Properties of the chiral-spin-liquid state

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It is shown that one of the class of variational spin-liquid wave functions recently derived by Wen, Wilczek, and Zee [Phys. Rev. B **39**, 11 413 (1989)] is identical to the fractional-quantum-Hall analog state proposed by Kalmeyer and Laughlin [Phys. Rev. Lett. **59**, 2095 (1987); Phys. Rev. B **39**, 11 879 (1989)] and Laughlin [Ann. Phys. (N.Y.) **191**, 163 (1989)], and that the neutral spin-$\frac{1}{2}$ excitations of the two states are also the same. The $\frac{1}{2}$ fractional statistics obeyed by these particles is demonstrated explicitly. The spin-spin correlation function and quasiparticle profile for the Wen-Wilczek-Zee states are calculated both numerically and by a hypernetted-chain procedure. Both are consistent with the idea that spontaneous breaking of time-reversal symmetry is an essential feature of any spin state lacking magnetic order. A version of this state for a three-dimensional spin system is reported and shown to have similar properties. A case is made that the neutral spin-$\frac{1}{2}$ excitations of the three-dimensional spin liquid behave like anisotropic monopoles.

I. INTRODUCTION

In a recent paper, Wen, Wilczek, and Zee[1] showed that the Affleck-Marston[2] variational procedure, when applied to a two-dimensional frustrated Heisenberg antiferromagnet, leads to a class of spin-liquid states lacking the parity and time-reversal invariance of the underlying Hamiltonian. While the accuracy and appropriateness of the procedure for the problem they studied is currently unclear, it is significant that two very different lines of reasoning, the Baskeran-Zou-Anderson[3] analogy with the Fermi sea and the Kalmeyer-Laughlin[4,5] analogy with the fractional-quantum-Hall state, have now led to ground states for this problem with similar properties. One of the purposes of this paper is to show that this similarity is more than cosmetic. One of the class of states found by Wen, Wilczek, and Zee is, in fact, equal to the Kalmeyer-Laughlin state. In addition, for this particular state, the Anderson[6] procedure of making a "spinon," the neutral spin-$\frac{1}{2}$ excitation of the spin liquid, namely, making a hole in the band and then Gutzwiller projecting,[7] is also equal to the Kalmeyer-Laughlin version of the spinon. We also wish to promote the point of view that the time-reversal symmetry-breaking characteristic of the class of states is actually essential for liquifaction of the spins. The main purpose, however, is to address one of the outstanding questions in the fractional quantum-Hall problem: What is the appropriate abstraction of fractional quantum-Hall behavior to a three-dimensional system? We find that the Wen-Wilczek-Zee calculation can be performed simply for a three-dimensional frustrated Heisenberg model, and that the solution is identical in most respects to the two-dimensional version. The ground state has broken parity and time-reversal invariance and exponentially decaying spin correlations. The elementary excitations are well-localized neutral spin-$\frac{1}{2}$ particles with a gap. However, the most peculiar feature of spinons in two dimensions, the $\frac{1}{2}$ fractional statistics obeyed by them, cannot be the same because fractional statistics is undefined in three dimensions. We shall make the case that the behavior we observe, and the appropriate analog of $\frac{1}{2}$ fractional statistics for a three-dimensional system, is the dynamics of an anisotropic Dirac monopole.

II. VARIATIONAL PROCEDURE

Let us begin by reviewing the Affleck-Marston[2] variational method for constructing the ground state and elementary excitations of the Heisenberg Hamiltonian

$$\mathcal{H}=\sum_{\langle j,k\rangle} J_{jk}\mathbf{S}_j\cdot\mathbf{S}_k \ , \tag{2.1}$$

where j and k denote sites on a square two-dimensional lattice, J_{jk} is the strength of the spin exchange coupling between these sites, and $\mathbf{S}=\frac{1}{2}\sigma$. As illustrated in Fig. 1, we are interested in the case of positive, i.e., frustrating, near-neighbor and second-neighbor interactions J and J'. This Hamiltonian is related to the electron Hamiltonian

$$\mathcal{H}_0=\tfrac{1}{2}\sum_{\langle j,k\rangle} J_{jk}\left[\sum_{s,s'} c_{js}^{\dagger}c_{ks'}^{\dagger}c_{ks}c_{js'}-\tfrac{1}{2}\right] , \tag{2.2}$$

where the c's are site annihilation operators and $s=(\uparrow,\downarrow)$ is a spin index, by

$$\mathcal{H}=\Pi_G\mathcal{H}_0\Pi_G \ , \tag{2.3}$$

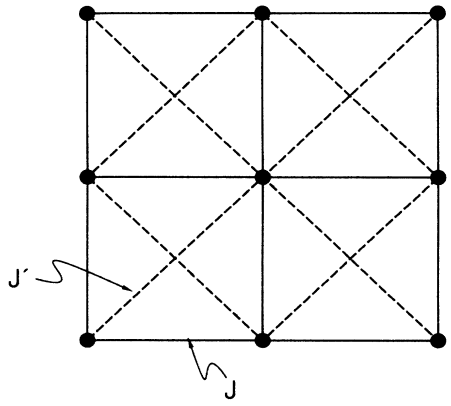

FIG. 1. Illustration of the frustrated Heisenberg Hamiltonian assumed by Wen, Wilczek, and Zee. The first- and second-neighbor couplings are J and J', respectively.

where Π_G is the Gutzwiller projection operator,[7] defined by

$$\Pi_G=\prod_j \left\{ \int_0^{2\pi} \exp\left[i\phi_j \left[\sum_\eta c_{j\eta}^\dagger c_{j\eta} - 1 \right] \right] d\phi_j \right\} . \quad (2.4)$$

The action of $\mathcal{H}_0$ on an allowed configuration, one for which each site is singly occupied, is the same as the action of $\mathcal{H}$. The Gutzwiller projector destroys all configurations except the allowed ones.

The Affleck-Marston procedure amounts to looking for a variational ground state for the system of the form

$$|\Psi\rangle=\Pi_G|\Psi_{\rm SD}\rangle , \quad (2.5)$$

where $\Psi_{\rm SD}$ is a single Slater-determinant electron wave function. Rather than satisfying the variational condition

$$\delta\left[\frac{\langle\Psi_{\rm SD}|\Pi_G\mathcal{H}_0\Pi_G|\Psi_{\rm SD}\rangle}{\langle\Psi_{\rm SD}|\Pi_G|\Psi_{\rm SD}\rangle} \right]=0 , \quad (2.6)$$

however, which is computationally difficult, one requires that

$$\delta\left[\frac{\langle\Psi_{\rm SD}|\mathcal{H}_0|\Psi_{\rm SD}\rangle}{\langle\Psi_{\rm SD}|\Psi_{\rm SD}\rangle} \right]=0 , \quad (2.7)$$

arguing that the two conditions are similar and that $|\Psi\rangle$ was only variational in the first place. The solution of Eq. (2.7) is a Slater determinant of orbitals $\varphi_l(j)$ satisfying the Hartree-Fock equations

$$\sum_k \langle j|\mathcal{H}_{\rm HF}|k\rangle\varphi_l(k)=E_l\varphi_l(j) , \quad (2.8)$$

where E_l is a Lagrange multiplier and

$$\langle j|\mathcal{H}_{\rm HF}|k\rangle=-J_{jk}\sum_s\langle c_{js}^\dagger c_{ks}\rangle = -2J_{jk}\sum_l \varphi_l^*(j)\varphi_l(k) . \quad (2.9)$$

These equations have many solutions, for if a set of orbitals $\{\varphi_l(j)\}$ comprises one solution then the set of orbitals $\{\varphi_l'(j)\}$ defined by

$$\varphi_l'(j)=e^{i\phi_j^{(0)}}\varphi_l(j) , \quad (2.10)$$

where $\phi_j^{(0)}$ is any function of position, also satisfies it and is manifestly different from the original set, i.e.,

$$|\langle\Psi_{\rm SD}'|\Psi_{\rm SD}\rangle|\neq 1 . \quad (2.11)$$

However, the state made by Gutzwiller projecting $|\Psi_{\rm SD}'\rangle$ is the same as that made by projecting $|\Psi_{\rm SD}\rangle$, since

$$\Pi_G|\Psi_{\rm SD}'\rangle=\prod_j \left\{ \exp(i\phi_j^{(0)}) \int_0^{2\pi} \exp\left[i(\phi_j+\phi_j^{(0)}) \left[\sum_\eta c_{j\eta}^\dagger c_{j\eta} - 1 \right] \right] d\phi_j \right\} |\Psi_{\rm SD}\rangle = \exp\left[i\sum_j \phi_j^{(0)} \right] \Pi_G|\Psi_{\rm SD}\rangle . \quad (2.12)$$

Thus, the transformation $\varphi_l(j)\rightarrow\varphi_l'(j)$ has no physical meaning, and we refer to it as a gauge transformation. Its relationship to an ordinary gauge transformation is actually central to this problem,[8] but will not be discussed here.

III. GROUND-STATE WAVE FUNCTIONS

Following Wen, Wilczek, and Zee,[1] we look for a solution of Eq. (2.9) of the form

$$\langle j|\mathcal{H}_{\rm HF}|k\rangle = i^{(m'+m)(l'-l)} \times \begin{cases} T, & j \text{ and } k \text{ near neighbors} \\ T', & j \text{ and } k \text{ second neighbors} \\ 0, & \text{otherwise} , \end{cases} \quad (3.1)$$

where the ordered pair (l,m) locates the kth site, in the manner

$$\mathbf{R}_k=(l\hat{\mathbf{x}}+m\hat{\mathbf{y}})b , \quad (3.2)$$

with b the bond length, and where T and T' are self-consistently determined parameters. The complex factor in this expression, which was originally discovered to be appropriate by Affleck and Marston,[2] mimics the effect of a uniform magnetic field threading half a magnetic flux quantum through each plaquette. We have specifically

$$i^{(m'+m)(l'-l)}=\exp\left[i\int_k^j \mathbf{A}\cdot d\mathbf{s} \right] , \quad (3.3)$$

where

$$\mathbf{A}=\frac{\pi}{b^2}y\hat{\mathbf{x}} . \quad (3.4)$$

The configuration of phases given by Eq. (3.1) is shown in Fig. 2.

Let us now demonstrate the self-consistency, and thus the extremal nature, of this solution. We first need to solve Eq. (2.8). Picking a unit cell as shown in Fig. 2 and writing

$$\varphi_{\mathbf{q}}(j)=e^{i\mathbf{q}\cdot\mathbf{R}_j}u_{\mathbf{q}}(j)\ , \tag{3.5}$$

where $\mathbf{q}$ is the crystal momentum and $u_{\mathbf{q}}$ is periodic, we have

$$\mathcal{H}_{\mathbf{q}}\begin{bmatrix}u_{\mathbf{q}}(1)\\ u_{\mathbf{q}}(2)\end{bmatrix}=E_{\mathbf{q}}\begin{bmatrix}u_{\mathbf{q}}(1)\\ u_{\mathbf{q}}(2)\end{bmatrix}\ , \tag{3.6}$$

where

$$\mathcal{H}_{\mathbf{q}}=2T[\cos(q_x b)\alpha_x+\cos(q_y b)\alpha_y +m_0\sin(q_x b)\sin(q_y b)\beta]\ , \tag{3.7}$$

with

$$\alpha_x=\begin{bmatrix}1&0\\0&-1\end{bmatrix},\quad \alpha_y=\begin{bmatrix}0&1\\1&0\end{bmatrix},\quad \beta=\begin{bmatrix}0&i\\-i&0\end{bmatrix}\ , \tag{3.8}$$

and

$$m_0=2\frac{T'}{T}\ . \tag{3.9}$$

Since these matrices satisfy the anticommutation relations

$$\{\alpha_\mu,\alpha_\nu\}=2\delta_{\mu\nu},\quad \{\alpha_\mu,\beta\}=0,\quad \beta^2=1\ , \tag{3.10}$$

Eq. (3.6) is a two-dimensional Dirac equation, for which we may write immediately

$$E_{\mathbf{q}}=\pm 2T[\cos^2(q_x b)+\cos^2(q_y b) +m_0^2\sin^2(q_x b)\sin^2(q_y b)]^{1/2}\ . \tag{3.11}$$

Note that the states at $\mathbf{q}$ and $\mathbf{q}+(\pi/b)\hat{\mathbf{x}}$ are distinct, even though they have the same energy. The Brillouin zone and dispersion relation are shown in Fig. 3. Writing the Green's function in the manner

$$\mathcal{G}_{\mathbf{q}}(E)=(E-\mathcal{H}_{\mathbf{q}}+i\eta_E)^{-1}=\frac{E+\mathcal{H}_{\mathbf{q}}}{E^2-E_{\mathbf{q}}^2+i\eta}\ , \tag{3.12}$$

with the sign of the infinitesimal picked to occupy the lower band, we obtain for the self-consistency conditions, with $|1\rangle$ and $|2\rangle$ defined as in Fig. 2,

$$T/J=\chi=\frac{4}{N}\sum_{\mathbf{q}}\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty}\langle 2|\mathcal{G}(E)|1\rangle e^{i\eta E}dE\right]e^{-iq_y b} =\frac{1}{2\pi^2}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\frac{A}{(A+m_0^2B)^{1/2}}d\theta_1 d\theta_2\ , \tag{3.13}$$

and

$$T'/J'=\chi'=\frac{4}{N}\sum_{\mathbf{q}}\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty}\langle 2|\mathcal{G}(E)|1\rangle e^{i\eta E}dE\right]\times e^{-i(q_x+q_y)b} =\frac{1}{\pi^2}m_0\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\frac{B}{(A+m_0^2B)^{1/2}}\times d\theta_1 d\theta_2\ , \tag{3.14}$$

where

$$A=\cos^2(\theta_1)+\cos^2(\theta_2),\quad B=\sin^2(\theta_1)\sin^2(\theta_2)\ . \tag{3.15}$$

Combining these equations, we obtain a condition for the mass of the form

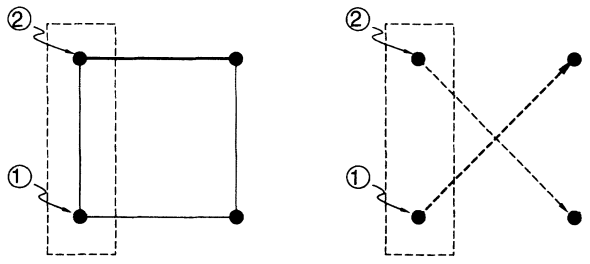

FIG. 2. Illustration of matrix elements of the Hartree-Fock Hamiltonian defined by Eq. (3.1) for the two-dimensional chiral spin liquid. Left: Near-neighbor interactions. All of these are positive except for the bold line, which is negative. Right: second-neighbor interactions. These are purely imaginary. The arrow points to $|j\rangle$ when $\mathrm{Im}(\langle j|\mathcal{H}_{\mathrm{HF}}|k\rangle)$ is positive.

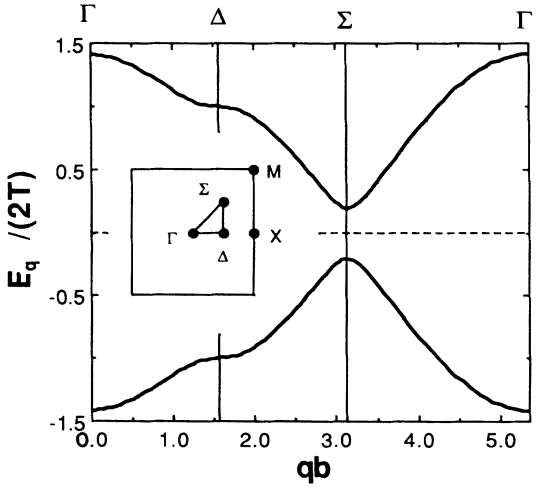

FIG. 3. Illustration of Brillouin zone for two-dimensional chiral spin liquid state and Lagrange multiplier spectrum $E_{\mathbf{q}}$, as defined by Eq. (3.11), for the case of $m_0=0.2$. Either band may be interpreted as the energy to make a spinon with a given crystal momentum.

$$\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\frac{A-4(J'/J)B}{(A+m_0^2B)^{1/2}}d\theta_1 d\theta_2=0\ , \tag{3.16}$$

which has a nontrivial solution for $J'/J>0.4454$. Self-consistency is actually demonstrated by satisfying Eq. (3.16) since the eigenstates of $\mathcal{H}_{\rm HF}$ do not depend on the magnitude of T. The numerical aspects of this solution are further discussed in Sec. XII.

IV. GUTZWILLER PROJECTION

In order for a wave function of this form to be useful we must be able to do computations with its Gutzwiller projection.[7] Let us begin the discussion of this by showing that the projected state is, up to a sign, the square of a spinless electron Slater determinant.

When the number of sites N is even, the unprojected state is a function of N electron positions j and spin coordinates η of the form

$$\Psi_{\rm SD}(j_1,\ldots,j_N|\eta_1,\ldots,\eta_N)=\frac{1}{\sqrt{N!}}\sum_p \mathrm{sgn}(p)[\varphi_1(j_{p(1)})\delta(\uparrow,\eta_{p(1)})][\varphi_1(j_{p(2)})\delta(\downarrow,\eta_{p(2)})]$$
$$\times\cdots[\varphi_{N/2}(j_{p(N-1)})\delta(\uparrow,\eta_{p(N-1)})][\varphi_{N/2}(j_{p(N)})\delta(\downarrow,\eta_{p(N)})]\ , \tag{4.1}$$

where the φ's are the negative-energy solutions of Eq. (2.8), δ denotes a spin δ function, and p is a permutation of N things. Since this wave function is antisymmetric under simultaneous interchange of position and spin variables, it suffices to know its value when the first $N/2$ spin variables are up and the second $N/2$ down, given by

$$\Psi_{\rm SD}(j_1,\ldots,j_N|\uparrow,\ldots,\uparrow,\downarrow,\ldots,\downarrow)=\begin{pmatrix}N\\N/2\end{pmatrix}^{-1/2}\Phi(j_1,\ldots,j_{N/2})\Phi(j_{N/2+1},\ldots,j_N)\ , \tag{4.2}$$

where

$$\Phi(j_1,\ldots,j_{N/2})=\frac{1}{\sqrt{(N/2)!}}\sum_{p'}\mathrm{sgn}(p')\varphi_1(j_{p'(1)})\cdots\varphi_{N/2}(j_{p'(n/2)})\ , \tag{4.3}$$

and p' is a permutation of $N/2$ things. With the wave functions expressed in this way, Gutzwiller projection may be understood as forcing the set $\{j_1,\ldots,j_{N/2}\}$ to be distinct from the set $\{j_{N/2+1},\ldots,j_N\}$.

The two factors on the right-hand side of Eq. (4.2) are equal, up to a sign, because the positive- and negative-energy solutions of Eq. (2.8) are related [cf. Eq. (3.5)] in the manner

$$\varphi^{(+)}_{(\pi/b)\hat{\mathbf{x}}-\mathbf{q}}(j)e^{i(\pi/b)\hat{\mathbf{y}}\cdot\mathbf{R}_j}=[\varphi^{(-)}_{\mathbf{q}}(j)]^*\ . \tag{4.4}$$

Since occupying both the upper and lower bands with spinless fermions is tantamount to occupying all the sites, we have

$$\langle 0|(c_{j_{N/2+1}}\cdots c_{j_N})(c_{j_1}\cdots c_{j_{N/2}})\left[\prod_{\mathbf{q}}c^{(+)}_{\mathbf{q}}\right]^\dagger\left[\prod_{\mathbf{q}}c^{(-)}_{\mathbf{q}}\right]^\dagger|0\rangle=(-1)^M \tag{4.5}$$

for some integer M. This phase is commonly referred to as the Marshall sign.[9] The first factor in Eq. (4.2) may be written

$$\Phi(j_1,\ldots,j_{N/2})=\left\langle 0\left|c_{j_1}\cdots c_{j_{N/2}}\left[\prod_{\mathbf{q}}c^{(-)}_{\mathbf{q}}\right]^\dagger\right|0\right\rangle\ . \tag{4.6}$$

The second factor may be written

$$\begin{aligned}\Phi(j_{N/2+1},\ldots,j_N)&=\left\langle 0\left|c_{j_{N/2+1}}\cdots c_{j_N}\left[\prod_{\mathbf{q}}c^{(-)}_{\mathbf{q}}\right]^\dagger\right|0\right\rangle\\&=\left\langle 0\left|\left[\prod_{\mathbf{q}}c^{(-)}_{\mathbf{q}}\right](c_{j_{N/2+1}}\cdots c_{j_N})^\dagger\right|0\right\rangle^*\\&=(-1)^M\left\langle 0\left|\left[\prod_{\mathbf{q}}c^{(-)}_{\mathbf{q}}\right](c_{j_1}\cdots c_{j_{N/2}})\left[\prod_{\mathbf{q}}c^{(+)}_{\mathbf{q}}\right]^\dagger\left[\prod_{\mathbf{q}}c^{(-)}_{\mathbf{q}}\right]^\dagger\right|0\right\rangle^*\\&=(-1)^M\left\langle 0\left|c_{j_1}\cdots c_{j_{N/2}}\left[\prod_{\mathbf{q}}c^{(+)}_{\mathbf{q}}\right]^\dagger\right|0\right\rangle^*\\&=(-1)^M\exp\left[i\frac{\pi}{b}\hat{\mathbf{y}}\cdot(\mathbf{R}_{j_1}+\cdots+\mathbf{R}_{j_{N/2}})\right]\Phi(j_1,\ldots,j_{N/2})\ ,\end{aligned} \tag{4.7}$$

which proves the assertion.

Let us now pick an ordering for the sites, as illustrated in Fig. 4, and define $G_0(j)$ to be $+1$ if the site is even and -1 if the site is odd. With an ordering thus established we may define the *Bose* wave function $\Psi_B(j_1,\ldots,j_{N/2})$ to be the Fermi wave function of Eq. (4.2) with the arguments $\{j_1,\ldots,j_{N/2}\}$ and $\{j_{N/2+1},\ldots,j_N\}$ placed separately in ascending order. In light of Eq. (4.7), $\Psi_B(j_1,\ldots,j_{N/2})$ must equal $\Phi^2(j_1,\ldots,j_{N/2})$ up to a sign. This sign need not be determined absolutely, but only up to the change induced by modifying the arguments $\{j_1,\ldots,j_{N/2}\}$. By definition there is no change in sign when the arguments are permuted. It suffices, therefore, to consider the sign change resulting from moving one of the boson coordinates, say j_1, to a location different from $\{j_1,\ldots,j_{N/2}\}$. This new location must be one of the $\{j_{N/2+1},\ldots,j_N\}$. Let us assume for the sake of discussion that it is $j_{N/2+1}$. Then the wave function with the boson moved is

$$\Psi_B(j_1,\ldots,j_{N/2})=\pm\,\Phi(j_{N/2+1},j_2,\ldots,j_{N/2})\times\Phi(j_1,j_{N/2+2},\ldots,j_N)\,, \tag{4.8}$$

where the sign counts the number of elementary exchanges required to place $\{j_{N/2+1},j_2,\ldots,j_{N/2}\}$ and $\{j_1,j_{N/2+2},\ldots,j_N\}$ separately in ascending order. As illustrated in Fig. 4, however, this is simply the number of sites "between" j_1 and $j_{N/2+1}$, which is counted by the factor $G_0(j_1)G_0(j_{N/2})$. Thus, taking into account the fact that j_1 and $j_{N/2+1}$ might be in different rows by defining

$$G_L(j)=\exp\left[i\frac{\pi}{b}\hat{\mathbf{y}}\cdot\mathbf{R}_j\right]G_0(j)=(-1)^{l+m}\,, \tag{4.9}$$

with l and m defined as in Eq. (3.2), we have finally

$$\Psi_B(j_1,\ldots,j_{N/2})=G_L(j_1)\cdots G_L(j_{N/2})\times\Phi^2(j_1,\ldots,j_{N/2})\,. \tag{4.10}$$

$G_L(j)$ is the Landau gauge version of the "gauge" factor used by Kalmeyer and Laughlin.[4,5]

Let us now discuss the condition that Ψ_B is a spin singlet. This must be the case because Gutzwiller projection commutes with total spin. It is obviously an eigenstate of S^z with eigenvalue 0 since the numbers of up and down spins are equal. The singlet condition thus reduces to showing that the lowering operator S^- destroys the state. The action of S^- is to flip each spin down, annihilating any configuration for which the spin is already down. For the original Fermi wave function, this gives

$$(S^-\Psi_{\rm SD})(j_1,\ldots,j_N|\eta_1,\ldots,\eta_N)=\sum_{i=1}^{N}\delta(\downarrow,\eta_i)\Psi_{\rm SD}(j_1,\ldots,j_N|\eta_1,\ldots,\bar{\eta}_i,\ldots,\eta_N)\,, \tag{4.11}$$

where $\bar{\eta}$ denotes the opposite of η. Because of the antisymmetry of $\Psi_{\rm SD}$ under interchange of spin and position coordinates, Eq. (4.11) may be expressed as the "Fock" condition[10] on Φ

$$\Phi(j_1,\ldots,j_{N/2})\Phi(j_{N/2+1},\ldots,j_N)=\Phi(j_{N/2+1},j_2,\ldots,j_{N/2})\Phi(j_1,j_{N/2+2},\ldots,j_N)+\Phi(j_{N/2+2},j_2,\ldots,j_{N/2})\times\Phi(j_{N/2+1},j_1,j_{N/2+3},\ldots,j_N)+\cdots\,, \tag{4.12}$$

which is satisfied whenever Φ is a Slater determinant. However, since each term on the right-hand side of this expression is of the form

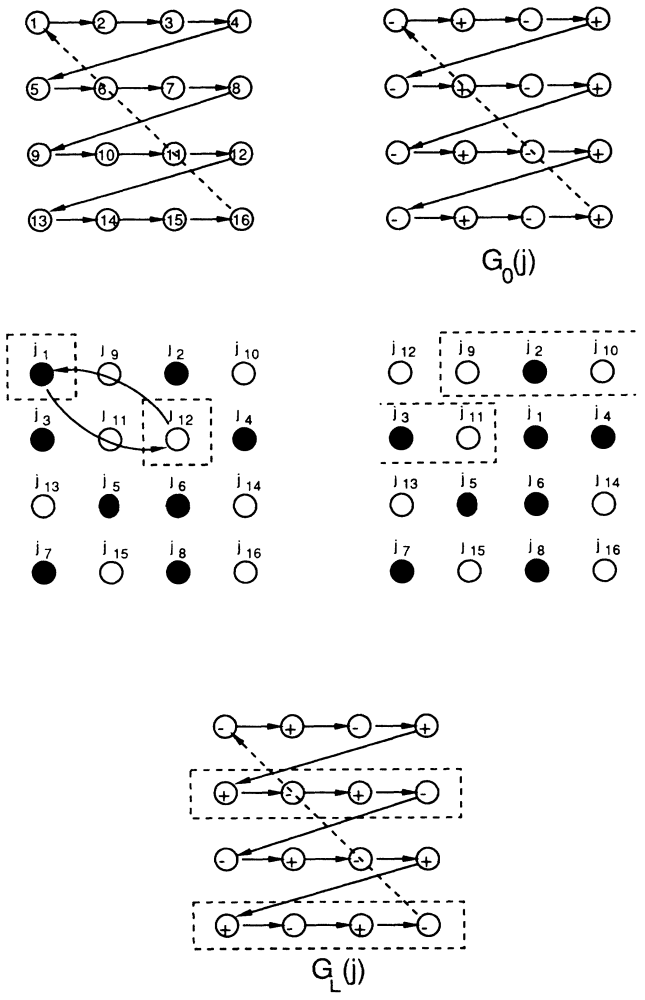

FIG. 4. Top: Site ordering assumed in discussion of Bose version of chiral-spin-liquid wave function. $G_0(j)$ is ±1 according as whether the site is even or odd. Middle: Moving the boson at j_1 to an unoccupied site is tantamount to interchanging the "up" coordinate j_1 with the "down" coordinate j_M. The number of elementary exchanges required to put the up and down coordinates back into ascending order is the number of sites "between" j_1 and j_M, which is counted by $G_0(j_1)G_0(j_M)$. Bottom: $G_L(j)$ is made from $G_0(j)$ by negating every other row.

$$-G_L(j_1)G_L(j)\Phi^2(j,j_2,\ldots,j_{N/2}) ,$$

we have

$$\sum_j G_L(j)\Phi^2(j,j_2,\ldots,j_N)=0 , \tag{4.13}$$

which is the singlet sum rule found by Kalmeyer and Laughlin,[4,5] and Zou, Doucot, and Shastry.[11]

V. PLASMA ANALOGY

In light of these considerations the problem of calculating with the projected state reduces to finding an efficient way to generate Φ^2. Let us do this by appealing to the theory of liquids,[12] which tells us that many properties of a system of this kind may be calculated by approximating its probability distribution function as the product of pair factors. This is appropriate because the function is vanishingly small in most of configuration space, so that Taylor expansion near the extrema, which for most liquids amount to the crystalline configurations, converges rapidly. Thus, let us make a fit to $|\Phi|^2$ of the form

$$|\Phi(\mathbf{r}_1,\ldots,\mathbf{r}_{N/2})|^2 \cong \prod_{\mu<\nu} e^{-v(\mathbf{r}_\mu-\mathbf{r}_\nu)} \prod_\tau e^{-u(\mathbf{r}_\tau)} , \tag{5.1}$$

where $\mathbf{r}_1$ denotes the position of j_1, and so forth. Again borrowing from the theory of liquids, we use as fitting criteria that the density, defined by

$$\rho(\mathbf{r}_1)=\frac{N}{2}\frac{1}{Z}\sum_{\mathbf{r}_2}\cdots\sum_{\mathbf{r}_{N/2}}|\Phi(\mathbf{r}_1,\ldots,\mathbf{r}_{N/2})|^2 , \tag{5.2}$$

where

$$Z=\sum_{\mathbf{r}_1}\cdots\sum_{\mathbf{r}_{N/2}}|\Phi(\mathbf{r}_1,\ldots,\mathbf{r}_{N/2})|^2 , \tag{5.3}$$

and two-point correlation function, given by

$$g(\mathbf{r}_1-\mathbf{r}_2)=\frac{N(N-1)}{Z}\sum_{\mathbf{r}_3}\cdots\sum_{\mathbf{r}_{N/2}}|\Phi(\mathbf{r}_1,\ldots,\mathbf{r}_{N/2})|^2 , \tag{5.4}$$

be correct. Since Φ is a Slater determinant, the quantities to be fit may be evaluated easily: The density is $\frac{1}{2}$ and the two-point function is the exchange hole of Φ, or

$$g(\mathbf{r})=1-4\left|\sum_{\mathbf{q}}\varphi_{\mathbf{q}}^*(0)\varphi_{\mathbf{q}}(\mathbf{r})\right|^2 , \tag{5.5}$$

where $\varphi_{\mathbf{q}}$ is defined as in Eq. (3.5) and normalized in the manner

$$\sum_{\mathbf{r}}|\varphi_{\mathbf{q}}(\mathbf{r})|^2=1 . \tag{5.6}$$

The fitting of v to g, while straightforward, is complicated by the fact that the exchange hole contains exactly one particle, i.e.,

$$\tfrac{1}{2}\sum_{\mathbf{r}}[g(\mathbf{r})-1]=-1 , \tag{5.7}$$

while sum rule defining the "coupling constant" Γ, that is

$$\tfrac{1}{2}\sum_{\mathbf{r}}|\mathbf{r}|^2[g(\mathbf{r})-1]=-\frac{2b^2}{\pi\Gamma} , \tag{5.8}$$

is not zero. This requires that v be of the form

$$v(\mathbf{r})=-2\Gamma\ln|\mathbf{r}|+\delta v(\mathbf{r}) , \tag{5.9}$$

where $\delta v(\mathbf{r})$ is short ranged. This is most easily understood in the context of the hypernetted chain approximation[4,12] for $g(\mathbf{r})$, defined by the equations

$$g(\mathbf{r})=\exp[-v(\mathbf{r})+h(\mathbf{r})-c(\mathbf{r})] , \tag{5.10}$$

$$h(\mathbf{r})=g(\mathbf{r})-1 , \tag{5.11}$$

$$\hat{h}(\mathbf{q})=\hat{c}(\mathbf{q})+\tfrac{1}{2}\hat{h}(\mathbf{q})\hat{c}(\mathbf{q}) , \tag{5.12}$$

with Fourier transforms defined in the manner

$$\hat{h}(\mathbf{q})=\sum_{\mathbf{r}}h(\mathbf{r})e^{-i\mathbf{q}\cdot\mathbf{r}} , \tag{5.13}$$

which is appropriate for this problem. The short-range nature of $h(\mathbf{r})$, together with Eq. (5.10), requires that c and v cancel at large r:

$$c(\mathbf{r})\cong -v(\mathbf{r}), \quad |\mathbf{r}|\to\infty . \tag{5.14}$$

This, in turn, requires that $\hat{c}$ and $\hat{v}$ cancel at small q:

$$\hat{c}(\mathbf{q})\cong -\hat{v}(\mathbf{q}), \quad |\mathbf{q}|\to 0 . \tag{5.15}$$

Thus, if v takes the form

$$\hat{v}(\mathbf{q})=\lim_{q_0\to 0}\frac{4\pi\Gamma}{b^2(|\mathbf{q}|^2+q_0^2)}+\hat{v}_s(\mathbf{q}) , \tag{5.16}$$

where $\hat{v}_s(\mathbf{q})$ does not diverge at small $\mathbf{q}$, then we have

$$\hat{h}(\mathbf{q})=-2+\frac{(|\mathbf{q}|b)^2}{\pi\Gamma}+O(|\mathbf{q}|^4) , \tag{5.17}$$

regardless of v_s, which is consistent with Eqs. (5.7) and (5.8). If the singularity at $\mathbf{q}\to 0$ is weaker than $|\mathbf{q}|^{-2}$ (it must be even) then Eq. (5.7) is not satisfied. If the divergence is stronger than $|\mathbf{q}|^{-2}$, on the other hand, then Eq. (5.9) is not satisfied. Thus, this form of v is the only possibility. In the limit of $q_0\to 0$, $u(\mathbf{r})$ must be of the form

$$u(\mathbf{r})=\Gamma\frac{\pi}{4}\frac{|\mathbf{r}|^2}{b^2} , \tag{5.18}$$

to neutralize the plasma.

The short-range correction may be evaluated in the hypernetted chain approximation by solving the equations

$$v_s(\mathbf{r})=-\ln[h(\mathbf{r})+1]-c_s(\mathbf{r})+h(\mathbf{r})-2\Gamma K_0(Q|\mathbf{r}|) , \tag{5.19}$$

and

$$\hat{c}_s(\mathbf{q})=\frac{\hat{h}(\mathbf{q})}{1+\frac{1}{2}\hat{h}(\mathbf{q})}+4\pi\Gamma\sum_{\mathbf{G}}\frac{Q^2}{|\mathbf{q}+\mathbf{G}|^2(|\mathbf{q}+\mathbf{G}|^2+Q^2)} , \tag{5.20}$$

where K_0 is a modified Bessel function of the second kind, $\mathbf{G}$ is a reciprocal lattice vector, and Q is a regulari-

TABLE I. Chiral-spin-liquid correlation for $m_0=1.0$ as evaluated using Eqs. (5.23)–(5.26) and by direct Monte Carlo integration of Eq. (5.21). The quantity $\langle \mathbf{S}\cdot\mathbf{S}\rangle$ is $\frac{3}{4}$ of $h_G(\mathbf{r})$. $h(\mathbf{r})$ is the correlation function of the unprojected state. Monte Carlo evaluation involved 7400 sweeps through an 8×8 lattice. The coupling constant of the unprojected state is $\Gamma=0.99$.

Shell	l	m	$v_s(\mathbf{r})$	$h(\mathbf{r})$	$h_G(\mathbf{r})$	Monte Carlo
1	1	0	0.026	−0.189	−0.269	−0.272±0.002
2	1	1	0.016	−0.059	−0.028	−0.023±0.001
3	2	0	−0.004	0.000	0.037	0.039±0.001
4	2	1	0.002	−0.001	0.009	0.007±0.002
5	2	2	0.003	0.000	−0.001	0.002±0.002
6	3	0	−0.002	−0.001	0.009	0.007±0.001
7	3	1	−0.001	0.000	−0.002	−0.001±0.001
8	3	2	0.001	0.000	0.000	0.000±0.002
9	4	0	−0.001	0.000	0.001	0.003±0.002

TABLE II. Chiral-spin-liquid correlation function for $m_0=0.5$ as evaluated using Eqs. (5.23)–(5.26) and by direct Monte Carlo integration of Eq. (5.4). The quantity $\langle \mathbf{S}\cdot\mathbf{S}\rangle$ is $\frac{3}{4}$ of $h_G(\mathbf{r})$. $h(\mathbf{r})$ is the correlation function of the unprojected state. Monte Carlo evaluation involved 5800 sweeps through an 8×8 lattice. The coupling constant of the unprojected state is $\Gamma=0.95$.

Shell	l	m	$v_s(\mathbf{r})$	$h(\mathbf{r})$	$h_G(\mathbf{r})$	Monte Carlo
1	1	0	0.191	−0.213	−0.320	−0.336±0.001
2	1	1	−0.003	−0.028	0.047	0.061±0.001
3	2	0	0.031	0.000	0.048	0.061±0.001
4	2	1	−0.004	−0.002	−0.010	−0.015±0.001
5	2	2	−0.015	0.000	0.003	0.006±0.001
6	3	0	0.018	−0.003	−0.011	−0.010±0.001
7	3	1	0.004	0.001	0.002	−0.001±0.001
8	3	2	−0.008	0.000	−0.001	−0.003±0.001
9	4	0	0.010	0.000	0.002	0.004±0.002

TABLE III. Chiral-spin-liquid correlation for $m_0=0.1$ as evaluated using Eqs. (5.23)–(5.26) and by direct Monte Carlo integration of Eq. (5.4). The quantity $\langle \mathbf{S}\cdot\mathbf{S}\rangle$ is $\frac{3}{4}$ of $h_G(\mathbf{r})$. The fifth column is the correlation function of the unprojected state. Monte Carlo evaluation involved 2000 sweeps through an 8×8 lattice. The coupling constant of the unprojected state is $\Gamma=0.69$.

Shell	l	m	$v_s(\mathbf{r})$	$h(\mathbf{r})$	$h_G(\mathbf{r})$	Monte Carlo
1	1	0	0.567	−0.228	−0.357	−0.380±0.001
2	1	1	0.124	−0.002	0.103	0.123±0.002
3	2	0	0.136	−0.000	0.062	0.082±0.001
4	2	1	0.038	−0.003	−0.030	−0.034±0.001
5	2	2	−0.005	0.000	0.012	0.008±0.001
6	3	0	0.060	−0.006	−0.021	−0.017±0.002
7	3	1	0.019	0.000	0.010	−0.005±0.001
8	3	2	−0.005	−0.001	−0.006	0.007±0.001
9	4	0	0.022	0.000	0.006	0.011±0.001

zation parameter. With v and u fit in this manner, the two-point function of the projected state, namely,

$$g_G(\mathbf{r}_1-\mathbf{r}_2)=\frac{N(N-1)}{\mathcal{Z}_G}\sum_{\mathbf{r}_3}\cdots\sum_{\mathbf{r}_{N/2}}|\Phi(\mathbf{r}_1,\ldots,\mathbf{r}_{N/2})|^4, \tag{5.21}$$

where

$$\mathcal{Z}_G=\sum_{\mathbf{r}_1}\cdots\sum_{\mathbf{r}_{N/2}}|\Phi(\mathbf{r}_1,\ldots,\mathbf{r}_{N/2})|^4 , \tag{5.22}$$

is given approximately by the solution of Eqs. (5.10)–(5.12) with v doubled. Written in their regulated[4] form, these become

$$g_G(\mathbf{r})=\exp[-2v_s(\mathbf{r})+h_G(\mathbf{r})-c_s^G(\mathbf{r})-4\Gamma K_0(Q|\mathbf{r}|)] . \tag{5.23}$$

$$h_G(\mathbf{r})=g_G(\mathbf{r})-1 , \tag{5.24}$$

$$\hat{h}_G(\mathbf{q})=\hat{c}_G(\mathbf{q})+\tfrac{1}{2}\hat{h}_G(\mathbf{q})\hat{c}_G(\mathbf{q}) , \tag{5.25}$$

$$\hat{c}_G(\mathbf{q})=c_s^G(\mathbf{q})-8\pi\Gamma\sum_{\mathbf{G}}\frac{Q^2}{|\mathbf{q}+\mathbf{G}|^2(|\mathbf{q}+\mathbf{G}|^2+Q^2)} . \tag{5.26}$$

The details of the solution of these equations have been discussed previously.[4]

Tables I–III compare a calculation of the spin-spin correlation function, given by

$$\langle\mathbf{S}(0)\cdot\mathbf{S}(j)\rangle=\tfrac{3}{4}h_G(j) , \tag{5.27}$$

using Eqs. (5.23)–(5.26) with a direct semiclassical Monte Carlo integration of Eq. (5.21) on an 8×8 lattice using a numerical Slater determinant routine. The agreement is extremely good for the first two values of m_0 and acceptably good for the third. There is a clear trend in the tables for the analytic calculation of the near-neighbor correlation to be 1–10% too positive. This trend was also found by Kalmeyer and Laughlin,[4] however, and thus may be attributed to the hypernetted chain approximation, rather than to Eq. (5.1). For the smaller values of m_0, there is also a trend for the Monte Carlo calculation to give small antiferromagnetic correlations of the wrong sign at large $\mathbf{r}$. Given that this is an artifact of the periodic boundary conditions, which is likely, these results show that Eq. (5.1) provides an efficient and quantitatively correct procedure for calculating properties of the Gutzwiller projected wave function.

VI. CHIRAL SYMMETRY BREAKING

One of the most significant features of the Wen-Wilczek-Zee state is its spontaneous breaking of time-reversal and parity invariance. From the discussion of Sec. III it is clear that the extremal condition is met whether m_0 is positive or negative, and thus that the system has two ground states, transformed into one another by complex conjugation or reflection about the x or y axis. (This degeneracy is different from the two-fold "center-of-mass" degeneracy associated with the quantum-Hall wave function on a torus.[5]) The ability of handedness to be acquired spontaneously is particularly important in light of the assertion by one of us[13] that the occurrence of $\frac{1}{2}$ fractional statistics, which requires handedness to make sense,[14] is a necessity whenever the ground state of the spins is disordered. This reasoning also says[13] that such a state should be characterized by an energy gap, that collapse of the gap as the Hamiltonian is modified should reflect the divergence of the spin-spin correlation length, and that both of these should signal the transition to the antiferromagnetically ordered state. Let us now show that all of these ideas are consistent with properties of the chiral spin-liquid state.

We first observe that no "magnetic" Hartree-Fock Hamiltonian of the form

$$\langle j|\mathcal{H}_{\rm HF}|k\rangle=\langle j|T|k\rangle\exp\left[i\int_j^k\mathbf{A}\cdot d\mathbf{s}\right] , \tag{6.1}$$

where $\langle j|T|k\rangle$ is a real-valued function of the separation $\mathbf{R}_j-\mathbf{R}_k$ and $\mathbf{A}$ is given by Eq. (3.4), can have a gap in its Lagrange multiplier spectrum if it is time-reversal invariant. In light of Eq. (3.3), $\mathcal{H}_{\rm HF}$ is obviously time-reversal invariant if $\langle j|T|k\rangle$ is zero whenever the x and y components of $\mathbf{R}_j-\mathbf{R}_k$ are both odd. This sufficient condition is also necessary whenever any of the other matrix elements are nonzero, as it then becomes the condition that the flux enclosed by any loop of tunneling matrix elements is integral. Invoking the Bloch condition for this problem as we did in Eqs. (3.5) and (3.6), we obtain a reduced Hamiltonian of the form

$$\mathcal{H}_{\mathbf{q}}=\begin{bmatrix}\alpha(\mathbf{q}) & \gamma(\mathbf{q})\\ \gamma^*(\mathbf{q}) & \beta(\mathbf{q})\end{bmatrix} , \tag{6.2}$$

with

$$\alpha(\mathbf{q})=\sum_{l,\nu}\langle 0,0|\mathcal{H}_{\rm HF}|l,2\nu\rangle\exp[i(lq_xb+2\nu q_yb)] , \tag{6.3}$$

$$\beta(\mathbf{q})=\sum_{l,\nu}\langle 0,1|\mathcal{H}_{\rm HF}|l,2\nu+1\rangle\exp[i(lq_xb+2\nu q_yb)] , \tag{6.4}$$

$$\gamma(\mathbf{q})=\sum_{l,\nu}\langle 0,0|\mathcal{H}_{\rm HF}|l,2\nu+1\rangle \times\exp\{i[lq_xb+(2\nu+1)q_yb]\} , \tag{6.5}$$

where $|l,m\rangle$ is defined as in Eq. (3.2). Since

$$\langle 0,1|\mathcal{H}_{\rm HF}|l,2\nu+1\rangle=(-1)^l\langle 0,0|\mathcal{H}_{\rm HF}|l,2\nu\rangle , \tag{6.6}$$

$\alpha(\mathbf{q})$ and $\beta(\mathbf{q})$ must be equal when $\pm q_x=\pm q_y=\frac{1}{2}\pi/b$. Letting

$$q_x=\frac{\pi}{2b}+\Delta q_x,\quad q_y=\frac{\pi}{2b}+\Delta q_y , \tag{6.7}$$

we have to lowest order in $\Delta\mathbf{q}$,

$$\frac{\alpha(\mathbf{q})+\beta(\mathbf{q})}{2}\cong\sum_{\mu,\nu}(-1)^{\mu+\nu}\langle 0,0|\mathcal{H}_{\rm HF}|2\mu,2\nu\rangle , \tag{6.8}$$

and

$$\frac{\alpha(\mathbf{q})-\beta(\mathbf{q})}{2}\cong\sum_{\mu,\nu}(-1)^{\mu+\nu}(2\mu+1)\times\langle 0,0|\mathcal{H}_{\rm HF}|2\mu+1,2\nu\rangle\Delta q_x b\ . \tag{6.9}$$

Thus, unless the coefficients

$$\langle 0,0|\mathcal{H}_{\rm HF}|2\mu+1,2\nu\rangle$$

conspire so as to make the right-hand side of Eq. (6.9) zero, or unless the matrix $\langle j|T|k\rangle$ is so long-ranged that the sum fails to converge, the deviation from equality is linear in Δq_x. Reasoning in a similar way for $\gamma(\mathbf{q})$, we obtain

$$\gamma(\mathbf{q})\cong\sum_{\mu,\nu}(-1)^{\mu+\nu}[\langle 0,0|\mathcal{H}_{\rm HF}|2\mu+1,2\nu+1\rangle+(2\nu+1)\langle 0,0|\mathcal{H}_{\rm HF}|2\mu,2\nu+1\rangle\Delta q_y b]\ . \tag{6.10}$$

The constant term in this expression is pure imaginary and thus identically zero whenever $\mathcal{H}_{\rm HF}$ is time-reversal invariant. The second term is equivalent to Eq. (6.8). Thus, the eigenvalue spectrum near the zone corner is characterized by an offset

$$E_0=\sum_{\mu,\nu}(-1)^{\mu+\nu}\langle 0,0|\mathcal{H}_{\rm HF}|2\mu,2\nu\rangle\ , \tag{6.11}$$

a velocity

$$v=\left|\sum_{\mu,\nu}(-1)^{\mu+\nu}(2\mu+1)\langle 0,0|\mathcal{H}_{\rm HF}|2\mu+1,2\nu\rangle\right|b\ , \tag{6.12}$$

and a gap

$$\Delta=\left|\sum_{\mu,\nu}(-1)^{\mu+\nu}\langle 0,0|\mathcal{H}_{\rm HF}|2\mu+1,2\nu+1\rangle\right|\ , \tag{6.13}$$

in the manner

$$E_{\mathbf{q}}\cong E_0\pm(v^2|\Delta\mathbf{q}b|^2+\Delta^2)^{1/2}\ . \tag{6.14}$$

It follows that any reasonable Hartree-Fock Hamiltonian respecting time-reversal invariance must produce light-like dispersion at the point $(\frac{1}{2},\frac{1}{2})\pi/b$ and cannot have a gap.

We next observe that the spin-spin correlation length of the Gutzwiller projected state diverges as the mass gap goes to zero. This follows in part from the discussion of the preceding section, in which it was established that the spin-spin correlation length of the projected state is *longer* than that of the unprojected state, since projection effectively halves the temperature of the classical statistical mechanics problem to which the square of the wave function corresponds. It suffices, therefore, to show that the correlation length of the unprojected state diverges, which is straightforward. Returning to Eq. (5.5), we obtain

$$\sum_{\mathbf{q}}\varphi_{\mathbf{q}}^*(0)\varphi_{\mathbf{q}}(\mathbf{r}_j)=\frac{1}{8\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{C}{(A+m_0^2B)^{1/2}}\times e^{i(l\theta_1+m\theta_2)}d\theta_1 d\theta_2\ , \tag{6.15}$$

where A and B are defined as in Eq. (3.15), l and m are defined as in Eq. (3.2), and

$$C=(A+m_0^2B)^{1/2}-[\cos(\theta_1)+\cos(\theta_2)+im_0\sin(\theta_1)\sin(\theta_2)]\ . \tag{6.16}$$

In the $m_0\to 0$ limit, the integrand of Eq. (6.15) develops a weak singularity which causes $h(\mathbf{r})$ to acquire a long-range tail. Let us evaluate this by regularizing the integral, in the manner

$$\frac{C}{(A+m_0^2B)^{1/2}}=\left[\frac{C}{(A+m_0^2B)^{1/2}}-D\right]+D\ , \tag{6.17}$$

where

$$D=\sum_{\mu,\nu}\{(-1)^{\mu}[\theta_1-(\mu+\tfrac{1}{2})\pi]+(-1)^{\nu}[\theta_2-(\nu+\tfrac{1}{2})\pi]+(-1)^{\mu+\nu}im_0\}\times\frac{\exp[-\alpha(\{[\theta_1-(\mu+\frac{1}{2})\pi]^2+[\theta_2-(\nu+\frac{1}{2})\pi]^2+m_0^2\}^{1/2}-m_0)]}{\{[\theta_1-(\mu+\frac{1}{2})\pi]^2+[\theta_2-(\nu+\frac{1}{2})\pi]^2+m_0^2\}^{1/2}}\ . \tag{6.18}$$

The quantity in square brackets in Eq. (6.17) contains no singularity and thus generates only short-range correlations. Evaluating the Fourier transform of D analytically, we obtain

$$\frac{1}{8\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}D\exp[i(l\theta_1+m\theta_2)]d\theta_1 d\theta_2=\frac{i^{l+m+1}}{2\pi}\frac{\exp\{-m_0[(l^2+m^2+\alpha^2)^{1/2}-\alpha]\}}{(l^2+m^2+\alpha^2)^{1/2}}\times\begin{cases}-m_0,\ l\text{ and }m\text{ both odd}\\ 0,\ l\text{ and }m\text{ both even}\\ F,\ \text{otherwise}\ ,\end{cases} \tag{6.19}$$

where

TABLE IV. Monte Carlo evaluation of $h_G(\mathbf{r})$ for $m_0=0$ compared with values reported by Gros (Ref. 16). The present calculation involved 2000 sweeps through an 8×8 lattice. The calculation of Gros was performed on a lattice with 122 sites.

Shell	l	m	Monte Carlo	Gros
1	1	0	-0.382 ± 0.001	-0.419 ± 0.014
2	1	1	0.133 ± 0.001	0.186 ± 0.021
3	2	0	0.089 ± 0.001	0.133 ± 0.022
4	2	1	-0.049 ± 0.001	-0.101 ± 0.023
5	2	2	0.020 ± 0.001	0.065 ± 0.024
6	3	0	-0.018 ± 0.001	-0.068 ± 0.028

$$F=\left[\frac{m_0}{(l^2+m^2+\alpha^2)^{1/2}}+\frac{1}{l^2+m^2+\alpha^2}\right]\times\begin{cases}l,\ l\text{ even and }m\text{ odd}\\ m,\ m\text{ even and }l\text{ odd}\end{cases}. \tag{6.20}$$

Thus, in the $m_0\to0$ limit, antiferromagnetic correlations fall off as $|\mathbf{r}|^{-4}$ to a radius of m_0^{-1} and exponentially thereafter, so that the spin-spin correlation length ξ, defined in the manner

$$\xi^2=\left[\sum_{\mathbf{r}}h(\mathbf{r})|\mathbf{r}|^2\right]\Big/\left[\sum_{\mathbf{r}}h(\mathbf{r})\right]\cong-\ln(2m_0),\quad m_0\to0\ , \tag{6.21}$$

diverges.

The fact that the unprojected wave function at $m_0=0$ has power-law decay in its correlation function suggests that the classical liquid to which it corresponds lies at a critical point.[15] The exponent 4 is much larger than the Halperin-Nelson[15] value of 0.3, but this could be ascribed to the presence of long-range "potentials." If this is true, then the projected state, which corresponds to the same liquid with its temperature lowered by a factor of 2, probably possess magnetic order. The assertion of Gros[15] that the spin correlations of the $m_0\to0$ projected state decay as $r^{-4/3}$ implies algebraic order, as opposed to true long-range order, but this distinction is unimportant, since the spinon size is still undefined. It should be noted that the validity of Gros's exponent, which is an unusual number,[15] is not yet clear. The presence of order is also consistent with the tendency of the numerical results to be sensitive to lattice size. This is illustrated in Table IV, where we compare a Monte Carlo calculation on a lattice with 122 sites reported by Gros[16] and by Zhang *et al.*[17] Note that the "d-wave" state with $t=\Delta$ discussed in these papers is identical[17] to the $m_0=0$ chiral-spin-liquid state.[2]

VII. RELATION TO THE KALMEYER-LAUGHLIN STATE

Let us now show that one of the class of ground states proposed by Wen, Wilczek, and Zee[1] exactly equals the Kalmeyer-Laughlin[4,5] state, defined by

$$\Psi_{\mathrm{KL}}(z_1,\ldots,z_{N/2})=\prod_{\mu<\nu}^{N/2}(z_\mu-z_\nu)^2\times\prod_{\gamma=1}^{N/2}G(z_\gamma)\exp\left[-\frac{\pi}{b^2}|z_\gamma|^2\right], \tag{7.1}$$

where $z_\gamma=(l_\gamma+im_\gamma)b$ is a complex number locating the γth up spin considered as a boson, and the "gauge" function is defined in the manner

$$G(z)=(-1)^{l+m+lm+1}\ . \tag{7.2}$$

That this state was similar to the class of "flux" phases was first pointed out by Zou, Doucot, and Shastry.[11] To do this, we shall make use of Eq. (4.10) and the fact that the Vandermonde determinant, given by

$$\prod_{\mu<\nu}(z_\mu-z_\nu)=\sum_p\mathrm{sgn}(p)z_{p(1)}^0\cdots z_{p(N/2)}^{N/2-1}\ , \tag{7.3}$$

is the wave function of a filled Landau band.

We first recall that the eigenstates of the continuum Hamiltonian

$$\mathcal{H}_{\mathrm{mag}}=\frac{1}{2}\left|\frac{1}{i}\nabla+\mathbf{A}\right|^2, \tag{7.4}$$

with $\mathbf{A}$ as given by Eq. (3.4), take the form

$$\varphi_{q_x}^{(n)}(x,y)=\exp(iq_xx)\exp[\tfrac{1}{2}(y-q_x)^2]\left[\frac{\partial}{\partial y}\right]^n\times\exp[-(y-q_x)^2]\ , \tag{7.5}$$

with the length units picked so that $b^2=\pi$ and with

$$\mathcal{H}_{\mathrm{mag}}\varphi_{q_x}^{(n)}(x,y)=(n+\tfrac{1}{2})\varphi_{q_x}^{(n)}(x,y)\ . \tag{7.6}$$

Since the states with $n=0$, which comprise the lowest Landau level, may also be written

$$\varphi_{q_x}(x,y)=\exp(-\tfrac{1}{2}q_x^2)\exp\left[-\frac{i}{2}xy\right]\times\exp(\tfrac{1}{4}z^2+iq_xz)\exp(-\tfrac{1}{4}|z|^2)\ , \tag{7.7}$$

they are linear combination of states of the form

$$\varphi_m(x,y)=z^m\exp\left[-\frac{i}{2}xy\right]\exp(-\tfrac{1}{4}|z|^2)\ , \tag{7.8}$$

the totally occupied Slater determinant of which is given by Eq. (7.3) times exponential factors. We next observe that the number of these states, namely, $L^2/(2\pi)=N/2$, is exactly the number of Bloch states $\varphi_{\mathbf{q}}$ as defined in Eq. (3.5). Thus, it is meaningful to restrict their arguments to lattice sites and find the combination of them that satisfies the Bloch condition on the lattice. Written out with the length units reinstated, these become

$$\varphi_{\mathbf{q}}(x,y)=\mathcal{N}_{\mathbf{q}}\sum_{n=-\infty}^{\infty}\exp(i2nq_yb)\exp\left[i\left[q_x+\frac{n2\pi}{b}\right]x\right]\exp\left\{-\frac{\pi}{2b^2}\left[y-\frac{b^2}{\pi}\left[q_x+\frac{n2\pi}{b}\right]\right]^2\right\}, \tag{7.9}$$

with $\mathcal{N}_{\mathbf{q}}$ a normalization constant. Since these $N/2$ states are orthogonal under the lattice inner product and are not zero, they span the Landau level, and thus give a wave function of the form of Eq. (7.3) when totally occupied. Thus, in order to prove the Kalmeyer-Laughlin state equivalent to the Wen-Wilczek-Zee state, it suffices to find a tight-binding Hamiltonian of the form of Eq. (3.1) for which $\varphi_{\mathbf{q}}$ as defined in Eq. (7.9) is an eigenstate for all $\mathbf{q}$. The gauge factors defined by Eqs. (4.9) and (7.2) are related by

$$G(z)=-G_L(z)\exp\left[-i\frac{\pi}{b^2}xy\right] . \tag{7.10}$$

Returning to Eq. (3.6) with $\mathcal{H}_{\mathbf{q}}$ defined as in Eq. (6.2), let us find matrix elements $\alpha(\mathbf{q})$, $\beta(\mathbf{q})$, and $\gamma(\mathbf{q})$ for which the ratio of the amplitudes to be on sites 1 and 2 satisfies

$$\mathbf{z}=\frac{u_{\mathbf{q}}(2)}{u_{\mathbf{q}}(1)}=\frac{\varphi_{\mathbf{q}}(0,b)}{\varphi_{\mathbf{q}}(0,0)}e^{-iq_y b} . \tag{7.11}$$

Since a shift of the energy has no effect on the eigenstate, we can assume without loss of generality that $\beta(\mathbf{q})=-\alpha(\mathbf{q})$. When this is the case, the eigenvalues are

$$E=\pm(\alpha^2+|\gamma|^2)^{1/2} , \tag{7.12}$$

and the ratio of the amplitudes is

$$z=\frac{\pm(\alpha^2+|\gamma|^2)^{1/2}-\alpha}{\gamma} . \tag{7.13}$$

Since the numerator of the right-hand side of this expression is real, we must have

$$\frac{\gamma}{\alpha}=\epsilon z^* , \tag{7.14}$$

for some real ϵ. Substituting this into Eq. (7.13), we obtain

$$\epsilon|z|^2=\pm\frac{\alpha}{|\alpha|}(1+\epsilon^2|z|^2)^{1/2}-1 , \tag{7.15}$$

the solution of which is

$$\frac{\gamma}{\alpha}=\frac{2z^*}{1-|z|^2} , \tag{7.16}$$

or zero, depending on the sign of α. Since $\gamma/\alpha=0$ is unphysical, we must have $\alpha>0$ when $|z|^2>1$ and $\alpha<0$ when $|z|^2<1$, assuming that the negative eigenvalue is the appropriate one. In light of this, an obvious choice for $\mathcal{H}'(\mathbf{q})$ is given by

$$\alpha(\mathbf{q})=|\varphi_{\mathbf{q}}(0,0)|^2-|\varphi_{\mathbf{q}}(0,b)|^2 , \tag{7.17}$$

and

$$\gamma(\mathbf{q})=2\varphi_{\mathbf{q}}^*(0,b)\varphi_{\mathbf{q}}(0,0)e^{iq_y b} . \tag{7.18}$$

Evaluating these expressions using Eq. (7.9), we obtain

$$\alpha(\mathbf{q})=\sum_{\mu,\nu}(-1)^{\nu}\exp\{-(\pi/4)[(2\mu+1)^2+(2\nu)^2]\} \times\exp\{i[(2\mu+1)q_x b+(2\nu)q_y b]\} , \tag{7.19}$$

and

$$\gamma(\mathbf{q})=\sum_{l,\nu}(i)^{(2\nu+1)l}\exp\{-(\pi/4)[l^2+(2\nu+1)^2]\} \times\exp\{i[lq_x b+(2\nu+1)q_y b]\} . \tag{7.20}$$

Working backward using Eqs. (6.1)–(6.4), we find that the tight-binding Hamiltonian to which this corresponds is

$$\langle j|T|k\rangle=T_0\exp\left[-\frac{\pi}{4}[(l-l')^2+(m-m')^2]\right] , \tag{7.21}$$

with T_0 a constant. This expression is negligibly small beyond second neighbors, and has a ratio of the second- to-first neighbor values of $\exp(-\pi/4)=0.46$. Since the mass m_0 is twice this value, or 0.92, the Wen-Wilczek-Zee state with $m_0=1$ described in Table I is almost identical to the Kalmeyer-Laughlin state. This accounts for the similarity of their correlation functions.[5]

Let us now ask what exchange couplings J_{jk} as defined in Eq. (2.1) lead to this particular solution when used with the variational procedure described in Sec. III. While a precise answer to this question requires accurate numerical calculations beyond the scope of this work, an approximate answer may be obtained by making use of the continuum result[18]

$$\int\cdots\int\prod_{\gamma\neq 1}(z_1'^*-z_\gamma^*)^2(z_1-z_\gamma)^2\prod_{1<\mu<\nu}|z_\mu-z_\nu|^4\exp\left[-\frac{\pi}{4b^2}(|z_1'|^2+|z_1|^2)\right] \times\exp\left[-\sum_{\gamma\neq 1}\frac{\pi}{2b^2}|z_\gamma|^2\right]d^2z_2\cdots d^2z_{N/2}=\text{const}\exp\left[-\frac{\pi}{4b^2}(|z_1'|^2+|z_1|^2)\right]\exp\left[\frac{\pi}{2b^2}z_1'^*z_1\right] , \tag{7.22}$$

which is exact in the thermodynamic limit. Thus, the self-consistency requirement is that J_{jk} be roughly constant.

VIII. SPINON WAVE FUNCTIONS

Since the set of allowed spin configurations is considerably smaller than the number of ways of deploying N spin-$\frac{1}{2}$ electrons on N sites, the *basis* set consisting of Gutzwiller projections of ordinary electron and hole excitations of the Hartree-Fock ground state is capable of describing any excitation of the chiral spin liquid. Furthermore, the validity of the variational reasoning leading to the ground state implies that these wave functions should themselves approximate true elementary excitations, since pairs of them constitute the lowest-energy excitations that can be expressed as pro-

jected Slater determinants. These particles, termed "spinons" by Anderson,[6] carry no charge, since the Gutzwiller projector forces each site to have one electron, but must have spin-$\frac{1}{2}$, since the projector commutes with total spin. The destruction of the charge by the Gutzwiller projector does great violence to the wave function and causes the spinon to have properties that are very different from those of an ordinary electron or hole.

Let us begin the discussion of spinons by showing that[8]

$$\Pi_G (c^{(+)}_{\mathbf{q}_1\uparrow})^\dagger c^{(-)}_{\mathbf{q}_2\downarrow} |\Psi_{\rm SD}\rangle = \Pi_G c^{(-)}_{\mathbf{q}'_1\uparrow} (c^{(+)}_{\mathbf{q}'_2\downarrow})^\dagger |\Psi_{\rm SD}\rangle \ , \tag{8.1}$$

where

$$\mathbf{q}'_1 = \frac{\pi}{b}\hat{\mathbf{x}} - \mathbf{q}_1, \quad \mathbf{q}'_2 = \frac{\pi}{b}\hat{\mathbf{x}} - \mathbf{q}_2 \ . \tag{8.2}$$

In other words, an "up" spinon may be made either by projecting a hole or by projecting an electron, so that the basis is massively overcomplete. Returning to Eq. (4.2), we have

$$[(c^{(+)}_{\mathbf{q}_1\uparrow})^\dagger c^{(-)}_{\mathbf{q}_2\downarrow} \Psi_{\rm SD}](j_1,\ldots,j_N|\uparrow,\ldots,\uparrow,\downarrow,\ldots,\downarrow) = (-1)^{N/2+1} \begin{pmatrix} N \\ N/2 \end{pmatrix}^{-1/2} \Phi^{(+)}_{\mathbf{q}_1}(j_1,\ldots,j_{N/2+1}) \Phi^{(-)}_{\mathbf{q}_2}(j_{N/2+2},\ldots,j_N) \ , \tag{8.3}$$

where

$$\begin{aligned}
\Phi^{(+)}_{\mathbf{q}_1}(j_1,\ldots,j_{N/2+1}) &= \left\langle 0 \left| c_{j_1}\cdots c_{j_{N/2+1}} (c^{(+)}_{\mathbf{q}_1})^\dagger \left[\prod_{\mathbf{q}} c^{(-)}_{\mathbf{q}}\right]^\dagger \right| 0 \right\rangle \\
&= \left\langle 0 \left| \left[\prod_{\mathbf{q}} c^{(-)}_{\mathbf{q}}\right] (c^{(+)}_{\mathbf{q}_1}) (c_{j_1}\cdots c_{j_{N/2+1}})^\dagger \right| 0 \right\rangle^* \\
&= (-1)^{M+1} \left\langle 0 \left| \left[\prod_{\mathbf{q}} c^{(-)}_{\mathbf{q}}\right] (c^{(+)}_{\mathbf{q}_1}) (c_{j_{N/2+2}}\cdots c_{j_N}) \left[\prod_{\mathbf{q}} c^{(+)}_{\mathbf{q}}\right]^\dagger \left[\prod_{\mathbf{q}} c^{(-)}_{\mathbf{q}}\right]^\dagger \right| 0 \right\rangle^* \\
&= (-1)^{M+N/2+1} \left\langle 0 \left| (c_{j_{N/2+2}}\cdots c_{j_N}) (c^{(+)}_{\mathbf{q}_1}) \left[\prod_{\mathbf{q}} c^{(+)}_{\mathbf{q}}\right]^\dagger \right| 0 \right\rangle^* \\
&= (-1)^{M+N/2+1} \exp\left[i\frac{\pi}{b}\hat{\mathbf{y}}\cdot(\mathbf{R}_{j_{N/2+2}} + \cdots + \mathbf{R}_{j_N}) \right] \Phi^{-}_{\mathbf{q}'_1}(j_{N/2+2},\ldots,j_N) \ .
\end{aligned} \tag{8.4}$$

Similarly, we have

$$\Phi^{(-)}_{\mathbf{q}_2}(j_{N/2+2},\ldots,j_N) = (-1)^{M+N/2} \exp\left[i\frac{\pi}{b}\hat{\mathbf{y}}\cdot(\mathbf{R}_{j_1} + \cdots + \mathbf{R}_{j_{N/2+1}}) \right] \Phi^{(+)}_{\mathbf{q}'_2}(j_1,\ldots,j_{N/2+1}) \ , \tag{8.5}$$

which proves the assertion. This relation shows that a spinon state is uniquely indexed by its spin $\mathbf{s}$ and crystal momentum $\mathbf{q}$, and thus that the number of linearly independent states is N. Counting the degrees of freedom in the manner appropriate to fermions, we obtain 2^N, which is the correct number for the spin problem. However, if we use the same reasoning to estimate the maximum possible z component of spin, we obtain $(N/2)(\hbar/2)$, which is manifestly incorrect. Evidently the spinons can pack more tightly together than is allowed for spin-$\frac{1}{2}$ fermions in $N/2$ orbitals. This behavior is also exhibited by the quasiparticles in the fractional quantum Hall effect[18] and is an indication that the particles obey fractional statistics.[19]

Let us now show that for the particular chiral spin-liquid wave function discussed in Sec. VII, the spinon state as defined above is identical to the Kalmeyer-Laughlin spinon. To do this, we first superimpose the itinerant magnetic Bloch states $\varphi_{\mathbf{q}}$ defined in Eq. (7.9) into a magnetic Wannier state centered at location j_0 in the manner

$$\varphi_{j_0}(j) = \sum_{\mathbf{q}} \varphi^*_{\mathbf{q}}(j_0)\varphi_{\mathbf{q}}(j) \ . \tag{8.6}$$

We then observe that placing a spinon in such a state is the same thing as placing an ordinary hole in it before projecting, i.e.,

$$\sum_{\mathbf{q}_2} \varphi^{(-)}_{\mathbf{q}_2}(j_0) \Pi_G (c^{(+)}_{\mathbf{q}_1\uparrow})^\dagger c^{(-)}_{\mathbf{q}_2\downarrow} |\Psi_{\rm SD}\rangle = \Pi_G (c^{(+)}_{\mathbf{q}_1\uparrow})^\dagger \left[\sum_{\mathbf{q}_2} \varphi^{(-)}_{\mathbf{q}_2}(j_0) c^{(-)}_{\mathbf{q}_2\downarrow} \right] |\Psi_{\rm SD}\rangle \ . \tag{8.7}$$

We observe finally that since the states $\varphi_{\mathbf{q}}$ are complete, we have

$$\sum_{\mathbf{q}_2} \varphi^{(-)}_{\mathbf{q}_2}(j_0) \Phi^{(-)}_{\mathbf{q}_2}(j_{N/2+2},\ldots,j_N) = \Phi(j_0, j_{N/2+2},\ldots,j_N) \ . \tag{8.8}$$

In other words, placing a hole in the down band is tantamount to holding one of the electron coordinates at j_0. Thus, the wave function of a down spinon at j_0 and an up spinon at j_0', written in the manner of Eq. (4.2), is

$$|j_0,j_0'\rangle=\begin{bmatrix}N\\N/2\end{bmatrix}^{-1/2}\Phi(j_0',j_2,\ldots,j_{N/2})\Phi(j_0,j_{N/2+2},\ldots,j_N)\ , \tag{8.9}$$

with the set $\{j_0',j_2,\ldots,j_{N/2}\}$ forced to be distinct from the set $\{j_0,j_{N/2+2},\ldots,j_N\}$. For the particular orbitals φ_q described in Sec. VII, we have for a down spinon at location z_0 in the thermodynamic limit

$$|z_0\rangle=\prod_\gamma(z_\gamma-z_0)\prod_{\mu<\nu}(z_\mu-z_\nu)^2\prod_\gamma G(z_\gamma)\exp\left[-\frac{\pi}{b^2}|z_\gamma|^2\right]\ , \tag{8.10}$$

which is the Kalmeyer-Laughlin[4,5] wave function.

Equation (8.9) has the useful feature of the facilitating computation of spinon properties for any chiral-spin-liquid state. For example, suppose we wish to compute $\langle S_z(\mathbf{r})\rangle$ in the presence of a spinon pair. Per Eq. (5.21), this may be written in terms of the boson density in the manner

$$\langle S_z(\mathbf{r})\rangle=\rho_G(\mathbf{r})-\tfrac{1}{2}\ , \tag{8.11}$$

where

$$\rho_G(\mathbf{r}_2)=\frac{N}{2}\frac{1}{Z_G}\sum_{\mathbf{r}_3}\cdots\sum_{\mathbf{r}_{N/2}}|\Phi(\mathbf{r}_0,\mathbf{r}_2,\ldots,\mathbf{r}_{N/2})|^2|\Phi(\mathbf{r}_0',\mathbf{r}_{N/2+2},\ldots,\mathbf{r}_N)|^2\ . \tag{8.12}$$

Let us assume for simplicity that $\mathbf{r}_0$ and $\mathbf{r}_0'$ are far apart, so that we need consider only one of them at a time. In light of Eq. (5.1), we have approximately in the vicinity of $\mathbf{r}_0$

$$\rho_G(\mathbf{r}_2)\cong\frac{N}{2}\frac{1}{Z}\sum_{\mathbf{r}_3}\cdots\sum_{\mathbf{r}_{N/2}}\prod_\tau e^{-v(\mathbf{r}_\tau-\mathbf{r}_0)}\prod_{\mu<\nu}e^{-2v(\mathbf{r}_\mu-\mathbf{r}_\nu)}\prod_\tau e^{-2u(\mathbf{r}_\tau)}\ . \tag{8.13}$$

This may be interpreted as a correlation function of a classical plasma with two components and may thus be evaluated using the multicomponent hypernetted chain equations. In the vicinity of the down spinon we have

$$\langle S_z(\mathbf{r})\rangle=\tfrac{1}{2}h_s(\mathbf{r})\ , \tag{8.14}$$

where $h_s(\mathbf{r})$ satisfies

$$g_s(\mathbf{r})\cong\exp[-v(\mathbf{r})+h_s(\mathbf{r})-c_s(\mathbf{r})]\ , \tag{8.15}$$

$$h_s(\mathbf{r})=g_s(\mathbf{r})-1\ , \tag{8.16}$$

$$\hat{h}_s(\mathbf{q})=\hat{c}_s(\mathbf{q})+\tfrac{1}{2}\hat{h}_s(\mathbf{q})\hat{c}_G(\mathbf{q})\ . \tag{8.17}$$

TABLE V. Spin profile $h_s(\mathbf{r})$, as evaluated using Eqs. (8.15)–(8.17). $\langle S_z\rangle$ in the vicinity of a down spin is $\frac{1}{2}h_s(\mathbf{r})$.

Shell	l	m	$m_0=1.0$	$m_0=0.5$	$m_0=0.1$
0	0	0	−1.000	−1.000	−1.000
1	1	0	−0.008	−0.012	−0.015
2	1	1	0.006	0.010	0.012
3	2	0	0.002	0.005	0.007
4	2	1	0.000	−0.001	−0.003
5	2	2	0.000	0.000	0.001
6	3	0	0.000	−0.001	−0.001
7	3	1	0.000	0.000	0.001

The solution of Eqs. (8.15)–(8.17) for three values of m_0 is shown in Table V. We note that the spinon is effectively confined to a single site in all three cases, as is expected of an eigenstate of total spin. The 1% tails seen at $m_0=0.5$ and $m_0=0.1$ are errors attributable to the hypernetted chain procedure.

IX. FRACTIONAL STATISTICS

We shall now explicitly demonstrate that the spinons defined by Eq. (8.9) obey $\frac{1}{2}$ fractional statistics.[18–20] Following Arovas, Schrieffer, and Wilczek,[21] we shall do this by evaluating the Berry phase[22] for adiabatic interchange of two particles. Let us recall the definition of the Berry phase. Suppose that the spin Hamiltonian is modified so as to include a well that traps a spinon at location z_0. This might be accomplished, for example, by adding a Zeeman interaction of the form

$$\Delta\mathcal{H}(z_0)=\sum_\gamma\mathbf{B}(z-z_0)\cdot\mathbf{S}_\gamma\ , \tag{9.1}$$

where $\mathbf{B}(z)$ is large only near z_0. If $|z_0\rangle$ is the exact ground state in the presence of $\Delta\mathcal{H}(z_0)$, then changing the parameter z_0 causes the spinon to track with the well, always being in the state $|z_0\rangle$. However, viewed as a sequence of small perturbations

$$\Delta\mathcal{H}(z_0)\rightarrow\Delta\mathcal{H}(z_0+\delta z_0)\ ,$$

the evolution must at every step produce a change to the wave function that is orthogonal to it. This follows directly from the second-order perturbation formula

$$\delta|m\rangle=\sum_{n\neq m}\frac{\langle n|\sigma\mathcal{H}|m\rangle}{E_m-E_n}|n\rangle\ . \tag{9.2}$$

The orthogonality condition requires that $|z_0\rangle$ evolve to $|z_0+\delta z_0\rangle$ times a phase $\delta\phi$, given by

$$e^{i\delta\phi}\frac{\langle z_0|z_0+\delta z_0\rangle}{\sqrt{\langle z_0|z_0\rangle\langle z_0+\delta z_0|z_0+\delta z_0\rangle}}\cong 1\ . \tag{9.3}$$

Thus, if the parameter z_0 is evolved around a closed path P, the wave function goes back to itself up to a phase ϕ_B, given by

$$\phi_B=\int_P \mathbf{A}\cdot d\mathbf{s}\ , \tag{9.4}$$

where

$$\mathbf{A}(z_0)=\mathrm{Im}\left[\frac{\langle z_0|\nabla|z_0\rangle}{\langle z_0|z_0\rangle}\right]\ . \tag{9.5}$$

Let us now evaluate the Berry phase for evolving a single spinon around a closed loop. The vector potential defined in Eq. (9.5) may be calculated simply from the overlap integral

$$\langle z_0'|z_0\rangle=\sum_{z_2}\cdots\sum_{z_{N/2}}\prod_{\gamma}(z_0'^{*}-z_\gamma^{*})(z_0-z_\gamma)\prod_{\mu<\nu}|z_\mu-z_\nu|^4\prod_{\gamma}\exp\left[-\frac{\pi}{b^2}|z_\gamma|^2\right]\ . \tag{9.6}$$

This, in turn, may be obtained by analytic continuation from the normalization integral $\langle z_0|z_0\rangle$, since the coefficients a_{mn} of any function of the form

$$f(z_0)=\sum_{m,n}a_{mn}(z_0^{*})^m(z_0)^n\ , \tag{9.7}$$

are given by

$$a_{mn}=\frac{1}{(m+n)!}\left[\frac{\partial}{\partial r}\right]^{m+n}\left[\frac{1}{2\pi}\int_0^{2\pi}f(re^{i\phi})e^{i(m-n)\phi}d\phi\right]\Bigg|_{r=0}\ . \tag{9.8}$$

However, because $\langle z_0|z_0\rangle$ is the partition function of a plasma with one of its particles held fixed, it must take the form

$$\langle z_0|z_0\rangle=P(z_0)\exp\left[\frac{\pi}{2b^2}|z_0|^2\right]\ , \tag{9.9}$$

where $P(z_0)$ is a periodic function. Writing this as a Fourier series, in the manner

$$P(z_0)=\sum_{\mathcal{G}}a_{\mathcal{G}}e^{(1/2)(\mathcal{G}^{*}z_0-\mathcal{G}z_0^{*})}\ , \tag{9.10}$$

where $\mathcal{G}=(l+im)(2\pi/b)$, we obtain

$$\frac{\langle z_0'|z_0\rangle}{(\langle z_0|z_0\rangle\langle z_0'|z_0'\rangle)^{1/2}}=\left[\sum_{\mathcal{G}}a_{\mathcal{G}}e^{(1/2)(\mathcal{G}^{*}z_0-\mathcal{G}z_0'^{*})}\right][P(z_0)P(z_0')]^{-1/2}\exp\left[-\frac{\pi}{4b^2}(|z_0|^2+|z_0'|^2)\right]\exp\left[\frac{\pi}{2b^2}z_0'^{*}z_0\right]\ . \tag{9.11}$$

The vector potential is then

$$\mathbf{A}(z_0)=\left[-\frac{\pi}{2b^2}y_0-\frac{1}{2}\frac{\partial}{\partial y_0}\ln[P(z_0)]\right]\hat{\mathbf{x}}+\left[\frac{\pi}{2b^2}x_0+\frac{1}{2}\frac{\partial}{\partial x_0}\ln[P(z_0)]\right]\hat{\mathbf{y}}\ . \tag{9.12}$$

Up to an unimportant periodic contribution, this is the symmetric gauge version of Eq. (3.4). Thus, adiabatic evolution of a spinon around a closed loop yields a Berry phase equal to the enclosed flux of the background "magnetic field" found in the self-consistency procedure. This shows that the violation of time-reversal symmetry necessary to stabilize the liquid state has measurable consequences even for one particle.

Let us now repeat this calculation for a pair of "down" spinons described by the wave function

$$|z_A,z_B\rangle=\prod_{\gamma}(z_\gamma-z_A)(z_\gamma-z_B)\prod_{\mu<\nu}(z_\mu-z_\nu)^2\prod_{\gamma}G(z_\gamma)\exp\left[-\frac{\pi}{b^2}|z_\gamma|^2\right]\ . \tag{9.13}$$

If z_A and z_B are farther away from each other than a plasma screening length then the arguments leading to Eq. (9.9) give

$$\langle z_A,z_B|z_A,z_B\rangle\cong\mathrm{const}|z_A-z_B|P(z_A)\exp\left[\frac{\pi}{2b^2}|z_A|^2\right]P(z_B)\exp\left[\frac{\pi}{2b^2}|z_B|^2\right]\ . \tag{9.14}$$

It is not possible to continue the function globally without taking into account the correction when z_A and z_B are close together. Local continuation is possible, however, as may be seen by idealizing the core correction in the manner

$$|z_A - z_B| \cong \frac{\beta}{\pi} \int |z| \exp(-\beta |z - z_A + z_B|^2) d^2 z \ . \tag{9.15}$$

The limit of this expression when $z'_A \cong z_A$, $z'_B \cong z_B$, and $|z_A - z_B|^2 \gg 1/\beta$ is

$$\frac{\beta}{\pi} \int |z| \exp[-\beta (z^* - z'^*_A - z'^*_B)(z - z_A - z_B)] d^2 z \cong (z'^*_A - z'^*_B)^{1/2} (z_A - z_B)^{1/2} \ , \tag{9.16}$$

regardless of β. Thus, the vector potential associated with moving z_B in a loop about z_A is equal to that of Eq. (9.12) plus the correction

$$\Delta \mathbf{A}(z_B) = -\frac{1}{2} \left[\frac{(y_B - y_A)}{(x_B - x_A)^2 + (y_B - y_A)^2} \right] \hat{\mathbf{x}} + \frac{1}{2} \left[\frac{(x_B - x_A)}{(x_B - x_A)^2 + (y_B - y_A)^2} \right] \hat{\mathbf{y}} \ , \tag{9.17}$$

the line integral of which is

$$\int_P \Delta \mathbf{A} \cdot d\mathbf{s} = \pi \ . \tag{9.18}$$

Since this is twice the extra Berry phase associated with interchanging the particles, the latter must equal $\pi/2$.

X. INTERPOLATING MAGNETIC BLOCH STATES

In order to evaluate the Berry phase of spinons it is necessary to define the wave function $|z_0\rangle$ for z_0 different from a lattice site. Since Eq. (8.10) is defined between lattice sites and is obviously reasonable, let us ask if there is a generalization of this property that works for any chiral-spin-liquid state. Returning to Eq. (7.9), we observe that the Landau Bloch states are defined for all x and y, and satisfy the periodicity relations

$$\varphi_{\mathbf{q}}(x + b, y) = e^{ibq_x} \varphi_{\mathbf{q}}(x, y) \ , \tag{10.1}$$

and

$$\varphi_{\mathbf{q}}(x, y + 2b) = e^{i2bq_y} e^{i2\pi x/b} \varphi_{\mathbf{q}}(x, y) \ . \tag{10.2}$$

In addition, however, they satisfy the *momentum* periodicity relations

$$\varphi_{\mathbf{q} + (2\pi/b)\hat{\mathbf{x}}}(x, y) = e^{-i2bq_y} \varphi_{\mathbf{q}}(x, y) \ , \tag{10.3}$$

and

$$\varphi_{\mathbf{q} + (\pi/b)\hat{\mathbf{y}}}(x, y) = \varphi_{\mathbf{q}}(x, y) \ , \tag{10.4}$$

as well as the relation

$$\varphi_{\mathbf{q}}(x, y) = e^{iq_x x} \varphi_{\mathbf{q} + (\pi/b)[x\hat{\mathbf{y}} - y\hat{\mathbf{x}}]}(0, 0) \ , \tag{10.5}$$

which together are equivalent to Eqs. (10.1) and (10.2). Thus, any set of Bloch states satisfying Eqs. (10.3) and (10.4) for the case of $x = 0$ and $y = 0$ is interpolated in a natural way by Eq. (10.5). To make sense, such a wave function must also satisfy

$$\varphi_{\mathbf{q}}(0, b) = \varphi_{\mathbf{q} - (\pi/b)\hat{\mathbf{x}}}(0, 0) \ . \tag{10.6}$$

Let us now consider the set of magnetic Bloch states described by the general reduced Hamiltonian $\mathcal{H}_{\mathbf{q}}$ of Eq. (6.2). We may write

$$\begin{bmatrix} \varphi_{\mathbf{q}}(0,0) \\ \varphi_{\mathbf{q}}(0,b) \end{bmatrix} = \frac{e^{i\chi(\mathbf{q})}}{\mathcal{N}_{\mathbf{q}}} \begin{bmatrix} \beta_{\mathbf{q}} - E_{\mathbf{q}} \\ -\gamma^*_{\mathbf{q}} e^{iq_y b} \end{bmatrix} \ , \tag{10.7}$$

where

$$E_{\mathbf{q}} = \tfrac{1}{2}(\alpha_{\mathbf{q}} + \beta_{\mathbf{q}}) + [\tfrac{1}{4}(\alpha_{\mathbf{q}} - \beta_{\mathbf{q}})^2 + |\gamma_{\mathbf{q}}|^2]^{1/2} \ , \tag{10.8}$$

$$\mathcal{N}_{\mathbf{q}} = [(\beta_{\mathbf{q}} - E_{\mathbf{q}})^2 + |\gamma_{\mathbf{q}}|^2]^{1/2} \ , \tag{10.9}$$

and $\chi(\mathbf{q})$ is a phase we wish to choose to satisfy Eqs. (10.3)–(10.6). In light of definitions of $\alpha_{\mathbf{q}}$, $\beta_{\mathbf{q}}$, and $\gamma_{\mathbf{q}}$, Eq. (10.6) may be written

$$e^{i\chi[\mathbf{q} - (\pi/b)\hat{\mathbf{x}}]} = -\frac{\gamma^*_{\mathbf{q}}}{|\gamma_{\mathbf{q}}|} e^{iq_y b} e^{i\chi(\mathbf{q})} \ , \tag{10.10}$$

which, when iterated, gives Eq. (10.3). Equation (10.4) is satisfied if $\chi(\mathbf{q})$ is periodic in q_y. These conditions do not specify $\chi(\mathbf{q})$ uniquely, since any set of phases defined in the region $0 \le q_x \le \pi/b$ and $0 \le q_y \le \pi/b$ may be extended by Eqs. (10.4) and (10.10) to all values of $\mathbf{q}$, and is thus consistent. However, a convenient solution is

$$e^{i\chi(\mathbf{q})} = \frac{\varphi^{(0)}_{\mathbf{q}}(0,0)}{|\varphi^{(0)}_{\mathbf{q}}(0,0)|} \ , \tag{10.11}$$

where $\varphi^{(0)}_{\mathbf{q}}$ is one of the normalized Landau Bloch states

$$\varphi^{(m)}_{\mathbf{q}}(x,y) = \left[\left[\frac{2\pi}{b^2} \right]^{m+1} \pi^{3/2} m! \right]^{-1/2} \sum_{n=-\infty}^{\infty} \exp(i2nq_y b) \exp\left[i \left[q_x + \frac{n2\pi}{b} \right] x \right] \times \exp\left\{ \frac{\pi}{2b^2} \left[y - \frac{b^2}{\pi} \left[q_x + \frac{n2\pi}{b} \right] \right]^2 \right\} \left[\frac{\partial}{\partial y} \right]^m \times \exp\left\{ -\frac{\pi}{b^2} \left[y - \frac{b^2}{\pi} \left[q_x + \frac{n2\pi}{b} \right] \right]^2 \right\} \ . \tag{10.12}$$

TABLE VI. Landau-level expansion coefficients, as defined in Eq. (10.15), for Wen-Wilczek-Zee state at various values of the mass m_0. The coefficients are identically zero for odd m and satisfy the sum rule $\sum_m |c_m|^2 = 1$.

m_0	c_0	c_2	c_4	c_6	c_8	c_{10}
1.0	0.999	0.004	−0.036	0.003	−0.021	−0.007
0.5	0.998	−0.013	−0.013	−0.006	−0.041	0.019
0.1	0.993	−0.024	0.014	−0.015	−0.058	0.045

Since these functions also satisfy Eqs. (10.1)–(10.5), we have

$$\left[\frac{\pi}{b^2}\right]\int_0^{2b}\int_0^{b}[\varphi_{\mathbf{q}}^{(m')}(x,y)]^*[\varphi_{\mathbf{q}}^{(m)}(x,y)]dx\,dy=\int_0^{\pi/b}\int_0^{2\pi/b}[\varphi_{\mathbf{q}}^{(m')}(0,0)]^*[\varphi_{\mathbf{q}}^{(m)}(0,0)]d\mathbf{q}=\delta_{mm'}\ , \tag{10.13}$$

and thus

$$\varphi_{\mathbf{q}}=\sum_m c_m\varphi_{\mathbf{q}}^{(m)}\ , \tag{10.14}$$

where

$$c_m=\int_0^{\pi/b}\int_0^{2\pi/b}[\varphi_{\mathbf{q}}^{(m)}(0,0)]^*[\varphi_{\mathbf{q}}(0,0)]d\mathbf{q}\ . \tag{10.15}$$

Thus, every magnetic Bloch state $\varphi_{\mathbf{q}}$ may be interpreted as a superposition of Landau Bloch states. The coefficients c_m associated with the Wen-Wilczek-Zee state are listed in Table VI for various values of the mass m_0. The dominance of the lowest Landau level in all cases further supports the idea that the Wen-Wilczek-Zee state and the Kalmeyer-Laughlin state are one and the same.

XI. THREE-DIMENSIONAL SPIN-LIQUID STATE

Let us now investigate whether the reasoning discussed in Secs. II and III will go through in three spatial dimensions as well as two. The chiral spin liquid in two dimensions is associated with a uniform magnetic field threading $\frac{1}{2}$ flux quantum through each square unit cell. By analogy, we expect the corresponding phase in three dimensions to put $\frac{1}{2}$ flux quantum through each *face* of a cubic unit cell, i.e., with a vector potential of the form

$$\mathbf{A}=\frac{\pi}{b}(\pm x\hat{\mathbf{y}}\pm y\hat{\mathbf{z}}\pm z\hat{\mathbf{x}})\ . \tag{11.1}$$

The eight possible sign choices in this expression correspond to the eight body diagonals along which the net magnetic field may point. Let us concentrate on the one of these for which all the signs are positive. We further expect the extremal condition to give rise to a tight-binding Hamiltonian of the form of Eq. (6.1), where the coefficients $\langle j|T|k\rangle$ are real numbers that respect the point-group symmetries of the lattice, and where the integral is performed along the straight line between sites j and k. If we denote a lattice site with three integers in the manner

$$\mathbf{R}_j=(l\hat{\mathbf{x}}+m\hat{\mathbf{y}}+n\hat{\mathbf{z}})b\ , \tag{11.2}$$

then the phase in Eq. (6.1) is

$$\exp\left[i\int_j^k\mathbf{A}\cdot d\mathbf{s}\right]=(i)^{[(l+l')(m-m')+(m+m')(n-n')+(n+n')(l-l')]}\ . \tag{11.3}$$

Let us first consider the case for which $\langle j|T|k\rangle$ is zero except for near neighbors, where it is a constant T. This causes the hopping matrix elements to be positive or negative, according as Fig. 5. In this gauge, the unit cell contains eight sites. With these labeled as in Fig. 5 the reduced Hamiltonian at wave vector $\mathbf{q}$ is

$$\mathcal{H}_{\mathbf{q}}=2T[\cos(q_xb)\alpha_x+\cos(q_yb)\alpha_y+\cos(q_zb)\alpha_z]\ , \tag{11.4}$$

where

$$\alpha_x=\begin{bmatrix}0&1&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&0&0&-1&0&0\\0&0&0&0&-1&0&0&0\\0&0&0&0&0&0&0&-1\\0&0&0&0&0&0&-1&0\end{bmatrix}\ , \tag{11.5}$$

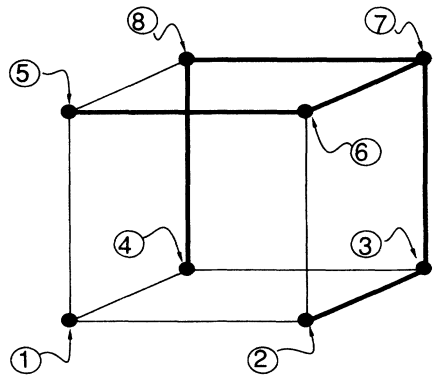

FIG. 5 Illustration of near-neighbor matrix elements of Hartree-Fock Hamiltonian for three-dimensional chiral spin liquid, as defined by Eqs. (6.1) and (11.3). As in Fig. 2, negative interactions are indicated as bold lines.

$$\alpha_y=\begin{bmatrix} 0&0&0&1&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&-1&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&1\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&-1&0&0\\ 0&0&0&0&1&0&0&0 \end{bmatrix}, \tag{11.6}$$

and

$$\alpha_z=\begin{bmatrix} 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&0&0&-1\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&-1&0&0&0&0&0\\ 0&0&0&-1&0&0&0&0 \end{bmatrix}. \tag{11.7}$$

The α's satisfy the Dirac anticommutation relations

$$\{\alpha_\mu,\alpha_\nu\}=2\delta_{\mu\nu}, \tag{11.8}$$

in light of which we have for the eigenvalues of $\mathcal{H}_q$

$$E_q=\pm 2T[\cos^2(q_x b)+\cos^2(q_y b)+\cos^2(q_z b)]^{1/2}. \tag{11.9}$$

Thus, as in the case of two dimensions, the Lagrange multiplier spectrum in the absence of time-reversal symmetry breaking has lightlike dispersion at the point $(\pi/b)(\frac{1}{2},\frac{1}{2},\frac{1}{2})$, consistent with the behavior of massless Dirac fermions.

Let us now consider the case in which an additional matrix element T' is added across the diagonals of the cube faces, as illustrated in Fig. 6. The reduced Hamiltonian becomes

$$\mathcal{H}_q=2T[\cos(q_x b)\alpha_x+\cos(q_y b)\alpha_y+\cos(q_z b)\alpha_z+m_0\beta'], \tag{11.10}$$

where

$$\beta'=\frac{1}{\sqrt{3}}\begin{bmatrix} 0&0&ia&0&0&ic&0&ib\\ 0&0&0&-ia&ic&0&-ib&0\\ -ia&0&0&0&0&ib&0&-ic\\ 0&ia&0&0&-ib&0&-ic&0\\ 0&-ic&0&ib&0&0&-ia&0\\ -ic&0&-ib&0&0&0&0&ia\\ 0&ib&0&ic&ia&0&0&0\\ -ib&0&ic&0&0&-ia&0&0 \end{bmatrix} \tag{11.11}$$

with

$$\begin{aligned} a&=\pm\sin(q_x b_0)\sin(q_y b_0),\\ b&=\pm\sin(q_y b_0)\sin(q_z b_0),\\ c&=\pm\sin(q_z b_0)\sin(q_x b_0), \end{aligned} \tag{11.12}$$

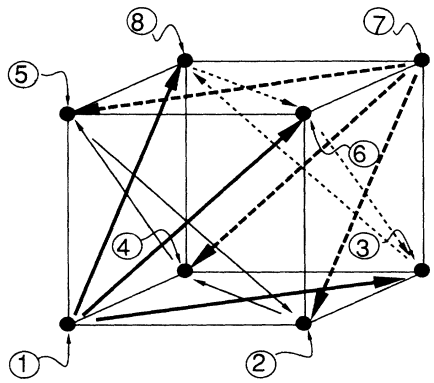

FIG. 6. Second-neighbor matrix elements of Hartree-Fock Hamiltonian for three-dimensional chiral spin state, as defined by Eqs. (6.1) and (11.3). As in Fig. 2, the arrow points to $|j\rangle$ when $\mathrm{Im}(\langle j|\mathcal{H}|k\rangle)$ is positive. Conversion of the mass matrix β' defined by Eq. (11.11) into the matrix β defined in Eq. (11.16) is accomplished by negating the dashed line.

and

$$m_0=2\sqrt{3}\frac{T'}{T}, \tag{11.13}$$

with the eight possible sign choices corresponding to the eight field directions. Unfortunately, the anticommutators of β' with the α's are not zero, but rather take the values

$$\{\alpha_x,\beta'\}=b\Delta,\quad \{\alpha_y,\beta'\}=c\Delta,\quad \{\alpha_z,\beta'\}=a\Delta, \tag{11.14}$$

where

$$\Delta=\frac{2}{\sqrt{3}}\begin{bmatrix} 0&0&0&0&0&0&-i&0\\ 0&0&0&0&0&0&0&i\\ 0&0&0&0&-i&0&0&0\\ 0&0&0&0&0&i&0&0\\ 0&0&i&0&0&0&0&0\\ 0&0&0&-i&0&0&0&0\\ i&0&0&0&0&0&0&0\\ 0&-i&0&0&0&0&0&0 \end{bmatrix}. \tag{11.15}$$

This reflects the fact that the phases specified by Eq. (11.3) single out a preferred direction, i.e., a particular body diagonal. The issue of degeneracy never arises when $T'=0$ because the eight configurations are then related by gauge transformations, which are unphysical. We now observe that this problem disappears if a vector potential generated by a lattice of monopoles, as illustrated in Fig. 7, is substituted for the uniform magnetic field vector potential in Eq. (11.3). This has the effect of negating the magnetic flux through all *odd* surfaces of the unit cell. The α's are unaffected by this change, but the mass matrix now becomes

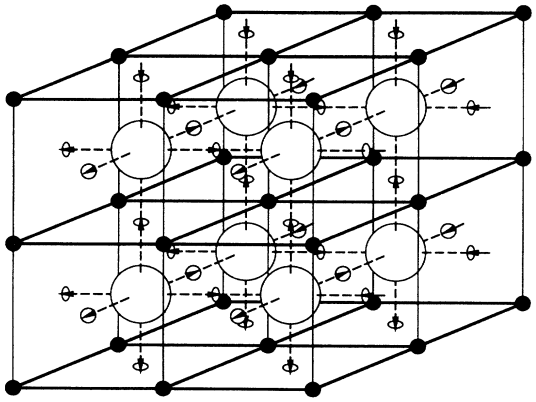

FIG. 7. Lattice of anisotropic monopoles associated with the mass matrix β defined in Eq. (11.16). Each of the eight possible ways flux $\pm\frac{1}{2}$ can thread the three cube faces is represented in the unit cell, which now has eight times the volume of a primitive cell. The cube at the lower left is equivalent to the one Fig. 6.

$$\beta=\frac{1}{\sqrt{3}}\begin{bmatrix} 0 & 0 & ia & 0 & 0 & ic & 0 & ib \\ 0 & 0 & 0 & -ia & ic & 0 & ib & 0 \\ -ia & 0 & 0 & 0 & 0 & -ib & 0 & ic \\ 0 & ia & 0 & 0 & -ib & 0 & ic & 0 \\ 0 & -ic & 0 & ib & 0 & 0 & ia & 0 \\ -ic & 0 & ib & 0 & 0 & 0 & 0 & -ia \\ 0 & -ib & 0 & -ic & -ia & 0 & 0 & 0 \\ -ib & 0 & -ic & 0 & 0 & ia & 0 & 0 \end{bmatrix} . \tag{11.16}$$

We then have

$$\{\alpha_x,\beta\}=0,\quad \{\alpha_y,\beta\}=0 ,$$
$$\{\alpha_z,\beta\}=0,\quad \beta^2=a^2+b^2+c^2 , \tag{11.17}$$

and thus an eigenvalue spectrum of the form

$$E_q=\pm 2T[\cos^2(q_x b)+\cos^2(q_y b)+\cos^2(q_z b)+m_0^2(a^2+b^2+c^2)]^{1/2} . \tag{11.18}$$

A total energy calculation is not necessary for determining which of these states has the lower variational energy. Using the fact that

$$\Delta^2=\tfrac{4}{3} , \tag{11.19}$$

we have immediately

$$E_q=\pm 2T\left[\left[\cos(q_x b)\pm\frac{m_0}{\sqrt{3}}a\right]^2+\left[\cos(q_y b)\pm\frac{m_0}{\sqrt{3}}b\right]^2+\left[\cos(q_z b)\pm\frac{m_0}{\sqrt{3}}c\right]\right]^{1/2} , \tag{11.20}$$

for the state characterized by β'. Thus, the lightlike singularity is not removed in this state, but is merely displaced from the zone corner, so that the cohesive energy associated with the opening of a gap is absent.

The self-consistency conditions for this state analogous to Eqs. (3.13)–(3.16) are

$$T/J=\chi=\frac{16}{N}\sum_q\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty}\langle 2|\mathcal{G}(E)|1\rangle e^{i\eta E}dE\right]e^{-iq_x b}$$
$$=\frac{1}{3\pi^3}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\frac{A}{(A+m_0^2B)^{1/2}}d\theta_1 d\theta_2 d\theta_3 , \tag{11.21}$$

and

$$T'/J'=\chi'=\frac{16}{N}\sum_q\left[\frac{1}{2\pi i}\int_{-\infty}^{\infty}\langle 3|\mathcal{G}(E)|1\rangle e^{i\eta E}dE\right]e^{-i(q_x+q_y)b}$$
$$=\frac{1}{\sqrt{3}\pi^2}m_0\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\frac{B}{(A+m_0^2B)^{1/2}}d\theta_1 d\theta_2 d\theta_3 , \tag{11.22}$$

where

$$A=\cos^2(\theta_1)+\cos^2(\theta_2)+\cos^2(\theta_3) , \tag{11.23}$$

and

$$B=\tfrac{1}{3}[\sin^2(\theta_1)\sin^2(\theta_2)+\sin^2(\theta_2)\sin^2(\theta_3)+\sin^2(\theta_3)\sin^2(\theta_1)] , \tag{11.24}$$

and thus

$$\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}\frac{A-6(J'/J)B}{(A+m_0^2B)^{1/2}}d\theta_1 d\theta_2 d\theta_3=0 , \tag{11.25}$$

which has a nontrivial solution for $J'/J>0.6704$. The Brillouin zone and Lagrange multiplier dispersion relation for the case of $m_0=0.2$ are shown in Fig. 8.

Let us now consider the properties of this state. Since the validity of the hypernetted chain procedure outlined

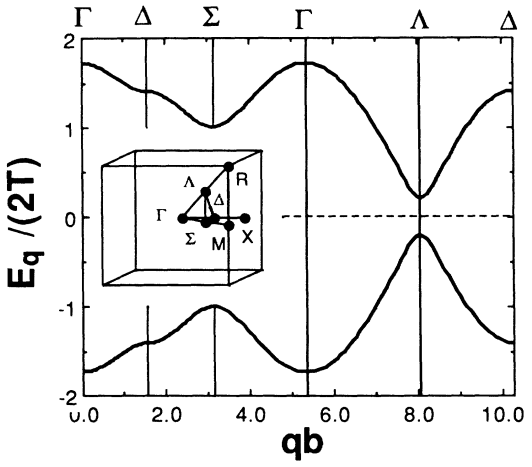

FIG. 8. Illustration of Brillouin zone for three-dimensional chiral spin liquid state and Lagrange multiplier spectrum E_q, as defined by Eq. (11.18), for $m_0=0.2$. Note the similarity to Fig. 3. Either band may be interpreted as the energy to make a spinon with a given crystal momentum.

in Sec. V is established in Tables I–III, particularly when the mass gap is large, it suffices to evaluate the spin-spin correlation of the projected state by solving Eqs. (5.19)–(5.26) with the substitution

$$2K_0(Q|\mathbf{r}|)\rightarrow\frac{e^{-Q|\mathbf{r}|}}{|\mathbf{r}|}\ , \tag{11.26}$$

as appropriate for a three-dimensional plasma. The result, together with the spinon profile evaluated by solving Eqs. (8.15)–(8.17), is shown in Table IX for the case of $m_0=1.0$. Note the similarity to Table I. As was the case with the two-dimensional system, the spin correlations become negligible beyond third neighbors, the near-neighbor correlation $\langle \mathbf{S}\cdot\mathbf{S}\rangle$ is of order -0.24, and the spinon is a compact object roughly one lattice constant in diameter.

XII. VARIATIONAL ENERGIES

Let us now consider the energetics of the chiral spin liquid in the context of the Hamiltonian of Eq. (2.1). Solving Eqs. (3.13)–(3.16) for the two-dimensional state and Eqs. (11.21)–(11.25) for the three-dimensional one, we obtain the results listed in Tables VII and VIII. The energy U, the total energy per *site* of the unprojected state, defined in the manner

$$U=\frac{1}{N}\frac{\langle\Psi_{\rm SD}|\mathcal{H}_0|\Psi_{\rm SD}\rangle}{\Psi_{\rm SD}|\Psi_{\rm SD}\rangle}=\begin{cases}-2J|\chi|^2-2J'|\chi'|^2, & d=2\\ -3J|\chi|^2-6J'|\chi'|^2, & d=3\ ,\end{cases} \tag{12.1}$$

is the quantity being minimized. The quantities m_0, χ, and χ' are also plotted in Figs. 9–11. The behaviors of the two systems are quantitatively similar. A minimum frustration of $J'/J\cong\frac{1}{2}$ is required in either case for the mass gap to open, i.e., for the magnetic order to be de-

TABLE VII. Self-consistent parameters for two-dimensional chiral spin liquid.

J'/J	m_0	χ	χ'	U/J
1.00	1.289	0.419	0.270	−0.497
0.98	1.234	0.422	0.265	−0.494
0.96	1.179	0.425	0.261	−0.491
0.94	1.126	0.427	0.256	−0.489
0.92	1.073	0.430	0.251	−0.486
0.90	1.020	0.433	0.245	−0.484
0.88	0.969	0.436	0.240	−0.481
0.86	0.918	0.439	0.234	−0.479
0.84	0.867	0.441	0.228	−0.477
0.82	0.817	0.444	0.221	−0.475
0.80	0.768	0.447	0.215	−0.473
0.78	0.720	0.449	0.207	−0.471
0.76	0.672	0.452	0.200	−0.469
0.74	0.625	0.455	0.192	−0.468
0.72	0.578	0.457	0.184	−0.467
0.70	0.532	0.460	0.175	−0.465
0.68	0.487	0.462	0.165	−0.464
0.66	0.442	0.464	0.155	−0.463
0.64	0.398	0.466	0.145	−0.462
0.62	0.354	0.469	0.134	−0.461
0.60	0.311	0.471	0.122	−0.461
0.58	0.269	0.472	0.110	−0.460
0.56	0.227	0.474	0.096	−0.460
0.54	0.186	0.475	0.082	−0.459
0.52	0.146	0.477	0.067	−0.459
0.50	0.106	0.478	0.051	−0.459
0.48	0.067	0.478	0.033	−0.459
0.46	0.034	0.479	0.018	−0.459

TABLE VIII. Self-consistent parameters for three-dimensional chiral spin liquid.

J'/J	m_0	χ	χ'	U/J
1.00	1.660	0.338	0.162	−0.499
0.98	1.552	0.342	0.157	−0.496
0.96	1.446	0.347	0.151	−0.493
0.94	1.343	0.352	0.145	−0.490
0.92	1.242	0.357	0.139	−0.488
0.90	1.144	0.361	0.132	−0.486
0.88	1.047	0.365	0.125	−0.484
0.86	0.952	0.370	0.118	−0.482
0.84	0.858	0.374	0.110	−0.480
0.82	0.767	0.378	0.102	−0.479
0.80	0.676	0.381	0.093	−0.478
0.78	0.586	0.385	0.083	−0.477
0.76	0.496	0.388	0.073	−0.476
0.74	0.406	0.391	0.062	−0.476
0.72	0.313	0.394	0.049	−0.475
0.70	0.216	0.396	0.035	−0.475
0.68	0.101	0.397	0.017	−0.475

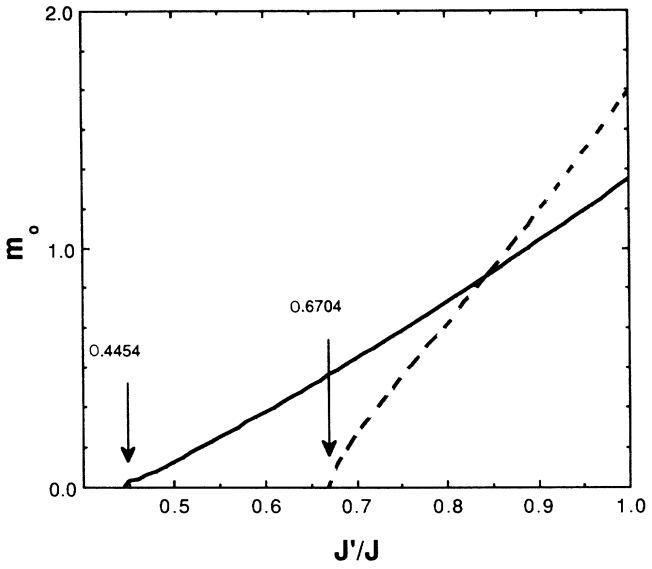

FIG. 9. Self-consistent mass gap m_0, as defined by Eqs. (3.16) and (11.25) for two-dimensional (solid) and three-dimensional (dashed) chiral-spin-liquid state, vs frustration parameter J'/J. These results are also listed in Tables VIII and IX.

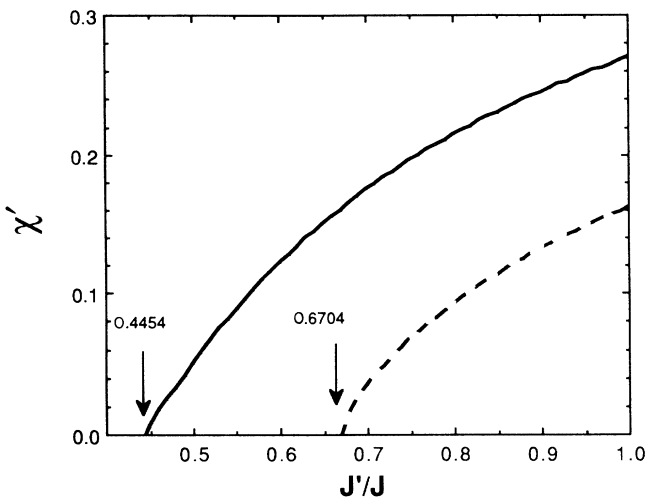

FIG. 11. Second-neighbor hopping matrix element χ', as defined by Eqs. (3.14) and (11.22), for two-dimensional (solid) and three-dimensional (dashed) chiral-spin-liquid state, vs frustration parameter J'/J. These results are also listed in Tables VIII and IX.

stroyed. The value of χ associated with the $m_0=0$ state is 0.4790 for two dimensions (2D) and 0.3979 for three. U is approximately $-0.48J$ over the entire range. The true variational energy per site, defined in the manner

$$U_G=\frac{1}{N}\frac{\langle\Psi_{\rm SD}|\Pi_G\mathcal{H}_0\Pi_G|\Psi_{\rm SD}\rangle}{\langle\Psi_{\rm SD}|\Pi_G|\Psi_{\rm SD}\rangle}=\begin{cases}\frac{3}{2}Jh_G(1)+\frac{3}{2}J'h_G(2), & d=2\\ \frac{9}{4}Jh_G(1)+\frac{9}{2}J'h_G(2), & d=3\end{cases} \tag{12.2}$$

is listed for the two-dimensional system in Table X. Three features of this result should be noted: (1) The overall magnitudes of U and U_G are comparable. (2) Frustration has the opposite effect on U and U_G. One goes up and the other goes down. (3) The amount of energy gained by opening the mass gap is comparable in the two cases. We define the latter in the manner

$$\delta U=\begin{cases}U+0.479J, & d=2\\ U+0.379J, & d=3\end{cases}, \tag{12.3}$$

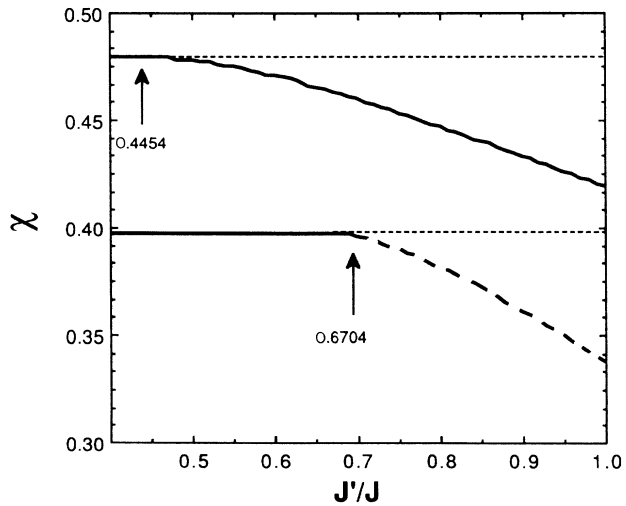

FIG. 10. Near-neighbor hopping matrix element χ, as defined by Eqs. (3.13) and (11.21), for two-dimensional (solid) and three-dimensional (dashed) chiral-spin-liquid state, vs frustration parameter J'/J. These results are also listed in Tables VIII and IX.

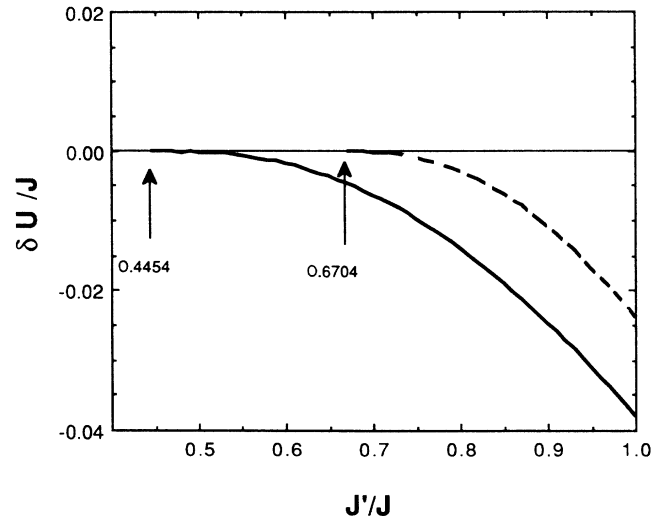

FIG. 12. Energy δU gained by opening the mass gap, as defined by Eq. (1.3), for two-dimensional (solid) and three-dimensional (dashed) chiral-spin-liquid state, vs frustration parameter J'/J. These results are also listed in Tables VIII and IX.

TABLE IX. Analog of Table I for three-dimensional chiral spin-liquid state with $m_0=1$. $h_G(\mathbf{r})$ is evaluated using Eqs. (5.19)–(5.26), with the substitution indicated in Eq. (11.26). $h(\mathbf{r})$ is the correlation function of the unprojected state. The coupling constant of the unprojected state, defined by Eq. (5.16), is $\Gamma=1.55$. $h_s(\mathbf{r})$ is the spinon profile, as given by the solution of Eqs. (8.15)–(8.17). $\delta h_G(\mathbf{r})$ is evaluated using Eqs. (12.7) and (12.8).

Shell	l	m	n	$v_s(\mathbf{r})$	$h(\mathbf{r})$	$h_G(\mathbf{r})$	$h_s(\mathbf{r})$	$N\delta h_G(\mathbf{r})$
0	0	0	0	0.000	−1.000	−1.000	−1.000	0.000
1	1	0	0	0.105	−0.125	−0.217	−0.006	0.365
2	1	1	0	−0.006	−0.015	−0.017	0.003	−0.095
3	1	1	1	−0.034	0.000	0.003	0.000	0.008
4	2	0	0	0.035	0.000	0.022	0.001	−0.072
5	2	1	0	0.006	0.000	−0.002	0.000	0.014
6	2	1	1	−0.007	0.000	−0.001	0.000	0.000
7	2	2	0	−0.001	0.000	0.000	0.000	−0.002
8	2	2	1	−0.005	0.000	0.000	0.000	0.000
9	3	0	0	0.012	0.000	−0.003	0.000	0.011

and for two dimensions only

$$\delta U_G = U_G + 0.600J = 0.245J' \ . \tag{12.4}$$

δU is also plotted in Fig. 12. The similarity of δU and δU_G is important in light of the inability of the hypernetted chain equations to accurately describe $h_G(\mathbf{r})$ when m_0 becomes small. It enables us to estimate that the energy gained in opening the mass gap of the three-dimensional system must be roughly twice δU, or $-0.05J$, when $m_0=1$. Table X also shows that the hypernetted chain gives a reliable value of U_G when m_0 is large. From Table IX we thus obtain $-0.56J$ for the three-dimensional system at $m_0=1$. Both of these numbers are comparable to the corresponding two-dimensional values listed in Table X.

It has been pointed out to us by Rokhsar[23] that the unprojected variational energy U, while an extremum, is not the absolute minimum because the energy of the "dimer" state is lower. To make the dimer state, we imagine pairing each site with exactly one of its near neighbors and then looking for a solution for which χ, as defined in Eq. (3.13), is a constant for the dimer neighbor and zero otherwise. Self-consistency is achieved when this constant is 1, giving $U^{\text{dimer}}=-0.5J$. However, when projected, this wave function gives an energy of $U_G^{\text{dimer}}=-0.375J$, which is higher than U_G. Even though this particular dimer state has an uncompetitive projected energy, the fact that its unprojected energy is lower than U casts doubt on the validity of the entire variational procedure, beginning with Eq. (2.7). It is conceivable that the approximation of Eq. (2.6) by Eq. (2.7) works well near some extrema but not others, but this is not clear at present.[24] One fact supporting the validity of the calculation is the similarity of the value of U_G at $J'/J=0$, namely $0.572J$, to the classical Néel energy of $-0.5J$ and the "exact" energy[25] of $-0.668J$.

Let us now consider the variational energy of the spinons. Kalmeyer and Laughlin[4] found a dispersion relation for spinons qualitatively similar to that of Fig. 3. Specifically, they reported a Brillouin zone half the linear dimension of the full zone, a center-of-mass of the band of roughly J, a dispersion of roughly $0.2J$ across the zone, and a minimum at the corner of the reduced zone. Because of the difficulty of the calculation, particularly in the absence of analyticity of the wave functions, we will not attempt to compute the entire dispersion relation, but rather only its center of mass, which is properly compared to the unprojected energy

$$\Delta = \frac{2T}{\pi^2}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2}(A+m_0^2B)^{1/2}d\theta_1 d\theta_2 \ , \tag{12.5}$$

in the two-dimensional system, with A and B defined as in Eq. (3.15). The projected energy is given by

$$\Delta_G = \tfrac{3}{2}JN\delta h_G(1)+\tfrac{3}{2}J'N\delta h_G(2) \ , \tag{12.6}$$

where $\delta h_G(\mathbf{r})$ is the change to $h_G(\mathbf{r})$ caused by the presence of a spinon localized at a site. Kalmeyer and Laughlin showed that the hypernetted chain approximation to $\delta h_G(\mathbf{r})$, obtained by solving the equations

$$\delta h_G(r)=[1+h_G(\mathbf{r})][\delta h_G(\mathbf{r})-\delta c_G(\mathbf{r})] \ , \tag{12.7}$$

TABLE X. Total energy per site U and U_G, as defined in Eqs. (12.1) and (12.2) and energy gained by opening the mass gap, as defined in Eqs. (12.3) and (12.4).

J'/J	m_0	U_G/J	$\delta U_G/J$	U/J	$\delta U/J$
0.892	1.0	−0.438±0.002	−0.060±0.002	−0.483	−0.024
0.686	0.5	−0.441±0.001	−0.012±0.001	−0.464	−0.005
0.497	0.1	−0.478±0.001	−0.002±0.001	−0.459	0.000

TABLE XI. $N\delta h_G(\mathbf{r})$ calculated using Eqs. (12.7) and (12.8).

Shell	l	m	$m_0=1.0$	$m_0=0.5$	$m_0=0.1$
1	1	0	0.404	0.510	0.586
2	1	1	−0.047	−0.208	−0.322
3	2	0	−0.106	−0.147	−0.201
4	2	1	−0.017	0.048	0.115
5	2	2	0.006	−0.015	−0.055
6	3	0	0.023	0.036	0.069
7	3	1	0.007	−0.013	−0.047
8	3	2	0.000	0.005	0.026
9	4	0	−0.005	−0.009	−0.026

and

$$[1-\tfrac{1}{2}\hat{c}_G(\mathbf{q})]\delta h_G(\mathbf{q})=[1+\tfrac{1}{2}\hat{h}_G(\mathbf{q})]+\frac{1}{N}\hat{c}_s(\mathbf{q})\hat{h}_s(\mathbf{q})\ , \tag{12.8}$$

with Fourier transforms defined by Eq. (5.13), is quantitatively correct. Evaluating these expressions, we obtain the results shown in Tables XI and XII. The similarity between Δ and Δ_G is surprising in light of the fact that the unprojected wave function strongly weights configurations for which the expected charged on the site is 1, configurations which are destroyed by Π_G. From Table IX and an appropriately modified version of Eq. (12.5) we obtain $\Delta=0.96J$ and $\Delta_G=0.45J$ at $m_0=1$ for the three-dimensional state.

It should be remarked that the chiral order parameter $\langle \mathbf{S}_1\cdot(\mathbf{S}_2\times\mathbf{S}_3)\rangle$, where sites 1, 2, and 3 form a near-neighbor triangle, is much larger for the projected state than the unprojected one. For the unprojected state, we have

$$\tfrac{1}{8}\sum_{\alpha,\beta}\sum_{\mu,\nu}\sum_{\rho,\sigma}\langle\alpha|\sigma_1|\beta\rangle\cdot(\langle\mu|\sigma_2|\nu\rangle\times\langle\rho|\sigma_3|\delta\rangle)\times\langle c^{\dagger}_{1\alpha}c_{1\beta}c^{\dagger}_{2\mu}c_{2\nu}c^{\dagger}_{3\rho}c_{3\delta}\rangle=\tfrac{3}{8}\chi^2\chi'\ , \tag{12.9}$$

which gives a value of 0.018 when $m_o=1$. This is appropriately compared with the value 0.074 obtained[5] for the Kalmeyer-Laughlin wave function on a square lattice.

XIII. MAGNETIC MONOPOLES

Let us now ask what the analog of $\frac{1}{2}$ fractional statistics might be in three dimensions. We know that the chiral-spin-liquid vacuum has correlations similar to those of the two-dimensional system. We know that wave functions for the spinons exist that these are compact spin-$\frac{1}{2}$ particles. If we assume the plasmon analogy to be valid, which is both reasonable and likely, we also know that the normalization integral of a spinon pair *grows* with separation. This growth is an effect of the Gutzwiller projector and is, in light of Eqs. (9.14) and (9.16), an indication that the set of localized spinon wave functions is overcomplete and possesses a complex overlap matrix. Overcompleteness is also indicated by the similarity between Figs. 3 and 8. Let us now imagine, as illustrated in Fig. 13, that the bonds between vertical layers become weaker as one approaches the sample edges, so that the layers effectively decouple. Since anisotropy in the coupling does not destroy the gap, at least at the mean-field level, adiabatic transport of the spinons to and from the decoupled region should be well defined. Accordingly, we can imagine transporting the particles to the boundary, interchanging them there, and transporting them back, resulting in a Berry phase of $\pi/2$. While it is not clear that the same phase results when the particles are interchanged in the interior, a reasonable guess is that it does. The qualitative behavior of the Berry phase inferred from this thought experiment is exactly that associated with a Dirac monopole of unit charge. As illustrated in Fig. 13, the Dirac quantization condition may

TABLE XII. Energy gained by adding a localized spinon, as defined by Eqs. (12.7) and (12.8) using the results of Tables VIII and XI.

m_0	Δ/J	Δ_G/J
1.0	0.969	0.543
0.5	0.929	0.550
0.1	0.918	0.638

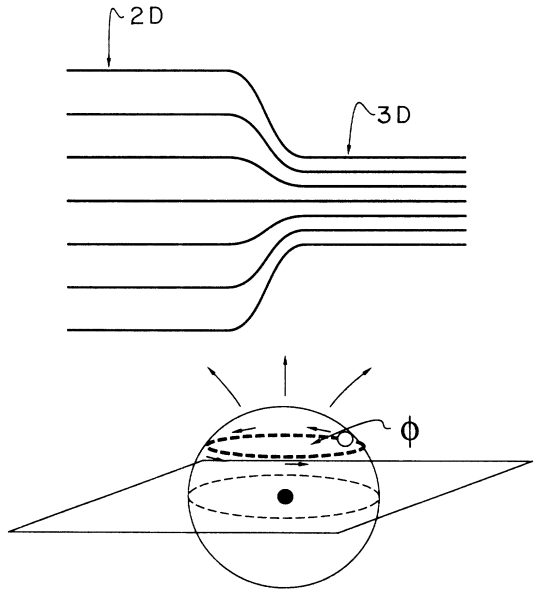

FIG. 13. Top: Illustration of thought experiment in which layers of three-dimensional chiral spin liquid are unleaved as one approaches the sample edge. A pair of spinons adiabatically transported to the edge, exchanged, and then transported back to the interior give a Berry phase of $\pi/2$. Bottom: Illustration of the Dirac condition for monopole charge. The Berry phase for evolving a test particle in a closed path is the flux enclosed by the path. Consistency requires that the flux through the upper cap equal the flux through the lower cap, modulo 1 flux quantum. When the monopole charge is 1, the Berry phase associated with an equatorial path is π, which amounts to $\pi/2$ for interchange along this path.

be understood[26] as requiring the Berry phase of an electrically charged test particle moving in a closed path to be well defined. This will be the case provided that the magnetic flux passing though any surface bounded by the closed path is the same, modulo a flux quantum. If the total flux emitted by the monopole is one quantum, then the flux passing through a surface bounding an equatorial path is $\frac{1}{2}$, which gives a Berry phase of π when one particle is moved in a circle about the other. This is the same result obtained in the two-dimensional case.

Since fractional statistics is known not to make sense in three dimensions, it is important to emphasize that even and odd planes of the three-dimensional chiral spin liquid have the opposite handedness. Thus, if spinons are monopoles, they are unusual in that the location of one spinon in the unit cell is sensed by a test spinon far away. While the meaning of this is not presently clear, one possibility is that the "potential" generated by a spinon is associated with a gauge group larger than U(1).[27] Behavior of this kind, broadly construed, has been suggested[28] by studies of nonabelian gauge theories with parity violating terms in their Lagrangians. There is also precedent in the high-energy literature[29] for condensation of charged monopoles, or "dyons,"[30] into superfluid phases. All of these features of the problem need to be clarified in future work.

XIV. SUMMARY

In this paper, we have given a detailed account of properties of the chiral spin liquid state proposed by Wen. Wilczek, and Zee. We have reviewed the variational reasoning leading to this state, emphasizing that its correctness is not yet established. We have shown that one of this class of states exactly equals the Kalmeyer-Laughlin state, and that the spinon excitations are also the same. We have demonstrated explicitly that spinons obey $\frac{1}{2}$ fractional statistics. We have introduced a simple technique for computing properties of these states and shown that the approximations inherent in the technique are valid. We have shown that variational energies associated with the true Gutzwiller projected version of the states are comparable in magnitude to those of the unprojected states. We have made the case that chiral symmetry breaking is essential for a spin liquid to make sense, and, in particular, that the Affleck-Marston flux phase is ordered. Finally, we have repeated the Wen-Wilczek-Zee calculation in three dimensions and found the properties of the three-dimensional state to be similar to those of the two-dimensional one. In particular, we find that three-dimensional spinons are well-defined spin-$\frac{1}{2}$ particles roughly one lattice constant in diameter when the mass gap is 1. We have argued that the analog of $\frac{1}{2}$ fractional statistics in this system is behavior associated with a Dirac monopole.

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