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Dynamical simulations of fractional giant Shapiro steps in two-dimensional Josephson arrays

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We have performed computer simulations of arrays of resistively shunted Josephson junctions in the presence of commensurate magnetic fields and alternating and direct currents. We have found *fractional* giant Shapiro steps in the simulated current-voltage characteristics in agreement with recent experimental measurements. The detailed motion of the magnetic-field-induced vortices is shown to be in agreement with a previously proposed phenomenological model of Benz *et al.* [Phys. Rev. Lett. **64**, 693 (1990)].

Experimental measurements[1] have shown that when a radio frequency (rf) current, $i_{\rm rf}\sin(2\pi\nu t)$, is applied to a square array of Josephson junctions in the presence of a commensurate perpendicular magnetic field, *fractional* giant Shapiro steps[2] occur in the dc I-V characteristics at voltages

$$\langle V_n\rangle = n\left(\frac{Nh\nu}{q2e}\right),\quad n=0,1,2,\ldots, \tag{1}$$

where $\langle V_n\rangle$ is the average voltage across the array, n is an integer, N is the number of junctions in the array in the direction of the applied current, and q is determined by the vortex superlattice unit-cell size $q\times q$. The vortex superlattice, in turn, is determined by the number of flux quanta per array unit cell $f=p/q$, where p and q are integers. Such a rational value of f is essential for observing fractional giant Shapiro steps because only for such fields are field-induced vortices arranged in ordered superlattices commensurate with the array of junctions.[3] Since the detailed motion of the vortices in response to the rf current is difficult to determine experimentally, we have performed simulations on two-dimensional arrays of overdamped resistively shunted junctions (RSJ) to gain insight into the physical origin of the fractional giant steps. The simulations show fractional giant steps in agreement with (1), and confirm the phenomenological model proposed by Benz *et al.*[1] (BRTL) to explain the steps.

Our numerical simulations were performed, as were previous simulations,[4] by solving the coupled differential equations found from current conservation at each node of the array. In the RSJ model, the current i_{ij} through an individual junction is given by

$$i_{ij} = i_c\sin\left[\phi_j-\phi_i-\left(\frac{2\pi}{\Phi_0}\right)\int_i^j \mathbf{A}\cdot d\mathbf{l}\right]+\frac{v_{ij}}{R} \tag{2}$$

where Φ_0 is the flux quantum, ϕ_i is the phase on node i, $\mathbf{A}$ is the vector potential, R is the junction resistance, and the voltage drop across the junction is given by the Josephson relation,

$$v_{ij} = \frac{\hbar}{2e}\frac{d\left[\phi_j-\phi_i-(2\pi/\Phi_0)\int_i^j \mathbf{A}\cdot d\mathbf{l}\right]}{dt}. \tag{3}$$

We use the Landau gauge $\mathbf{A}=Hx\mathbf{y}$ so that the integral vanishes for junctions along the x direction and equals $(2\pi H/\Phi_0)x(y_j-y_i)=\pm 2\pi fx/a$ along the y direction, where y_i and y_j are the coordinates of the ith and jth nodes. Current conservation at each node requires

$$\sum_j i_{ij} = i_i^{\rm ext}. \tag{4}$$

In general $i_i^{\rm ext}=0$, except at the boundaries, where $i_i^{\rm ext}=\pm[i_{\rm dc}+i_{\rm rf}\sin(2\pi\nu t)]$ is the current injected (+) or extracted (−), as in the experimental configuration.

The coupled first-order differential equations resulting from (2)-(4) were solved as in Ref. 4 by using a fourth-order Runge-Kutta method with uniform time steps. Starting with initial phases at each node, the equations were integrated forward one time step to generate new phases, which were then used to integrate forward another time step, etc. I-V curves were calculated by ramping the current from zero, where the phases were placed initially in the zero-temperature ground-state configuration,[3] in order to avoid boundary-related metastable states due to finite array size. The phases were allowed to relax over 200 rf periods before time averaging the voltage to obtain the average voltage $\langle V\rangle$.

The choice of boundary conditions is not arbitrary. Simulations must be done on small arrays, since it is not computationally feasible to simulate the 1000×1000 arrays used in our experiments. Periodic boundary conditions were chosen in the direction perpendicular to the current, rather than free boundaries, because free boundaries cause nonuniform vortex motion in small samples. The number of junctions in the periodic direction must be a multiple of q in order to accommodate a $q\times q$ vortex superlattice.

A more critical boundary problem that influences the dynamics of the array is the method of current injection. We originally used busbars in our simulations by tying the junctions on both ends to single nodes, and injecting and extracting the current from these nodes. However, busbars were found to strongly affect the simulated I-V curves because of the nonphysical phase constraints they introduce, leading, for example, to unreasonably small critical currents. Fractional steps were also observed in the simulations with busbars and periodic boundary conditions or free boundary conditions, but we found that uniform current injection and periodic boundary conditions in small arrays more closely simulated the behavior of our large experimental arrays. We note that the *dynamical* behavior on the steps was not significantly affected by boundary conditions, even though the *values* of the dc currents at which steps occur were affected.

With periodic boundary conditions perpendicular to the current and uniform current injection imposed at the ends of the array, the critical current per junction for the $f=\frac{1}{2}$ state was found to be $i_c(f=\frac{1}{2})=0.35i_c(f=0)$. This is in agreement with the critical current found by Mon and Teitel[5] for the same boundary conditions. It should be noted that this critical current is lower than that calculated (for infinite arrays) by other methods,[6] where it was found that $i_c(f=\frac{1}{2})=0.414i_c(f=0)$. This difference is due to the boundary conditions: Uniform current injection gives a pattern of currents that strongly deviates from the periodic ground state near the current injection (or extraction) nodes. By injecting nonuniform current values chosen to match the spatially periodic ground state, we were able to reproduce the $i_c(f=\frac{1}{2})=0.414i_c(f=0)$ value in our simulations.[7] The critical current that is found experimentally for the $f=\frac{1}{2}$ state agrees more closely with $0.414i_c(f=0)$.

Figure 1 shows representative dc current-voltage (I-V) curves computed for various values of external field f with an applied rf current. The first curve shows results for $f=0$ and $i_{rf}=i_c(f=0)$, at a normalized frequency $\Omega=\nu/\nu_c=1$, where $\nu_c=2ei_cR/h$ is the characteristic frequency. i is the applied dc current per junction parallel to the current flow, and $\langle V\rangle$ is the time-averaged voltage across the array. The constant-voltage Shapiro steps occur at the voltages given by (1) with $q=1$. This curve is the same as the response of a single junction to combined dc and rf currents because each junction responds in the same way to the external dc and rf currents.

The second curve in Fig. 1 shows an I-V curve for $f=\frac{1}{2}$, with the same rf current amplitude and frequency as the $f=0$ curve. Fractional giant Shapiro steps now appear at voltages given by (1), with $q=2$. The third curve shows an I-V curve for $f=\frac{1}{3}$. The voltages of the Shapiro steps are also given by (1), with $q=3$ in this case. All of these simulations produce results which are in agreement with experiment.

To gain further insight into the rf response of arrays, and to make a detailed comparison with the theoretical model discussed by BRTL, we have looked at the instantaneous voltage and phase difference across individual junctions in the array as a function of time. A 4×5 junction array was chosen so that one row of junctions would be symmetrically located between the current injection and removal edges of the array. Figure 2 shows the voltages across two adjacent junctions in this row oriented along the current direction on the lowest fractional step of the $f=\frac{1}{2}$ curve shown in Fig. 1. Two main features are worth noting. First, the two junctions have the same voltage wave form, but are out of phase by exactly one rf period. Second, the period of the voltage on *each* junction on this lowest step is twice the period of the external rf current, although the *spatial average* voltage retains the period of the drive.

Figure 3 shows the phases and supercurrents in two adjacent unit cells in the center of the 4×5 junction array (with $f=\frac{1}{2}$) at the times A, B, C indicated in Fig. 2, i.e., before, during, and after the phase slips which dominate the behavior of the array on the lowest-voltage step. [The dc current flows from top to bottom in this figure. The instantaneous voltage of the center junction (labeled 2 in Fig. 3) is plotted by the solid line and the junction to the left (labeled 1) is plotted by a dashed line in Fig. 2.] Figure 3(a) shows the junctions at a point in the drive cycle when the rf and dc currents nearly cancel, and the phase and current configuration resembles the $I=0$ ground state. In Fig. 3(b), the current in the center junction (la-

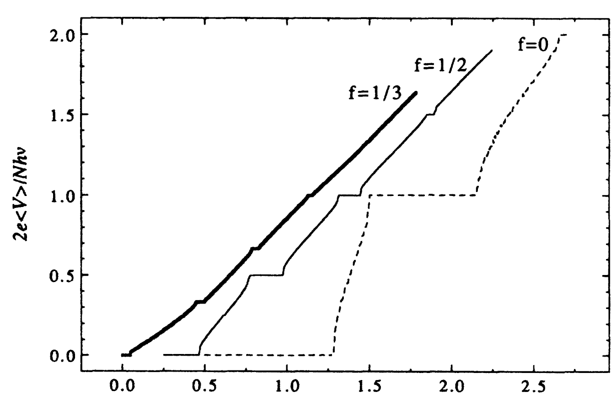

FIG. 1. Normalized time-average voltage vs normalized current per junction for simulated $M\times N$ arrays with $\Omega=1$, $i_{rf}/i_c=1$, and different magnetic fields: $f=0$, 4×5 junctions; $f=\frac{1}{2}$, 4×5 junctions; $f=\frac{1}{3}$, 3×6 junctions. Curves $f=\frac{1}{2}$ and $f=0$ are shifted from the origin along the current axis by successive 0.25 increments.

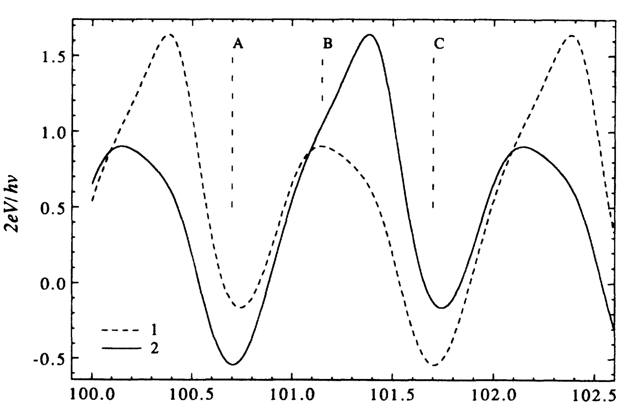

FIG. 2. Normalized instantaneous voltage vs normalized time across two adjacent junctions (labeled 1 and 2 in Fig. 3) parallel to the current for a 4×5 array on the $n=1$, $q=2$ fractional giant step, with $f=\frac{1}{2}$, $\Omega=1$, $i_{rf}/i_c=1$, and $i_{dc}/i_c=0.65$. Lines A, B, and C mark the times associated with the snapshots shown in Fig. 3. Time is normalized to the external rf current period, $1/\nu$.

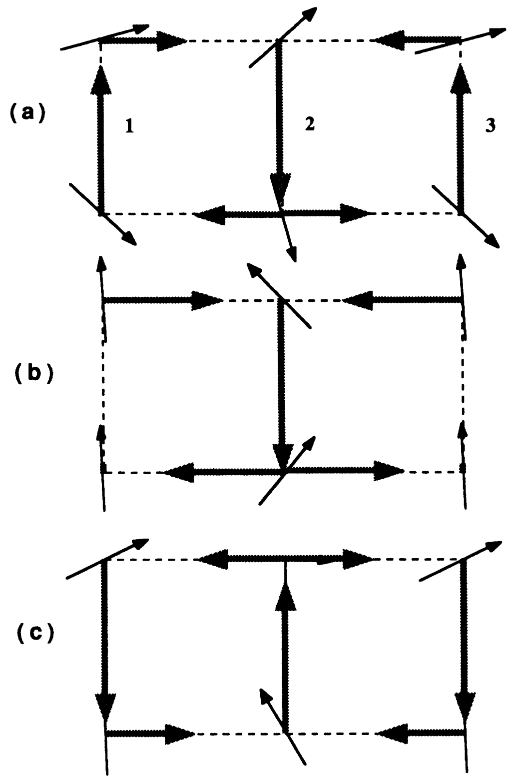

FIG. 3. Supercurrents (thick arrows) and phases (thin arrows) for two unit cells in the center of a 4×5 junction array under the same conditions as in Fig. 2. (a) $\nu t = 100.7$, (b) $\nu t = 101.15$, and (c) $\nu t = 101.7$. Times (a) and (c) are exactly one rf period apart and show all currents having exactly reversed direction.

beled 2 in Fig. 3) has increased toward the zero-field critical current, while the currents of opposite sense in the adjacent junctions (labeled 1 and 3) have decreased nearly to zero. The phases continue to evolve until the currents in the outer junctions go to zero and begin to increase in magnitude in the opposite direction, after which the current in the center junction goes through zero and changes direction. Finally, the currents in the transverse junctions also reverse, so that unit cells which started with clockwise (counterclockwise) currents in Fig. 3(a) end up with counterclockwise (clockwise) currents in Fig. 3(c). In effect, the $f=\frac{1}{2}$ vortex superlattice has moved one unit cell. The phase difference across each junction in the direction of the current has advanced by an average amount of π, which leads, via the Josephson voltage-frequency relation, to (1) for the voltage drop across the whole array, with $n=1$ and $q=2$. More generally, when each junction has an average phase slip of $n\pi$ per rf cycle, the nth step results. These results give detailed confirmation to the model proposed in Ref. 1.

Analogous behavior occurs for $f=\frac{1}{3}$. The period of the instantaneous voltage across a single junction as a function of time on the lowest step for $f=\frac{1}{3}$ is tripled with respect to the rf drive, instead of doubled as for $f=\frac{1}{2}$. The voltage wave forms across adjacent junctions are the same, but again are out of phase by one rf drive period. This shows that the vortex superlattice effectively moves one junction lattice unit cell per rf cycle.

We have also investigated the response of the arrays to lower frequencies, specifically $\Omega=0.1$. Since the characteristic response frequency of the junctions ν_c is ten times faster than the drive frequency in this regime, the vortices slip more quickly (in a time $1/\nu_c$) into adjacent cells. The junctions in the array still collectively lock to the same voltage so that the steps are given by (1).

In conclusion, we have performed RSJ model simulations on arrays of overdamped Josephson junctions and studied their dynamical response to applied rf currents in the presence of a magnetic field. The results show fractional giant Shapiro steps at voltages (1) in agreement with recent experimental observations.[1] By following the response of adjacent junctions in the array, we have shown that on fractional giant steps, the vortex superlattice does indeed slip perpendicular to the applied current in synchrony with the rf drive current as in the model proposed by BRTL.

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