the true state is strictly in the lowest LL. Since the product states may have finite amplitude in higher LL's, we project them onto the lowest LL to obtain trial states $\mathcal{P}\chi$ appropriate in the limit $\hbar\omega_c\to\infty$. We obtain the true state corresponding to a given trial state χ_p by exact diagonalization of the Coulomb Hamiltonian in the subspace defined by the appropriate total angular momentum and denote it by $\psi_p(1)$, where the argument 1 indicates that only the lowest LL is kept in the calculation. (The symbol ψ will be used to denote the true Coulomb states and the symbol χ will be reserved for the trial states.) The Table I shows the overlaps of $\mathcal{P}\chi_{1/3}^-$ and $\mathcal{P}\chi_{2/5}$ with the corresponding true states $\psi_{1/3}^-(1)$ and $\psi_{2/5}(1)$. The total angular momentum L of the four-particle quasielectron state is 14 for which there are 15 distinct configurations of electrons in the lowest LL, and the total angular momentum of the $\frac{2}{5}$ state is 12 for which there are nine distinct configurations. The phase space is therefore large enough that the near unity overlaps establish beyond any reasonable doubt that $\mathcal{P}\chi_p$ describe the physical system quite accurately and thus have precisely the correct correlations built in them. This confirms the basic validity of this approach in which *first* good correlations are built in keeping an appropriate number of lowest LL's and *then* this correlated state is projected onto the lowest LL to obtain a trial state to describe the physical state in the limit $\hbar\omega_c\to\infty$.

A. Role of higher Landau levels

Perhaps the most crucial aspect in which the present approach differs from all the previous approaches is in its use of the higher LL's in order to identify the correlations responsible for the FQHE. Further insight into the role of the higher LL's can be gained by considering a model system in which the Hilbert space is restricted to the lowest n LL's. We make the following assertions. (1)

TABLE I. This table shows the overlaps of the lowest LL projections of certain trial states with the corresponding true Coulomb states, defined by $\langle\psi|\mathcal{P}\chi\rangle/(\langle\psi|\psi\rangle\langle\mathcal{P}\chi|\mathcal{P}\chi\rangle)^{1/2}$. N is the number of electrons.

State	N	Overlap
$\chi_{\bar{1}/3}$	3	1
$\chi_{\bar{1}/3}$	4	0.996
$\chi_{2/5}$	4	0.998

For $p \leq 1$

$$\mathcal{P}\psi_p(n) \approx \psi_p(1) \tag{19}$$

i.e., the lowest LL projection of the true state $\psi_p(n)$ is an excellent approximation to $\psi_p(1)$ *independent of the specific choice of* n and the LL spacing $\hbar\omega_c$. (For $l<p\leq l+1$ the analogous equation is $\mathcal{P}\psi_p(n)\approx\psi_p(l+1)$, where the operator $\mathcal{P}$ projects $\psi_p(n)$ to the lowest $l+1$ LL's.) This is clearly true in the extreme quantum limit $\hbar\omega_c\rightarrow\infty$ but is far from obvious in the limit $\hbar\omega_c=0$. For the most dramatic effect, as well as for reasons that will become clear later, we take the LL's to be degenerate, i.e., set $\hbar\omega_c=0$. (2) The states χ_p are accurate representations of the true ground states $\psi_p(n)$ for appropriate values of n, and consequently their lowest LL projections ($\mathcal{P}\chi_p$) are good approximations of the true ground state $\psi_p(1)$.

We demonstrate the plausibility of these statements by considering simple examples. For the first assertion we calculate $\psi(2)$ (with $\hbar\omega_c=0$) and $\psi(1)$ for a three-electron system for several values of the total angular momentum. The overlaps of $\mathcal{P}\psi(2)$ with $\psi(1)$ are shown in Table II which clearly corroborate Eq. (19). This confirms that in order to obtain the lowest LL ground state, it is valid to *first* calculate the ground state keeping as many (a finite number of) LL's as one wishes, and *then* take its projection onto the lowest LL.

For the second assertion let us consider the special case of $n=2$. It can be proven that the states $\chi_{2/(2m-1)}=\chi_1^{m-1}\chi_2$ are the unique ground states in the presence of short-range repulsive interactions of the type $\nabla^{2(m-1)}\delta(\mathbf{r})$, first considered by Trugman and Kivelson.[27]

TABLE II. This table shows the overlaps of $\mathcal{P}\psi(2)$ with $\psi(1)$ for a three-electron system for several values of the total angular momentum, L. For $L<5$ the phase space in the lowest LL is so limited that the overlap is trivially 1.

L	Overlap
5	0.993
6	0.999
7	0.999
8	0.999
9	0.988
10	0.983
11	0.961
12	0.997
13	0.994

Since we have taken the LL's to be degenerate, these states are eigenstates of the kinetic energy, which explains our motivation for setting $\hbar\omega_c=0$. Since these states vanish as r^m, as the distance between two particles, r, approaches zero, the expectation value of the above interaction with respect to these states is zero, as can be seen easily by integration by parts. Therefore, these states are ground states. What remains to be shown is that $\chi_{2/(2m-1)}$ is the only state at $\nu=2/(2m-1)$ that vanishes as r^m. This can be shown as follows. Since only the states of the lowest two LL's are available, the largest power of a z_j^* in the polynomial multiplying the exponential is 1. This implies that the most general trial state that vanishes as r^m is

$$\chi_1^{m-1}\chi_\nu , \tag{20}$$

where $\nu\leq 2$. This state corresponds to a system at filling factor $\nu/[(m-1)\nu+1]$. Clearly, at $2/(2m-1)$ the only possibility is $\chi_{2/(2m-1)}$ [and no such state can be constructed for filling factors greater than $2/(2m-1)$]. This presumably implies, based on the intuition gained from the Laughlin states, that $\chi_{2/(2m-1)}$ is a legitimate approximation to the true ground state $\psi_{2/(2m-1)}(2)$ for physical interactions for this model system. Therefore, it is now no surprise that $\mathcal{P}\chi_{2/5}$ accurately describes $\psi_{2/5}(1)$.

In the extreme quantum limit one expects the electron density in the true system to be predominantly in the lowest LL, and therefore the lowest LL restriction seems to be quite natural. However, we use the subtle trick of starting with a bigger Hilbert space (defined by the parameter n) and in the end making use of Eq. (19) to extract the lowest LL state. The advantage is that different choices of n provide a simple and natural description of the FQH states at different filling factors, thus bringing out the otherwise hidden structure of these states. We have shown that just as $n=1$ is suitable for the description of the ground states at $1/m$, $n=2$ is suitable for extracting the ground states at $2/(2m-1)$. Similarly, other values of n will yield ground states at other filling factors.

Unfortunately, even though all the product states are ground states in the above model for proper choices of the short-range interaction, not all are *nondegenerate* ground states. For example, for $n=3$, the state $\chi_3\chi_1^{m-1}$ at $3/(3m-2)$ is degenerate with a large number of compressible states of the form $\chi_2^2\chi_1^{m-3}\chi_{3/4}$. However, surely this degeneracy is an artifact of the model, and we believe that for realistic interactions the highly correlated state $\chi_3\chi_1^{m-1}$ has lower energy than the other states which are not even homogeneous in general. Indeed, $\chi_3\chi_1^{m-1}$ is as natural as $\chi_2\chi_1^{m-1}$ in this scheme.

B. Validity of the unprojected states

We have thus plausibly demonstrated that $\mathcal{P}\chi_p$ describe the physical system accurately. Clearly, this implies that the states χ_p are themselves legitimate representations of the true fractional states in the sense that they have precisely those correlations built in them that are responsible for incompressibility at filling factor p.

Thus, these states can be used to calculate certain topological properties of the FQH states, e.g., the charge and the statistics of the quasiparticles. Of course, for a computation of several other quantities, e.g., the ground-state energy or the quasiparticle gap in the limit $\hbar\omega_c \to \infty$, one must use the projected states.

Some readers may find it slightly disturbing that χ_p in general have a finite density of electrons in the higher LL's. However, we would like to emphasize that whether one writes a trial state strictly within the lowest LL or a trial state largely in the lowest LL, one *must* in the end argue that the structure of the state is not very sensitive to small changes in the occupation of various LL's, because under the typical experimental conditions some amount of LL mixing is certainly expected. Laughlin has argued that his states describe the physics correctly even when the coupling to higher LL's is turned on so long as there is a gap of any size in the true state.[34] Similar reasoning can be applied to these states. That the physics of these states is insensitive to variations in the occupation of the higher LL's is in fact quite clear from our numerical calculations on the $\frac{2}{5}$ state in which we consider a model system with only the lowest two LL's accessible. We have shown that $\chi_{2/5}$ is a good approximation to the true state when $\hbar\omega_c = 0$, and the closely related state $\mathcal{P}\chi_{2/5}$ is a good approximation to the true state when $\hbar\omega_c = \infty$. This seems to suggest that there is no change in the character of the true state, i.e., no phase transition, as the occupation of the higher LL is varied by varying $\hbar\omega_c$. Consequently, $\chi_{2/5}$ is as legitimate a representation of the true state as $\mathcal{P}\chi_{2/5}$. (Notice that if all the LL's were available, $\chi_{2/5}$ would not remain a reasonable representation of the true state for small $\hbar\omega_c$ but only for large $\hbar\omega_c$.)

Thus, it seems that the states χ_p capture the physics responsible for incompressibility at fractional filling factors. We have, moreover, argued that they lie largely in the lowest LL in the thermodynamic limit,[20,21] and therefore do not do very badly energetically either. The argument briefly goes as follows. Expand the polynomial part of the product state χ_p. In each term the coordinates of a particle appear as $z_j^{s_j} z_j^{*t_j} \exp(-|z_j|^2/4)$ where t_j is of order 1 (assuming only a finite number of the lowest LL's are occupied) and s_j is typically very large, of order N ($s_j \gg 1$). Expressing it as a superposition of the single particle states, it can be shown (Appendix) that it has its amplitude predominantly in the lowest LL and the amplitude decreases by a factor of order $s_j^{-1/2}$ in each successive higher LL. Thus, dropping the higher ($l \neq 0$) LL states altogether in each such expression should lead to a state which is almost the same as the initial state χ_p; i.e., projection onto the lowest LL does not change the state χ_p significantly. This suggests that the states χ_p are largely in the lowest LL. The crucial untested assumption here is that there are no huge cancellations among the remaining terms. (Of course, we also assume that $p < 1$.) An exact evaluation of what fraction of electrons occupies the higher LL's in the thermodynamic limit seems to be a rather involved combinatorial problem. In any case, provided it is largely in the lowest LL, χ_p itself is not an unreasonable trial state under typical experimental conditions, for which some amount of LL mixing is certainly expected (the ratio of the Coulomb energy to $\hbar\omega_c$ is typically 0.2, which is not extremely small).

We again emphasize why it is useful to include several LL's in the theory of the FQHE. With the lowest LL restriction there is a simple description of the FQHE at the "special" filling factors $1/m$, as embodied elegantly in the Laughlin states. The structure of the other FQH states is, however, not as clear. We have argued here that with the use of higher LL states one can write equally simple states at other incompressible filling factors as well, and thereby provide a general understanding of the nature of the electronic correlations in the FQHE. The use of the higher LL's thus brings out the structure of the FQH states in a very simple and intuitive fashion.

V. ORDER OF STABILITY

We now show that the order of stability predicted by this scheme is completely consistent with the experiments.

Even though incompressibility and hence FQHE are possible at all rational filling factors with odd denominators, only a small number of fractions are observed experimentally. The reason is that only those states are observed that have a finite gap in the presence of disorder. Disorder reduces the quasiparticle gap; as disorder is increased the quasiparticle gap eventually disappears and the FQHE is destroyed.[35] Therefore, in the presence of disorder only a finite number of states have a finite gap, and as the experimental samples are made purer the number of such states increases. A state with a larger quasiparticle gap is more stable in the presence of disorder than a state with a smaller quasiparticle gap, and predicting the order of stability of various fractions requires a computation of the quasiparticle gaps.

Fortunately, as we now show, it is possible to predict, to an extent, the order of stability with the help of certain plausible rules without actually computing the quasiparticle gap. The most obvious rule is that if $\chi_{p'_\lambda}$ is less stable than χ_{p_λ}, the state $[p'_\lambda, 1, 1, \ldots]$ is less stable than $[p_\lambda, 1, 1, \ldots]$ for fixed m. When the states at p_λ and p'_λ are equally stable (e.g., when both are integers) we assume that, so long as the resulting states are sufficiently similar, the state with the smaller quasiparticle charge has smaller quasiparticle gap and is consequently less stable. This implies that the state $[p_1, 1, 1, \ldots]$ is more stable than $[p_1 + 1, 1, 1, \ldots]$ and the state $[p_1, 1, 1, \ldots]$ is more stable than $[p_1, 1, \ldots, 1, 1]$. In Fig. 1(a) we show[36] the filling factors of the fundamental FQH states $[p_1, 1, 1, \ldots]$ as a function of p_1 and m. Any given state is more stable than the one above it or the one on its right. In this and the following figures all fractions except the last one of a horizontal sequence have been experimentally observed, and the last fraction constitutes a prediction of theory. More fractions can be generated from the fundamental fractions of Fig. 1(a) which we generically refer to as p_a. Figure 1(b) shows the filling factors of the hole analog states of Fig. 1(a), i.e., $1 - p_a$.

Figure 1(c) contains the filling factors of states $[1-p_a,1,1]$, and Fig. 1(d) of the states $[1+p_a, 1,1]$. Figure 1(e) is the hole analog of Fig. 1(d). Also in all these figures any given state is again more stable than the one above it or the one on its right according to our rules. All of the observed filling factors (< 1) have thus been obtained in a very compact manner, and the predicted order of stability is in excellent agreement with experiments. In Figs. 1(a) and 1(b) the horizontal sequences are precisely the experimentally observed sequences converging to $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, etc. and appear very naturally in this theory. It is also worth pointing out that the higher LL states play an important role in all the states except those in the first columns of Figs. 1(a), 1(b), and 1(c).

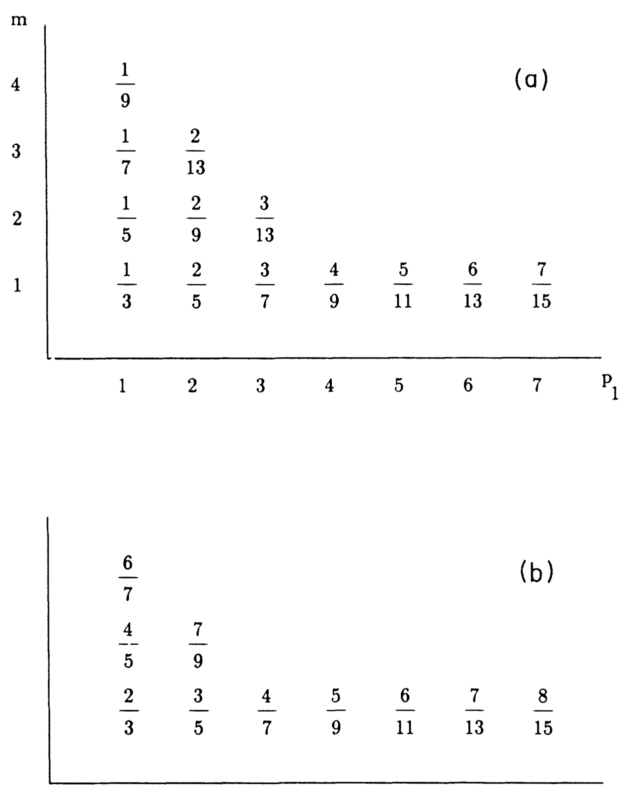
We are now in a position to answer a specific question raised in the beginning of this paper: Why is $\frac{5}{13}$ not observed even though $\frac{6}{13}$ is? The state at $\frac{6}{13}$ is [6,1,1] whereas the best state at $\frac{5}{13}$ is $[\frac{5}{3},1,1]$ where $\frac{5}{3}$ is either $1+\frac{2}{3}$ or $2-\frac{1}{3}$. Since it is reasonable to expect the state at $\frac{5}{3}$ to be less stable than the one at 6, it is not surprising that the state at $\frac{5}{13}$ is less stable than the one at $\frac{6}{13}$.

VI. GENERALIZED PRODUCT STATES

The most obvious generalization of the above states is to write[21]

$$\chi_p = \prod_{\lambda=1}^{m} \chi_{p_\lambda} \equiv [p_1, \ldots, p_m] , \tag{21}$$

where χ_{p_λ} are *any* known incompressible states. The exponential factor in a state χ_{p_λ} is again defined with



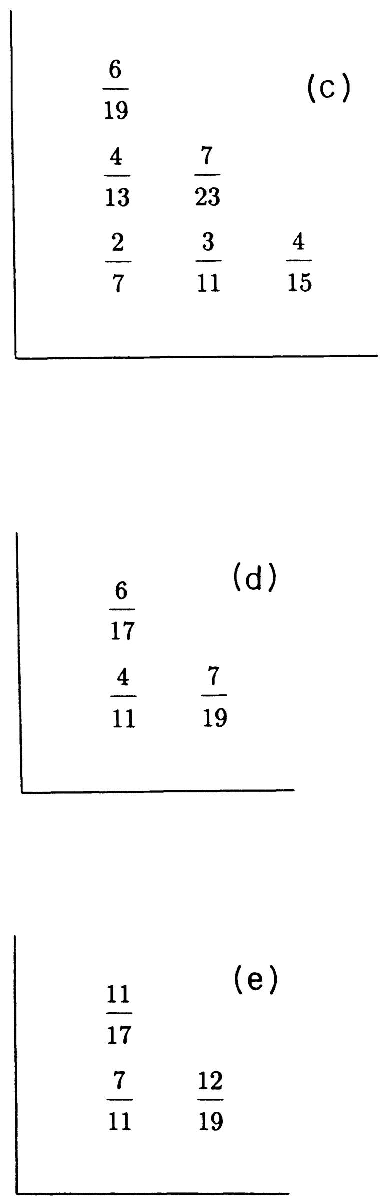


FIG. 1. The filling factors of the states $[p_1,1,1\ldots]$ denoted by p_a are shown in (a), the filling factors $1-p_a$ of the corresponding hole analog states are shown in (b), and the filling factors of the states $[p_1,1,1]$ obtained by choosing $p_1=1-p_a$ and $p_1=1+p_a$, are shown in (c) and (d), respectively. (e) is the hole analog of (d). All the fractions of a horizontal sequence except the rightmost one have been observed. No other odd-denominator fractions have been observed.

charge

$$e_\lambda = ep/p_\lambda \ , \tag{22}$$

and the filling factor of the product state is given by

$$p = \left[\sum_{\lambda=1}^{m} p_\lambda^{-1} \right]^{-1} . \tag{23}$$

Unless mentioned otherwise, we will assume p_λ's to be integers and $p_1 \geq p_2 \geq \cdots \geq p_m$.

Variational states for the quasiparticle excitations can be written as before. As before, the quasiparticles of the product state that correspond to quasiparticles of χ_{p_λ} have charge $e_\lambda = ep/p_\lambda$ and obey p/p_λ statistics.

We start by showing that the product states of the type given in Eq. (21) can be written at *all* rational filling factors, i.e., this theory predicts possibility of FQHE at all rational fractions. We prove this by explicit construction of *an* incompressible state for each rational fraction. A state at P/Q, where Q is odd, can be constructed by choosing in Eq. (23) $m = Q$ and $p_\lambda = P$. Similarly, a state at P/Q, where P is odd, Q is even, and r is an even integer so chosen that $P < rQ$, can be obtained by choosing $m = rQ + r - P$, r of the p_λ's equal to r^2, and the rest equal to rP. These states are obviously not unique. Given an incompressible state at a fraction p, one can construct states at $n \pm p$ where n is an integer. As before, more states can be generated by choosing χ_{p_λ} in Eq. (21) to be one of the states obtained above corresponding to a noninteger filling factor. In general, there will be several candidates for a given rational fraction, but usually the most stable one can be determined uniquely with the help of the rules outlined earlier.

Only rational filling factors are generated in this theory; incompressibility is not possible at nonrational filling factors. This results directly from the fact that all incompressible filling factors are generated starting from integers.

One interesting outcome of this generalization is that there is nothing in principle that forbids FQHE at even-denominator fractions,[37] which is an esthetically pleasing result, especially considering that an even-denominator fraction has been observed. Of course, we must explain why the even-denominator fractions are so rare. To this end we find it useful to classify the states according to m_0 which is defined to be the number of p_λ's different from 1. Presumably the strongest correlations are due to binding of the zeros of the wave function to the electrons, as in the Laughlin states, and since that happens strictly only in χ_1, the most strongly correlated states are those that have most of the p_λ's equal to unity. More quantitatively, a state $[\{p_\lambda\}]$ has vanishing interaction energy in Haldane's pseudopotential scheme[5] so long as pseudopotentials V_n vanish for $n \geq m - m_0$, where $m - m_0$ is the number of p_λ's equal to 1. Thus, the smaller the m_0, the more stable one would expect the state to be for a given m. Therefore, we make the plausible assumption that the more experimentally relevant states are those with small values of m_0. This assumption is consistent with the fact that only odd-denominator fractions have been observed, since for $m_0 \leq 1$ only odd-denominator fractions can be obtained. Indeed, this assumption has already been implicitly used in comparing the order of stability of the experimentally observed fractions with the states of the type $[p_1, 1, 1, \ldots]$.

Even-denominator fractions appear only when two or more of the p_λ's are different from 1. For example, the state

$$[2,2,1] = \chi_1 \chi_2^2 \tag{24}$$

corresponds to filling factor $\nu = \frac{1}{2}$. Thus even-denominator fractions are rare because they occur only when $m_0 \geq 2$, in which case the correlations are relatively weak. We emphasize here that in the present approach the FQHE at even-denominator fractions is in principle possible even for spinless electrons.

Even for $m_0 \geq 2$ there are relatively very few even-denominator states. Consider $m_0 = 2$ for example. The only even-denominator states in this case are $[4j+2, 4j+2, 1, 1, \ldots]$ where $j = 0, 1, 2, \ldots$, the filling factor is

$$p = \frac{2j+1}{(m-2)2j + (m-1)} . \tag{25}$$

Some examples of these states are [2,2,1] at $\frac{1}{2}$, [2,2,1,1,1] at $\frac{1}{4}$, and [6,6,1] at $\frac{3}{4}$. The state [6,6,1] is extremely unlikely to be observed, because the much stronger state [6,1,1] at $\nu = \frac{6}{13}$ is barely observable in the best available samples of the day.[16] Thus, there is little likelihood of the observation of any even-denominator fractions other than $\frac{1}{2}$, and maybe $\frac{1}{4}$. The most stable spin-polarized even-denominator state is predicted to be [2,2,1] at $\nu = \frac{1}{2}$. Since all the observed odd-denominator state can be written in the form $[p_1, 1, 1, \ldots]$, this theory is consistent with the fact that [2,2,1] has not been observed. Only detailed numerical calculations can tell for what values of p_1 the state $[p_1, 1, 1, \ldots]$ becomes less stable than [2,2,1].

In general the charge of the quasiparticles of the P/Q state is not e/Q. For example, the charge of the quasiparticles of the state [2,2,1] at $\nu = \frac{1}{2}$ is $e/4$.

VII. ROLE OF SPIN

Until now we have been considering spinless electrons. In sufficiently high magnetic fields when the Zeeman energy is very large one expects the states for $\nu < 1$ to be spin polarized and consequently the electron spin to be irrelevant to the problem. However, when Zeeman energy is small, spin could play an important role in producing new possible states.[16-19,38-40] At the simplest level spin can be incorporated in the present formalism by taking one of the χ_{p_λ}'s (say χ_{p_1}) to describe electrons with spin. The resulting state is

$$\chi_p \equiv [q_\uparrow, q_\downarrow; p_2, \ldots, p_m] = \chi_{q_\uparrow, q_\downarrow} \prod_{\lambda=2}^{m} \chi_{p_\lambda} , \tag{26}$$

where $\chi_{q_\uparrow, q_\downarrow}$ has $q_\uparrow$ ($q_\downarrow$) up-spin (down-spin) Landau bands occupied, and χ_{p_λ} are formed from the N particle coordinates without regard to their spin degree of freedom. The filling factor of the product state is given by Eq. (23) with $p_1=q_\uparrow+q_\downarrow$. Let us now consider some specific examples.

The state $[1,1;1,1\ldots]$ is given by

$$\chi_{2/(2m-1)}=\chi_{1,1}\chi_1^{m-1}\;, \tag{27}$$

$$\chi_{1,1}=\left[\prod_{1\le i<j\le N/2}(z_i-z_j)\right]\left[\prod_{1\le s<t\le N/2}(z_s'-z_t')\right]\;, \tag{28}$$

$$\chi_1=\prod_{i<j}(z_i-z_j)\prod_{s<t}(z_s'-z_t')\prod_{i,s}(z_i-z_s')\;, \tag{29}$$

where z and z' denote the coordinates of spin-down and spin-up electrons, respectively, the indices i,j,s,t run from 1 to $N/2$ where N is the total number of electrons, and the filling factor can be easily shown to be $2/(2m-1)$. (The exponential factors have been omitted for simplicity.) Since the number of spin-up and spin-down electrons is equal, the states $\chi_{2/(2m-1)}$ are spin unpolarized and are precisely the spin-unpolarized states considered by Halperin[18] and Haldane.[41]

The state $[q_\uparrow,q_\downarrow;1,1,\ldots]$ has filling factor

$$p=\frac{p_1}{p_1(m-1)+1} \tag{30}$$

and is given by

$$\chi_p=\prod_{i<j}(z_i-z_j)^{m-1}\times\prod_{s<t}(z_s'-z_t')^{m-1}\prod_{i,s}(z_i-z_s')^{m-1}\chi_{q_\uparrow,q_\downarrow}\;, \tag{31}$$

where i,j run from 1 to $Nq_\downarrow/q$ and s,t run from 1 to $Nq_\uparrow/q$. When $q_\downarrow=q_\uparrow$ these states are spin unpolarized and when $q_\downarrow\neq q_\uparrow$ they are partially spin polarized. To the best of our knowledge this is the first time that explicit wave functions have been written to describe incompressibility in the presence of *partial* polarization. By analogy with the spin unpolarized states these may also be experimentally relevant for small Zeeman energies.

At filling factor $\frac{1}{2}$, besides the fully spin-polarized state [2,2,1], a spin-unpolarized state

$$[1,1;2,1]=\chi_{1,1}\chi_2\chi_1 \tag{32}$$

is also possible. This is similar in spirit to a state considered in Ref. 11(b).

As emphasized by Haldane,[41] a physically acceptable state must not only be an eigenstate of S_z, but also of the total spin operator, or, in other words, must satisfy Fock's cyclic condition. The states χ_p given here satisfy this criterion if $\chi_{q_\uparrow,q_\downarrow}$ does, because the factor multiplying it is symmetric under the exchange of any two particles regardless of their spin.

In general one could put spin in more than one of the states in the product. For instance, one can write the state [1,1;1,1;1] at $\nu=\frac{1}{2}$, which is the same as one of the states proposed by Halperin[18] and later numerically studied by MacDonald, Yoshioka, and Girvin.[19]

The only even-denominator state observed to date is $\frac{5}{2}$ (i.e., $\frac{1}{2}$ in the second LL) and there is experimental evidence that it is not completely spin polarized and probably spin unpolarized.[16,42] FQHE at even-denominator fractions does not appear naturally in the hierarchy scheme, but trial states have been constructed[17–19] to explain the experimental observation of $\frac{5}{2}$. We, on the other hand, obtain even- and odd-denominator fractions in the same theoretical scheme. As shown earlier, there are very few even-denominator states in the favorable parameter range, which explains the experimental scarcity of even-denominator fractions. Also, $\frac{1}{2}$ appears in the present framework to be the most likely even-denominator candidate. One can write both spin-polarized and spin-unpolarized states at $\frac{1}{2}$. When the Zeeman energy is sufficiently small, a spin unpolarized state is expected to be more favorable because of its smaller m_0, and because of the analogy with the numerical calculations that show that the spin-unpolarized states of the type $[1,1;1,1,\ldots]$ are more favorable than the corresponding spin-polarized states for sufficiently small Zeeman energy.[38] Thus, it is qualitatively clear why the spin-unpolarized $\frac{1}{2}$ state is the only even-denominator state observed so far.

VIII. COMPARISON WITH THE STANDARD HIERARCHY APPROACH

It is instructive to spend some time in comparing the present theory with the standard theory of the FQHE. There are several differences as well as similarities between the two approaches, which we now consider.

In the hierarchy theory the daughter state at each step is obtained when the quasiparticles of the parent state condense into a low-energy Laughlin-type state. Even though the quasiparticles do not play any role in the construction of the FQH states presented in this paper, a hierarchylike interpretation is also possible for these states. For concreteness we consider the states of the form

$$\chi_1^2\chi_\nu\;. \tag{33}$$

The $\frac{1}{3}$ state is obtained with the choice $\nu=1$, the $\frac{2}{5}$ state is obtained with $\nu=2$, the $\frac{3}{7}$ state is obtained with $\nu=3$, and so on. Let us now start with the $\frac{1}{3}$ state (i.e., $\nu=1$) and slowly increase ν while keeping the number of particles, N, fixed. Taking a disk shaped sample and choosing circular gauge, we fill $N_0=N/\nu$ single particle states closest to the center in the lowest LL of the χ_ν (so that the lowest LL is completely occupied) and distribute "uniformly" the remaining $N-N_0=N(1-\nu^{-1})$ electrons in the corresponding N_0 states of the higher (second) LL. The electrons in the partially filled LL in χ_ν can be considered as the quasielectrons in the state $\chi_1^2\chi_\nu$ of the underlying "incompressible" state comprised of the filled LL's. In general there are a large number of ways of distributing the quasielectrons into the available states. However, at $\nu=2$, when $N/2$ quasielectrons are created, they have a unique configuration, and the resulting state, $\chi_1^2\chi_2$, is nondegenerate, indicating incompressibility. One can now similarly create quasielectrons of the $\frac{2}{5}$ state to obtain the next nondegenerate state that occurs at $\frac{3}{7}$. Thus at first sight the trial states proposed here seem to be a microscopic realization of the hierarchy ideas. However, there are several significant differences between the two approaches which we discuss below.

The main difference is in the very physics of incompressibility. In the scheme presented here the quasielectrons acquire their incompressible arrangement from analogy to the IQH states and not by mimicking the Laughlin states. Thus, the "fundamental" states here are the IQH states and not the Laughlin states. [Note that we have also used the term "fundamental" for the simplest FQH states shown in Fig. 1(a).]

Also, as is clear from the above discussion, if a state A can be obtained from a state B by creation of quasielectrons, the state B can be obtained from the state A by creation of quasiholes. Therefore, it is inappropriate to call one of them the "parent" state and the other the "daughter" state. In fact, the FQH states are more intimately related to the corresponding IQH states than to each other.

For the same reason, some of the states that appear naturally in the hierarchy theory do not emerge naturally here. For example, in the hierarchy theory, starting from the $\frac{2}{5}$ state, one can obtain the $\frac{3}{7}$ state by creating quasielectrons and the $\frac{5}{13}$ state by creating quasiholes. In the present scheme, starting from the $\frac{2}{5}$ state one can reach the $\frac{3}{7}$ state by creating quasielectrons, but creation of quasiholes leads back to the $\frac{1}{3}$ state (at the simplest level). Thus, here the hierarchical fractions obtained through creation of quasielectrons occur more naturally than the ones obtained through creation of quasiholes. *The important point is that these are also the observed fractions.* For example, the $\frac{3}{7}$ state has been observed whereas the $\frac{5}{13}$ state has not. Thus, while the hierarchy scheme produces along with the observed fractions a large number of fractions that are not observed, the present scheme produces the observed fractions in a very compact manner. We take this to be a strong evidence in favor of this approach, in which the FQH states are constructed by analogy with the IQH states rather than the Laughlin states.

Moreover, this theory can be generalized to describe the incompressible states at even-denominator filling factors. We find this gratifying even though we have not yet investigated how well the generalized product states do in describing the true system.

Despite these differences there are various similarities between the two approaches. The charges of the quasiparticles agree in the two approaches at the simplest level (of Sec. III) which covers all the odd-denominator states observed to date. In the example of the transition from $\frac{1}{3}$ to $\frac{2}{5}$ to $\frac{3}{7}$ considered above, the $\frac{2}{5}$ state is obtained from the $\frac{1}{3}$ state after creating $N/2$ quasiparticles, and the $\frac{3}{7}$ state is similarly obtained from the $\frac{2}{5}$ state after creating $N/3$ quasiparticles. This is also in exact agreement with the hierarchy theory.

Now we use some ideas of the hierarchy theory to demonstrate that the trial states proposed here are in fact very natural for describing the physics of the FQHE. Even in the hierarchy approach, when a quasielectron is created by inserting a flux quantum through an infinitely thin magnetic solenoid piercing the sample, it has some amplitude in the higher LL. However, since in the high-field limit the amplitude of the true quasielectron in the higher LL is insignificant, the terms containing the z^*'s are thrown away to obtain a quasielectron state which is strictly in the lowest LL.[4,34] It is interesting to ask what would happen if one used the "unprojected" Laughlin quasielectrons to build the hierarchical states, deferring the projection to the end. In this case the hierarchical states would also have a finite amplitude in the higher LL's. For example the $\frac{2}{5}$ state thus obtained would have some amplitude *at least* in the lowest two LL's. As shown earlier, the state $\chi_{2/5}=\chi_1^2\chi_2$ at $\nu=\frac{2}{5}$ is the unique state which has amplitude *only* in the lowest two LL's, and which vanishes in the same way as the Laughlin state $\chi_{1/3}$ as two electrons approach each other. This shows that the $\frac{2}{5}$ state proposed here is in some sense the best state that one can build with $N/2$ unprojected quasielectrons.[43] This example demonstrates three crucial points of the present theory in a very forceful manner. (i) Higher LL's appear very naturally in the theory of the FQHE. (ii) The FQH states are very intimately related to the IQH states. (iii) Despite some fraction of electrons in the higher LL's, the states χ_p correctly capture the physics of the FQHE.

The other hierarchical result that is embodied very naturally in the trial states proposed here is the following. If one applies the hierarchy construction to the Laughlin state with $m=1$, it should produce the states with 2, 3, ... filled LL's.[41] The sequence 1,2,3, ... is thus analogous to the sequences $\frac{1}{3},\frac{2}{5},\frac{3}{7},\ldots$ and $\frac{1}{5},\frac{2}{9},\frac{3}{13},\ldots$. Unfortunately, it has not attracted much attention in the hierarchy theory even though it is undoubtedly the best understood sequence and ought to serve as the prototype when one attempts to understand the other sequences. The analogy between the $\frac{1}{3}$ sequence, the $\frac{1}{5}$ sequence, and so on, to the 1 sequence, which is implicit in the hierarchy approach, is in fact very transparent in present trial states: just as one goes from χ_1 to χ_2 to χ_3, one can go from $\chi_1\chi_1^2=\chi_{1/3}$ to $\chi_2\chi_1^2=\chi_{2/5}$ to $\chi_3\chi_1^2=\chi_{3/7}$, and from $\chi_1\chi_1^4=\chi_{1/5}$ to $\chi_2\chi_1^4=\chi_{2/9}$ to $\chi_3\chi_1^4=\chi_{3/13}$.

IX. CONCLUSION

We have provided a theory which combines into a single coherent conceptual framework various pieces of the QHE, which were previously more or less unrelated. The most appealing feature of this theory is that it unifies the IQHE and the FQHE. It constructs new possible incompressible states by simply taking products of the known incompressible states. Our claim is that legitimate trial wave functions at all rational filling factors can be obtained starting from the IQH states in this way. Thus, at the most fundamental level, incompressibility at fractional filling factors is due to a combination of the correlations that are responsible for QHE at integer filling factors.

The Laughlin states are a special example of such product states and are obtained by taking odd powers of the $\nu=1$ IQH state. Numerical calculations show that the lowest LL projection of the simplest new trial state, namely $\chi_{2/5}=\chi_1^2\chi_2$, also describes the true state very accurately for few particle systems. The quasiparticle excitations of the FQH states can be understood elegantly by analogy to the quasiparticle excitations of the underlying IQH states. This provides a simple way of "deriving" the trial state for the quasihole proposed by Laughlin. The quasielectron wave function obtained in analogous manner is new and our numerical results show that its lowest LL projection is also a very good approximation of the true quasielectron state.

Trial states have been proposed in the past to describe the FQH states at fractions other than $1/m$.[11,12] However, this is the first time that it has been realized that remarkably simple states for the FQHE can be written by making use of the IQH states, which are known to have correlations that lead to incompressibility.

We conclude by enumerating the main features of the theory described above. (i) It unifies various aspects of the phenomenon of the quantum Hall effect. (ii) It produces explicit wave functions for all the fractional quantum Hall states—spin polarized or otherwise. (iii) It also produces all the quasiparticle wave functions. (iv) It predicts possibility of incompressibility, and hence of quantum Hall effect, at *all* rational filling factors. (v) The predicted order of stability is completely consistent with experiments. (vi) It permits an adiabatic calculation of statistics and the charge of the quasiparticles in general. In short, we believe that the approach presented in this paper provides a fairly complete, satisfying and consistent picture of the FQHE.

Note added in proof. (i) The fractions $\frac{3}{13}$ and $\frac{4}{15}$ have also been observed recently,[44] which is consistent with the predictions of Fig. 1. (ii) The lowest LL projection of the incompressible spin-polarized $\frac{1}{2}$ state in Eq. (24) does not have a large overlap with the corresponding true Coulomb state for a four-electron system. This is consistent with the fact that FQHE is not observed at $\nu=\frac{1}{2}$. (iii) The lowest LL projection of the spin-unpolarized state at $\nu=\frac{1}{2}$ [Eq. (32)] also does not have a large overlap with the corresponding true Coulomb state. These results suggest that the *generalized* trial states of Sec. VI may not be physically relevant.

ACKNOWLEDGMENTS

The author has greatly benefited from discussions with N. Read, S. Kivelson, D. Stone, A. Szafer, V. Goldman, and N. Trivedi.

APPENDIX

In this Appendix we derive some of the results used in the text. The single particle states in circular gauge are given by[4]

$$\eta_{l,s}=(-1)^{l+s}[2\pi 2^{s+2l}l!(l+s)!]^{-1/2} \times e^{(x^2+y^2)/4}D^{*l}D^{l+s}e^{-(x^2+y^2)/2}, \quad (A1)$$

where $l=0,1\ldots$ is the LL index, $s=-l,-l+1,\ldots$ is the angular-momentum index, $D=\partial/\partial x+i\partial/\partial y$, and x and y are expressed in units of the magnetic length. Taking $z=x+iy$ and $z^*=x-iy$ to be the independent variables one can replace D and D^* by $2\partial/\partial z^*$ and $2\partial/\partial z$, respectively, and write

$$\eta_{l,s}=[2\pi 2^s l!(l+s)!]^{-1/2}e^{zz^*/4} \times\left[\frac{\partial}{\partial z}\right]^l z^{l+s}e^{-zz^*/2}. \quad (A2)$$

This can also be written in the form

$$\eta_{l,s}=\frac{e^{is\theta}}{\sqrt{2\pi}}\left[\frac{l!}{(s+l)!}\right]^{1/2}e^{-t/2}t^{s/2}L_l^s(t), \quad (A3)$$

where $t=zz^*/2$, $e^{i\theta}=z/|z|$, and

$$L_l^s(t)=\frac{1}{l!}t^{-s}\left[\frac{\partial}{\partial t}\right]^l t^{s+l}e^{-t}. \quad (A4)$$

The orthonormality of the single particle states follows from that of L_l^s because the volume element $r\,dr\,d\theta=dt\,d\theta$. This form is especially suitable for calculations involving several LL's.

In the state χ_p the coordinates of a particle appear as $z^{*l}z^{l+s}\exp(-zz^*/4)$, which can be easily expressed as a superposition of the states $\eta_{l',s}$:

$$z^{*l}z^{l+s}\exp(-zz^*/4)=\sum_{l'=0}^{l}a_{l'}\eta_{l',s}, \quad (A5)$$

where

$$a_{l'}=(-1)^{l'}\left[\frac{2\pi 2^{s+2l}}{(l'+s)!\,l'!}\right]^{1/2}\frac{(s+l)!\,l!}{(l-l')!}. \quad (A6)$$

For large s

$$\frac{a_{l'+1}}{a_{l'}}\sim O\left[\frac{1}{\sqrt{s}}\right], \quad (A7)$$

implying that the amplitude is the largest in the lowest LL and decreases by a factor of order $s^{-1/2}$ in each successive higher LL. Thus, for large s the state $z^{*l}z^{l+s}\exp(-zz^*/4)$ is almost entirely in the lowest LL. The projection of this state onto the lowest LL is given by

$$a_0\eta_{0,s}=\exp(-zz^*/4)\left[2\frac{\partial}{\partial z}\right]^l z^{s+l}. \quad (A8)$$

Thus, in order to obtain the lowest LL projection of any given many-electron state, one first expands the polynomial multiplying the exponential, writes each term in an ordered form where all the z^*'s appears to the left of the z's, and then replaces z_j^* by $2\partial/\partial z_j$ with the understanding that the derivatives do not act on the exponential.

[1]D. C. Tsui, H. L. Störmer, and A. C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982).
[2]For a review of experimental as well as theoretical work, see *The Quantum Hall Effect*, edited by R. E. Prange and S. M. Girvin (Springer-Verlag, New York, 1987).
[3]R. B. Laughlin, Phys. Rev. B **23**, 5632 (1981).
[4]R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
[5]F. D. M. Haldane, Phys. Rev. Lett. **51**, 605 (1983).
[6]B. I. Halperin, Phys. Rev. Lett. **52**, 1583 (1984).
[7]R. B. Laughlin, Surf. Sci. **141**, 11 (1984); in *The Quantum Hall Effect* (Ref. 2), pp. 233–302.
[8]S. T. Chui, T. M. Hakim, and K. B. Ma, Phys. Rev. B **33**, 7110 (1986).
[9]S. Kivelson, C. Kallin, D. P. Arovas, and J. R. Schrieffer, Phys. Rev. Lett. **56**, 873 (1986); Phys. Rev. B **36**, 1620 (1987); D. H. Lee, G. Baskaran, and S. A. Kivelson, Phys. Rev. Lett. **59**, 2467 (1987).
[10]S. M. Girvin, in *The Quantum Hall Effect* (Ref. 2), pp. 353–399; S. M. Girvin and A. H. MacDonald, Phys. Rev. Lett. **58**, 1252 (1987); E. H. Rezayi and F. D. M. Haldane, *ibid.* **61**, 1985 (1988); S. C. Zhang, T. H. Hansson, and S. Kivelson, *ibid.* **62**, 82 (1989); N. Read, *ibid.* **62**, 86 (1989).
[11](a) A. H. MacDonald, G. C. Aers, and M. W. C. Dharma-wardana, Phys. Rev. B **31**, 5529 (1985); (b) D. Yoshioka, A. H. MacDonald, and S. M. Girvin, *ibid.* **38**, 3636 (1988); (c) **39**, 1932 (1989).
[12]B. I. Halperin, Helv. Phys. Acta **56**, 75 (1983); R. Morf, N. D'Ambrumenil, and B. I. Halperin, Phys. Rev. B **34**, 3037 (1986); N. D'Ambrumenil and R. Morf, *ibid.* **40**, 6108 (1989).
[13]D. P. Arovas, J. R. Schrieffer, and F. Wilczek, Phys. Rev. Lett. **53**, 722 (1984).
[14]K. von Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. **45**, 494 (1980).
[15]B. I. Halperin, Phys. Rev. B **25**, 2185 (1982).
[16]R. Willett *et al.*, Phys. Rev. Lett. **59**, 1776 (1987).
[17]F. D. M. Haldane and E. H. Rezayi, Phys. Rev. Lett. **60**, 956 (1988).
[18]B. I. Halperin, Helv. Phys. Acta **56**, 75 (1983).
[19]A. H. MacDonald, D. Yoshioka, and S. M. Girvin, Phys. Rev. B **39**, 8044 (1989).
[20]J. K. Jain, Phys. Rev. Lett. **63**, 199 (1989).
[21]J. K. Jain, Phys. Rev. B **40**, 8079 (1989).
[22]See, A. M. Chang, in *The Quantum Hall Effect* (Ref. 2), pp. 175–232.
[23]R. Willett *et al.*, Surf. Sci. **196**, 257 (1988); A. M. Chang *et al.*, Phys. Rev. Lett. **53**, 997 (1984).
[24]R. G. Clark *et al.*, Surf. Sci. **196**, 219 (1988); **196**, 257 (1988).
[25]V. J. Goldman, M. Shayegan, and D. C. Tsui, Phys. Rev. Lett. **61**, 881 (1988).
[26]V. J. Goldman, D. C. Tsui, and M. Shayegan, in *Proceedings of the Nineteenth International Conference on the Physics of Semiconductors, Warsaw, Poland, August 1988*, edited by W. Zawadzki (Institute of Physics, Polish Academy of Science, Warsaw, 1988).
[27]S. A. Trugman and S. A. Kivelson, Phys. Rev. B **31**, 5280 (1985).
[28]V. L. Pokrovosky and A. L. Talapov, J. Phys. C **18**, L691 (1985).
[29]D. P. Arovas, J. R. Schrieffer, F. Wilczek, and A. Zee, Nucl. Phys. B **251**, 117 (1985); R. B. Laughlin, Phys. Rev. Lett. **60**, 2677 (1988).
[30]S. M. Girvin, Phys. Rev. B **29**, 6012 (1984); A. H. MacDonald, *ibid.* **30**, 3550 (1984).
[31]I am indebted to N. Read (private communication) for pointing out this possibility.
[32]A. H. MacDonald and S. M. Girvin, Phys. Rev. B **33**, 4414 (1986); R. Morf and B. I. Halperin, *ibid.* **33**, 2221 (1986).
[33]J. K. Jain (unpublished).
[34]R. B. Laughlin, in *The Quantum Hall Effect* (Ref. 2), pp. 233–302.
[35]R. B. Laughlin *et al.*, Phys. Rev. B **32**, 1311 (1985).
[36]Since we do not get any states analogous to the states $\chi_{-p}^{2m'}$ of Ref. 20 (where p and m' are integers) only the right half of the figure in Ref. 20 is contained in Fig. 1(a).
[37]R. Tao, J. Phys. C **18**, L1003 (1985); R. Tao and Y. S. Wu, Phys. Rev. B **31**, 6859 (1985).
[38]T. Chakraborty and F. C. Zhang, Phys. Rev. B **29**, 7032 (1984); **30**, 7320 (1984); **34**, 7076 (1986); F. C. Zhang and T. Chakraborty, *ibid.* **30**, 7320 (1984); X. C. Xie, Y. Guo, and F. C. Zhang (unpublished).
[39]R. G. Clark, *et al.*, Phys. Rev. Lett. **62**, 1536 (1989).
[40]J. P. Eisenstein, Phys. Rev. Lett. **62**, 1540 (1989).
[41]F. D. M. Haldane, in *The Quantum Hall Effect* (Ref. 2), pp. 303–352.
[42]J. P. Eisenstein *et al.*, Phys. Rev. Lett. **61**, 997 (1988).
[43]J. K. Jain, S. A. Kivelson, and N. Trivedi, Phys. Rev. Lett. (to be published).
[44]H. W. Jiang *et al.* (unpublished).