PHYSICAL REVIEW B VOLUME 41, NUMBER 10 1 APRIL 1990

One-dimensional generalized Fibonacci tilings

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(Received 18 July 1989)

Polynomial (nonsingular) dynamical trace maps of generalized Fibonacci tilings $(A,B \rightarrow A^m B^n, A)$ are derived for arbitrary values of m and n. It is shown that these sequences can be grouped into two distinct classes. The sequences in class I correspond to $n=1$ and arbitrary m. They are shown to have volume-preserving and invertible trace maps with an invariant the same as that of the golden-mean sequence. The class-II sequences correspond to $n>1$ and arbitrary m and are shown to be associated with volume-nonpreserving and noninvertible trace maps with a common pseudoinvariant which is of the form of the invariant of class-I maps. Furthermore, it is shown for the class-II case that if $n=m+1$ the trace maps are two dimensional.

I. INTRODUCTION

In recent years, much theoretical effort has been devoted to the study of one-dimensional quasiperiodic (QP) crystals. These studies were stimulated both by experimental indication[1] that there might exist in nature structures intermediate between the periodic and random ones, and by the recently developed experimental techniques to manufacture arbitrarily ordered layered structures.[2] Quasicrystals are nonperiodic systems with well-defined long-range positional order such that their Fourier transforms contain δ peaks (unlike in periodic crystals, these peaks should lie at all possible integer combinations of at least two intervals whose ratio is irrational). All other nonperiodic systems with long-range positional order whose Fourier transforms do not contain δ peaks are usually called aperiodic and are deemed to constitute a link between quasiperiodic and random systems.

Kohmoto *et al.*[3] introduced a dynamical-systems-theory approach in which the investigation of the QP Fibonacci model is reduced to studying a volume-preserving map. This approach has been successful in explaining the energy spectrum and some of the scaling properties of the system. The Fibonacci (golden-mean) lattice is quasiperiodic and has become by far the most studied one-dimensional nonperiodic system with well-defined order. Nevertheless, there have also been several attempts to investigate other lattices in an effort to shed some light on the problem of the properties of all possible intermediate structures between periodic and random ones. Among others, Gumbs and Ali[4,5] have introduced a class of generalized Fibonacci sequences (GFS), derived the dynamical trace maps of several of them, and have studied electronic tight-binding models in which the interaction follows the GFS. It was found that in some of these sequences the dynamical trace maps are volume nonpreserving (dissipative) and noninvertible. In our earlier work,[6] we have studied the spectra of magnetic excitations in generalized Fibonacci superlattices with dissipative maps. The study of the general properties of such dissipative maps themselves[7] has provided a good understanding of the electronic and magnetic spectra. One of the important results of these studies is that the spectra seem to be mixed, containing both critical and extended states. That is similar to the results of Riklund *et al.*[8] for the electronic spectrum of the Thue-Morse lattice. On this basis, the GFS with dissipative maps investigated in Refs. 4–6 have been classified as in between periodic and QP sequences. However, the preliminary study of their Fourier transforms suggest that they are aperiodic, i.e., in between QP and random. This apparent contradiction is one of the reasons why further investigation of these lattices is important.

GFS are defined[4,6] by the following inflation scheme:

$$S_{L+1}=S_L^m S_{L-1}^n \,, \tag{1}$$

where $S_0 \equiv B$, $S_1 \equiv A$, m and n are integers, and S_L^m represents m adjacent repetitions of the string S_L. The inflation scheme of Eq. (1) is equivalent to the substitution rule $A \rightarrow A^m B^n$, $B \rightarrow A$, where A^m represents a string of m A's. The total number of A's and B's in S_L is equal to the generalized Fibonacci number F_L given by the recurrence relation $F_L = mF_{L-1} + nF_{L-2}$, $F_0 = F_1 = 1$. The ratio of A's to B's for $L \rightarrow \infty$ of these lattices is $\tau = \sigma / n$, where[6]

$$\sigma = \lim_{L\rightarrow\infty} F_L / F_{L-1} = \tfrac{1}{2}[m + (m^2+4n)^{1/2}] \,.$$

(The integer value of "incommensurability" $\tau - n$ for $n=m+1$ does not mean that we have periodic lattices in this special case. But they are probably not quasiperiodic either.) Setting $m=n=1$ gives the standard Fibonacci sequence with *the golden mean,* $\sigma = \sigma_g = \frac{1}{2}(1+\sqrt{5})$; $m=2, n=1$ gives the sequence with the *silver mean,* $\sigma_s = 1+\sqrt{2}$; $m=3, n=1$ the sequence with the *bronze mean,* $\sigma_b = \frac{1}{2}(3+\sqrt{13})$; $m=1, n=2$ the sequence with the *copper mean* $\sigma_c = 2$; and $m=1, n=3$ the sequence with the *nickel mean* $\sigma_n = \frac{1}{2}(1+\sqrt{13})$.[4–6] The $n=1$ case is identical with one of the three classes of generalizations of the Fibonacci lattice introduced recently by Holzer[9,10] who calls these "precious means" (PM) lattices. For this special PM case Holzer derived polynomial trace maps that all have the same polynomial invariant as the original Fibonacci ($n=m=1$) case. The main purpose of this paper is to present polynomial maps for all values of n and m, and show that all the cases with $n>1$ belong to the same class of volume nonpreserving noninvertible maps.

The Lth generalized Fibonacci generation S_L can be considered as a unit cell of a periodic lattice that becomes the Lth periodic approximant of the respective infinite generalized Fibonacci sequence. As in the previous studies,[3-7,9,10] we will limit ourselves to the class of physical properties that can be studied in terms of 2×2 transfer matrices of unit determinant. Denoting the whole transfer matrix of S_L as $\mathcal{M}_L$, the matrix equivalent of Eq. (1) is

$$\mathcal{M}_{L+1}=\mathcal{M}_{L-1}^n\mathcal{M}_L^m , \tag{2}$$

where $\mathcal{M}_0$ and $\mathcal{M}_1$ are the transfer matrices of the two basic building blocks B and A, respectively. The allowed energies for frequencies are then given by the condition

$$-2\le x_L\le 2 , \tag{3}$$

where $x_L=\mathrm{Tr}(\mathcal{M}_L)$ is the trace of $\mathcal{M}_L$. It is to be noted that, unlike in Refs. 3–7, 9, and 10, here we have dropped the factor of $\frac{1}{2}$ from the definition of x_L to simplify the formulas involved.

II. SOME PROPERTIES OF THE UNIMODULAR MATRICES

Our task is to express x_{L+1} in terms of the previous three traces x_L, x_{L-1}, and x_{L-2}. It will be shown in the next section that for all m and n, at most, three previous traces are sufficient to determine the next one. Thus, the recursion relation for the traces can always be cast in the form of, at most, three-dimensional maps.

Because of the form of Eq. (2), we will need formulas for the traces of powers and products of unimodular matrices. Let us present the relevant relations for convenience in the derivation of the trace maps and their properties. Let $\underline{a}$ be a 2×2 matrix such that $\det\underline{a}=1$. Then it is straightforward to prove that $\underline{a}^2=x\underline{a}-\underline{I}$, where $x=\mathrm{Tr}\underline{a}$ and $\underline{I}$ is a unit matrix. Applying this formula repeatedly, one gets the relation

$$\underline{a}^k=d_k(x)\underline{a}-d_{k-1}(x)\underline{I} , \tag{4}$$

where $d_k(x)$ is a polynomial in x such that

$$\begin{aligned} &d_{k+1}(x)=xd_k(x)-d_{k-1}(x) ,\\ &d_0(x)\equiv 0 ,\\ &d_1(x)\equiv 1 , \end{aligned} \tag{5}$$

and k is an arbitrary integer (positive or negative). Thus, $d_2(x)=x$, $d_3(x)=x^2-1$, $d_4(x)=x^3-2x$, etc. For positive k, we have

$$d_k(x)=S_{k-1}(x)=U_{k-1}(x/2) ,$$

where $S_j(\xi)$ and $U_j(\xi)$ are Chebyshev polynomials of the first and second kind. Using mathematical induction, one can derive from Eq. (5), the following relations which hold for arbitrary k and l:

$$\begin{aligned} &d_{-k}=-d_k ,\\ &d_kd_{l-1}-d_{k-1}d_l=d_{l-k} ,\\ &d_kd_{l+1}-d_{k-1}d_l=d_{l+k} ,\\ &d_{2k}=d_k(d_{k+1}-d_{k-1}) ,\\ &d_{l+k}-d_{l-k}=d_k(d_{l+1}-d_{l-1}) ,\\ &d_{l+k}+d_{l-k}=d_l(d_{k+1}-d_{k-1}) ,\\ &d_{k+l}d_{k-l}=d_k^2-d_l^2 ,\\ &d_{k+l}^2+d_k^2=d_l^2+d_kd_{k+l}(d_{l+1}-d_{l-1}) ,\\ &d_{k+l}d_{k+1-l}=d_{k+1}d_k-d_ld_{l-1} ,\\ &d_{k+1}d_k-d_ld_{l-1}+d_{k-l+1}d_{k-l}\\ &\qquad -d_kd_{k-l+1}(d_{l+1}-d_{l-1})=0 ,\\ &\frac{d}{dx}(d_{k+1}-d_{k-1})=kd_k . \end{aligned} \tag{6}$$

All the d polynomials in Eq. (6) have identical argument. As a test for the validity of these formulas one can show that they are in agreement with the identity $d_n(2)=n$. Using Eq. (4), one can immediately get the following two relations:

$$\begin{aligned} \mathrm{Tr}(\underline{a}^k)&=xd_k(x)-2d_{k-1}(x)\\ &=d_{k+1}(x)-d_{k-1}(x) \end{aligned} \tag{7}$$

and

$$\mathrm{Tr}(\underline{a}^k\underline{b})=\mathrm{Tr}(\underline{b}\underline{a}^k)=d_k(x)\mathrm{Tr}(\underline{a}\underline{b})-d_{k-1}(x)\mathrm{Tr}\underline{b} , \tag{8}$$

where $x=\mathrm{Tr}\underline{a}$.[11] Note that for positive k, $\mathrm{Tr}(\underline{a}^k)=C_k(x)=2T_k(x/2)$, where $T_j(\xi)$ and $C_j(\xi)$ are another set of Chebyshev polynomials of the first and second kind.

III. THE TRACE MAPS

Let us now proceed with the derivation of the recursion relations for the traces. In this section we will present three types of trace maps of the GFS. We assume that we know $x_{L-2}=\mathrm{Tr}(\mathcal{M}_{L-2})$ and $x_{L-1}=\mathrm{Tr}(\mathcal{M}_{L-1})$. Then from Eq. (2) we have

$$x_L=\mathrm{Tr}(\mathcal{M}_L)=\mathrm{Tr}(\mathcal{M}_{L-2}^n\mathcal{M}_{L-1}^m) . \tag{9}$$

Using Eq. (8) with $\underline{a}=\mathcal{M}_{L-1}$, $\underline{b}=\mathcal{M}_{L-2}^n$, and $k=m$, we get

$$x_L=d_m(x_{L-1})\mathrm{Tr}(\mathcal{M}_{L-1}\mathcal{M}_{L-2}^n)-d_{m-1}(x_{L-1})\mathrm{Tr}(\mathcal{M}_{L-2}^n) .$$

Using again Eqs. (8) and (7) for the calculation of the two traces occurring on the right-hand side of the last equation, we get

$$\begin{aligned} x_L=d_m(x_{L-1})[&d_n(x_{L-2})\mathrm{Tr}(\mathcal{M}_{L-1}\mathcal{M}_{L-2})\\ &-d_{n-1}(x_{L-2})x_{L-1}]\\ &-d_{m-1}(x_{L-1})[d_{n+1}(x_{L-2})-d_{n-1}(x_{L-2})] . \end{aligned} \tag{10}$$

Substituting in Eq. (10) $L+1$ for L, one gets

$$x_{L+1}=d_m(x_L)[d_n(x_{L-1})\mathrm{Tr}(\mathcal{M}_L\mathcal{M}_{L-1})-d_{n-1}(x_{L-1})x_L]-d_{m-1}(x_L)[d_{n+1}(x_{L-1})-d_{n-1}(x_{L-1})] . \quad (11)$$

Using Eq. (2), one can write

$$\mathrm{Tr}(\mathcal{M}_L\mathcal{M}_{L-1})=\mathrm{Tr}(\mathcal{M}_{L-1}^{m+1}\mathcal{M}_{L-2}^{n}) .$$

Comparing this result with Eq. (9), one can see that it is sufficient to substitute $m+1$ for m in Eq. (10) to get the following expression for $\mathrm{Tr}(\mathcal{M}_L\mathcal{M}_{L-1})$:

$$\mathrm{Tr}(\mathcal{M}_L\mathcal{M}_{L-1})=d_{m+1}(x_{L-1})[d_n(x_{L-2})\mathrm{Tr}(\mathcal{M}_{L-1}\mathcal{M}_{L-2})-d_{n-1}(x_{L-2})x_{L-1}]-d_m(x_{L-1})[d_{n+1}(x_{L-2})-d_{n-1}(x_{L-2})] . \quad (12)$$

Finally, from Eq. (10) it is easy to express $\mathrm{Tr}(\mathcal{M}_{L-1}\mathcal{M}_{L-2})$ in terms of x_{L-2}, x_{L-1}, and x_L, which is used in Eq. (12). The result is then substituted into Eq. (11), and Eqs. (5) and (6) are used to simplify the final expression to obtain

$$x_{L+1}=\frac{d_m(x_L)d_n(x_{L-1})}{d_m(x_{L-1})}[x_Ld_{m+1}(x_{L-1})-d_{n+1}(x_{L-2})+d_{n-1}(x_{L-2})]-d_{m+1}(x_L)d_{n-1}(x_{L-1})-d_{m-1}(x_L)d_{n+1}(x_{L-1}) . \quad (13)$$

Equation (13) is our *first trace map* of the GFS. It is clear from Eq. (13) that x_{L+1} has been expressed as a function of x_L, x_{L-1}, and x_{L-2} only. Defining

$$\underline{r}\equiv(x,y,z)\equiv(x_{L-2},x_{L-1},x_L)$$

and

$$\underline{r}'\equiv(x',y',z')\equiv(x_{L-1},x_L,x_{L+1}) ,$$

we have a Kohmoto-type three-dimensional map in the space of the three consecutive traces:

$$\begin{aligned} x'&=y ,\\ y'&=z ,\\ z'&=\frac{d_m(z)d_n(y)}{d_m(y)}[zd_{m+1}(y)-d_{n+1}(x)+d_{n-1}(x)]-d_{m+1}(z)d_{n-1}(y)-d_{m-1}(z)d_{n+1}(y) . \end{aligned} \quad (14)$$

For $n=1$, Eq. (14) reduces correctly to Eq. (6) of Ref. 10. Equations (14) or (13) give a polynomial map only for $m=n$, or $m=1$ and n arbitrary. Thus, for $m=n=2$ we get the "mixed-case" map as given by Eq. (33) of Ref. 5 (there is a misprint in Ref. 5, the first x_{l-1} in Eq. (33) should be replaced by x_l). For $m=1$ and $n=2$, Eq. (14) gives the three-dimensional version of the copper-mean map as given by Eq. (2) of Ref. 7.

For general m and n, the map given by Eq. (14) can be singular for some initial conditions. This situation can never happen for the initial conditions (those corresponding to the traces of the first three transfer matrices) we are interested in here because the original six-dimensional matrix map of Eq. (2) is regular. Thus, it should be possible to find regular, polynomial maps for all m and n. Equation (10) suggests how to achieve that. As it gives x_L as a polynomial in x_{L-2}, x_{L-1}, and $\mathrm{Tr}(\mathcal{M}_{L-1}\mathcal{M}_{L-2})$, let us define a new "coordinate system" in the following way:

$$\begin{aligned} \tilde{x}&=x\equiv x_{L-2} ,\\ \tilde{y}&=y\equiv x_{L-1} ,\\ \tilde{z}&=\mathrm{Tr}(\mathcal{M}_{L-1}\mathcal{M}_{L-2}) . \end{aligned} \quad (15)$$

Then Eq. (12) immediately gives us the recursion relation for the new z coordinate, $\tilde{z}$, and Eq. (10) gives the relation between the old and new z coordinates that can be substituted into the second formula of Eq. (14) to get the recursion relation for the new y coordinate. Using Eq. (6), our *second trace map* can be written in the form

$$\begin{aligned} \tilde{x}'&=\tilde{y} ,\\ \tilde{y}'&=d_m(\tilde{y})[d_n(\tilde{x})\tilde{z}-d_{n-1}(\tilde{x})\tilde{y}]-d_{m-1}(\tilde{y})[d_{n+1}(\tilde{x})-d_{n-1}(\tilde{x})] ,\\ \tilde{z}'&=d_{m+1}(\tilde{y})[d_n(\tilde{x})\tilde{z}-d_{n-1}(\tilde{x})\tilde{y}]-d_m(\tilde{y})[d_{n+1}(\tilde{x})-d_{n-1}(\tilde{x})] . \end{aligned} \quad (16)$$

The complete relations between the new and old coordinates are

$$\begin{aligned} \tilde{x}&=x ,\\ \tilde{y}&=y ,\\ \tilde{z}&=\frac{1}{d_n(x)}\left\{\frac{z+d_{m-1}(y)[d_{n+1}(x)-d_{n-1}(x)]}{d_m(y)}+d_{n-1}(x)y\right\} . \end{aligned} \quad (17)$$

Again, for $n=1$, Eq. (16) reduces to Eq. (16) of Ref. 10. For $m=2$ and $n=1$, Eq. (16) gives the silver-mean map as given by Eq. 11 of Ref. 5, and for $m=3$ and $n=1$, the bronze-mean map of Eq. (18) of the same reference provided that the roles of x and y are exchanged which is a trivial transformation of coordinates.

Neither Eq. (14) nor Eq. (16) gives the nickel-mean map as investigated in Refs. 5–7. This can be obtained after transformation to yet other coordinates in which the obtained map has an especially simple form for all m and n. This transformation reads as

$$\bar{x}=d_{n+1}(\tilde{x})-d_{n-1}(\tilde{x}) ,$$
$$\bar{y}=\tilde{y} , \tag{18}$$
$$\bar{z}=d_{n+1}(\tilde{x})\tilde{y}-d_n(\tilde{x})\tilde{z} .$$

Note that unlike Eq. (17), Eq. (18) does not represent a one-to-one transformation of coordinates because of the way $\tilde{x}$ is transformed into $\bar{x}$. It is obvious from Eq. (15) that $\tilde{x}$ has still the meaning of the trace of the transfer matrix $\mathcal{M}_{L-2}$. Then according to Eq. (7), $\bar{x}$ is the trace of the nth power of $\mathcal{M}_{L-2}$. The condition (3) for the allowed bands remains valid in all three coordinate systems discussed above because if the modulus of trace of an unimodular matrix is bounded by two, the same is true for the trace of its arbitrary power. Using Eq. (6), we obtain our *third trace map* in the form

$$\bar{x}'=d_{n+1}(\bar{y})-d_{n-1}(\bar{y}) ,$$
$$\bar{y}'=d_{m+1}(\bar{y})\bar{x}-d_m(\bar{y})\bar{z} , \tag{19}$$
$$\bar{z}'=d_{n-m}(\bar{y})\bar{z}-d_{n-m-1}(\bar{y})\bar{x} .$$

This third map gives the nickel-mean map as given by Eq. (28) of Ref. 5 provided that another trivial coordinate transformation is applied: $\bar{x}$ and $\bar{y}$ coordinates exchanged and $\bar{z}$ replaced by $-\bar{z}$. We note that, unlike the previously discussed maps, all the d polynomials occurring in Eq. (19) have the same argument, $\bar{y}$. The beauty of this map consists in its simplicity and certain symmetry of the three equations.

IV. DISCUSSION

It is immediately obvious that for all lattices with an integer ratio τ (those with $n=m+1$) the trace map of Eq. (19) reduces to a two-dimensional map as the last line of Eq. (19) reduces to $\bar{z}'=\bar{z}$. Thus, for $n=m+1$, $\bar{z}$ becomes an invariant. Using Eqs. (18) and (17), one can immediately express it in terms of the traces x_0, x_1, and x_2 of the first three transfer matrices:

$$\bar{z}=[d_{n+1}(x_0)-d_{n-1}(x_0)]\left[x_1-\frac{d_{n-2}(x_1)}{d_{n-1}(x_1)}\right]-\frac{x_2}{d_{n-1}(x_1)} .$$

For the copper-mean case ($n=2$), this gives an expression identical to that for -2γ [cf. Eq. (3) of Ref. 7]. Thus, $\bar{z}$ directly plays the role of the γ parameter of the two-dimensional copper-mean map as discussed in Refs. 4–7.

A very important feature of the maps of Eqs. (16) and (19) is the manner of transformation of the expression

$$\tilde{I}=\tilde{x}^2+\tilde{y}^2+\tilde{z}^2-\tilde{x}\tilde{y}\tilde{z}-4$$

and

$$\bar{I}=\bar{x}^2+\bar{y}^2+\bar{z}^2-\bar{x}\bar{y}\bar{z}-4 ,$$

respectively (four times Kohmoto's invariant for the golden-mean case[3]). Using Eq. (6), one can show that

$$\tilde{I}'=d_n(\tilde{x})^2\tilde{I} , \tag{20}$$

and

$$\bar{I}'=d_n(\bar{y})^2\bar{I} . \tag{21}$$

This implies that all GFS are divided into two classes. Class I corresponds to $n=1$ and arbitrary m. As $d_n(x)=1$, both $\tilde{I}$ and $\bar{I}$ are true invariants as in the case of the golden mean. The same result was obtained previously by Holzer.[10] Both maps of Eqs. (16) and (19) are volume preserving and invertible for this case. Class II consists of all other cases ($n>1$, m arbitrary). Both $\tilde{I}$ and $\bar{I}$ behave as pseudoinvariants[6,7] (their sign is conserved in the sense that nonnegative I remains nonnegative, and nonpositive I remains nonpositive). If $d_n(\tilde{x})=0$ [$d_n(\bar{y})=0$], $\tilde{I}=0$ ($\bar{I}=0$) in all future interactions. Thus, the surface $\tilde{I}=0$ ($\bar{I}=0$) plays the role of an *attractor* as discussed in Ref. 7 for the copper- and nickel-mean cases. Eqs. (20) and (21) indicate that all cases with $n>1$ behave essentially in the same way as the copper- and nickel-mean cases. This is further supported by the fact that for $n>1$ both maps of Eqs. (16) and (19) are volume nonpreserving and noninvertible. Namely, the Jacobians of the maps of Eqs. (16) and (19) are $-nd_n(\tilde{x})^2$ and $-nd_n(\bar{y})^2$, respectively. The only difference from Ref. 7 is that the Lissajous curves (copper-mean map attractors) of Ref. 7 would become simpler ellipses (sections through the Kohmoto's surface $\bar{I}=0$) in the coordinate system (18). It could very well be that the above division ($n=1$: map is volume preserving, invertible, has invariant; $n>1$: volume nonpreserving, noninvertible, has only pseudoinvariant) will be preserved under an arbitrary coordinate transformation that conserves the polynomial form of the map. However, a proof remains to be found. The two pseudoinvariants of Eqs. (20) and (21) are related in the following way:

$$\bar{I}=d_n(\tilde{x})^2\tilde{I} .$$

Thus, $\bar{I}$ is identical to the next iteration of $\tilde{I}$. There is no contradiction in the two seemingly different recursion formulas of Eqs. (20) and (21), remember that $\bar{y}=\tilde{x}'$. For comparison, the Jacobian of the coordinate transformation of Eq. (18) is equal to $-nd_n(\tilde{x})^2$.

The above division into two classes is somewhat surprising in the light of the probable division of the GFS into quasiperiodic and aperiodic sequences. The number-theoretic results of Bombieri and Taylor[12] indicate that the Fourier spectrum (structure factor) of a GFS is only composed of δ peaks if

$$1+m>n . \tag{22}$$

Thus, only those GFS satisfying Eq. (22) can be quasiperiodic. All others are aperiodic, i.e., the lattices which do not satisfy Eq. (22) are somewhere between QP and random. Equations (20) and (21) show that the trace maps behave in the same way for all lattices with the same n, irrespective of the fact whether there are quasiperiodic ($m > n-1$) or aperiodic ($m \leq n-1$) according to Eq. (22). Originally we had hoped that our division into classes I and II would correspond to the division into quasiperiodic and aperiodic lattices. However, there are exceptions to the validity of the Bombieri-Taylor criterion and further work is required to clarify this question.

ACKNOWLEDGMENTS

This research was supported by a grant to M.K.A. from the Natural Sciences and Engineering Research Council of Canada.

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[11]Using the properties of the Chebyshev polynomials, it is easy to find for $k>0$ the explicit formula

$$\mathrm{Tr}(\underline{a}^k)=k\sum_{j=0}^{[k/2]}\frac{(-1)^j}{k-j}\binom{k-j}{j}x^{k-2j}\,.$$

This must be identical with Eq. (A6) of Ref. 9. Comparing then this formula with Eqs. (A6) and (A5) of Ref. 9, one gets the following summation rule:

$$\delta_{l0}=k\sum_{j=0}^{l}\frac{(-1)^j}{k-j}\binom{k-j}{j}\binom{k-2j}{l-j}\,,\qquad k>0,\ l=0,\ldots,\left[\frac{k}{2}\right].$$

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