PHYSICAL REVIEW B VOLUME 41, NUMBER 10 1 APRIL 1990

Dependence of some electromagnetic properties of superconductors on coupling strength

F. Marsiglio,* J. P. Carbotte, and J. Blezius†
Physics Department, McMaster University, Hamilton, Ontario, Canada L8S 4M1
(Received 31 August 1989)

We have calculated select electromagnetic properties for many real superconductors based on tunneling-derived electron-phonon spectral densities. We use this data to fit coefficients in semi-phenomenological forms derived through a series of approximations to the exact microscopic expressions. It is found that the derived forms represent well the strong-coupling corrections.

I. INTRODUCTION

Expressions for the electromagnetic properties of a strong-coupling superconductor valid within Eliashberg theory were derived by Nam.[1] Perhaps because of the complexity of the formulas involved and the fact that finite-temperature solutions to the Eliashberg gap equations are needed as input, few numerical results have been obtained so far for specific systems. Nam[2] himself calculated the zero-temperature frequency dependent conductivity for Pb which was later reconsidered by Swihart and Shaw.[3,4] Blaschke and Blocksdarf[5] calculate the surface resistance of Sn, Pb, Nb, and amorphous Ga. The temperature dependence of the dc Josephson current in a superconductor-insulator-superconductor (SIS) tunnel junction was measured and calculated by Lim *et al.*[6] in Pb and calculated by Vashishta and Carbotte[7] in $Pb_{0.9}Bi_{0.1}$. The temperature dependent electromagnetic coherence length was estimated by Kerchner and Ginsberg[8] as well as its zero-temperature reduction over the BCS value. The zero-temperature reduction in the dc Josephson current was considered by Ginsberg *et al.*[9] and by Carbotte and Vashishta.[10] Blezius and Carbotte[11] have calculated the temperature dependent London penetration depth for several impurity concentrations in V_3Si.

In this paper, we wish to consider the London penetration depth, the electromagnetic coherence length, and the dc Josephson current for an SIS junction, a quantity also related to the local penetration depth. The aim is to calculate these quantities for many different materials from a knowledge of their spectral density $\alpha^2F(\omega)$ and Coulomb pseudopotential μ^*. These quantities are known from tunneling inversion.[12] A second aim is to derive, from microscopic theory, simple semiphenomenological formulas involving the single parameter $T_c/\omega_{\ln}$,[13,14] and fit the unknown coefficients in the resulting form to the real material data so as to provide a simple useful approximate formula for strong-coupling corrections.

In Sec. II, we introduce formalism and formulas for the various electromagnetic properties of interest which depend on solutions of the Eliashberg equations. In Sec. III, we consider the dc Josephson current as well as the local limit penetration depth quantities which are related by an appropriately constant factor. Section IV is concerned with the penetration depth in the London limit, while Sec. V. deals with the electromagnetic coherence length. Conclusions are found in Sec. VI.

II. FORMALISM

The equations for the Matsubara gaps $\Delta(i\omega_n)$ and renormalization factors $Z(i\omega_n)$ are[15,16]

$$\Delta(i\omega_n)Z(i\omega_n)=\pi T\sum_{m=-\infty}^{\infty}[\lambda(i\omega_n-i\omega_m)-\mu^*(\omega_c)\theta(\omega_c-|\omega_m|)]\times\frac{\Delta(i\omega_m)}{[\omega_m^2+\Delta^2(i\omega_m)]^{1/2}} \quad (1)$$

and

$$Z(i\omega_n)=1+\frac{\pi T}{\omega_n}\times\sum_{m=-\infty}^{\infty}\lambda(i\omega_n-i\omega_m)\frac{\omega_m}{[\omega_m^2+\Delta^2(i\omega_m)]^{1/2}}\ , \quad (2)$$

where the Matsubara frequencies are $i\omega_n=i\pi T(2n-1)$, $n=0,\pm1,\pm2,\ldots$. The electron-phonon spectral density $\alpha^2F(\nu)$ appears through the relation

$$\lambda(z)=\int_0^\infty\frac{2\nu\,d\nu\,\alpha^2F(\nu)}{\nu^2-z^2}\ , \quad (3)$$

and $\mu^*(\omega_c)$ is the Coulomb pseudopotential with cutoff ω_c.

Electromagnetic properties of superconductors were discussed in the original paper of Bardeen, Cooper, and Schrieffer[17] within pairing theory. The work was extended to strong-coupling form by Nam[1] who gave formulas for many electromagnetic properties and also gave limited numerical results for the specific case of Pb. The basic formulas are complex and a solution of Eq. (1) and (2) is fundamental to evaluating properties.

Within linear response theory, the Fourier transform of the current density[1]

$$J_\mu(q;\omega)=-K_{\mu\nu}(q;\omega)A^\nu(q;\omega)\ , \quad (4)$$

where $\mu=1,2,3$ corresponds to components $x\ y\ z$ of the vector $\underline{J}$ and q,ω are momentum and frequency, respectively. In Eq. (4), $A^{\nu}(q,\omega)$ is the ν'th component of the vector potential **A** describing the electromagnetic field. $K_{\mu\nu}(q,\omega)$ is a tensor which gives the current response function and is the Fourier transform of the current-current correlation function.

Electromagnetic properties of interest here are expressible in terms of the kernel $K_{\mu\nu}(q;\omega)$ of Eq. (4). For example, the frequency dependent penetration depth for the penetration of a magnetic field into the surface of a bulk superconductor is given, for the case of specular reflection, by the formula[1]

$$\lambda(T,\omega)=\frac{2}{\pi}\int_0^\infty dq\frac{1}{q^2+K(q;\omega)/4\pi}\ , \tag{5}$$

where we have assumed isotropy and have made explicit the temperature dependence. What is most often quoted as the penetration depth is the frequency $\omega\to 0$ limit of (6) which describes the static situation and is denoted by

$$\lambda(T)=\lim_{\omega\to 0}\lambda(T,\omega)\ . \tag{6}$$

Moreover, simple expressions for $\lambda(T)$ are possible in limiting cases. In a dirty superconductor, the mean free path (l) of an electron can be greatly reduced due to the increased scattering probability of impurities or other defects. In that instance, the electromagnetic response becomes local which implies that the $q\to 0$ limit of $K(q,\omega)$ is the important quantity. The integration in Eq. (5) weights most the $q=0$ part and we get[1]

$$\lambda_l(T)=\left[4\pi\sigma_N T\sum_{n=1}^{\infty}\frac{\Delta^2(i\omega_n)}{\omega_n^2+\Delta^2(i\omega_n)}\right]^{-1/2}\ , \tag{7}$$

a quantity that depends only on the Matsubara gaps $\Delta(I\omega_n)$. In Eq. (7), the subscript l stands for local limit and σ_N is the normal-state conductivity due to impurity scattering which is given by

$$\sigma_N=\tfrac{2}{3}N(0)e^2v_F^2\tau_N\ , \tag{8}$$

with $N(0)$ the single spin electronic density of states at the Fermi surface, e the charge on the electron, v_F the electron Fermi velocity, and τ_N the impurity lifetime. The local limit, for which Eq. (7) holds, is generally characterized by the condition $\xi(0)\gg l$ where $\xi(0)$ is the zero-temperature coherence length to be introduced later.

Except for an appropriate change of the numerical coefficient and proportionality constants in Eq. (7), $\lambda_l^{-2}(T)$ also gives the critical dc Josephson current $[J_c(T)]$ observed in a superconductor-insulator-superconductor (SIS) tunnel junction so that[18]

$$\frac{J_c(T)}{J_c(0)}=\left[\frac{\lambda_l(0)}{\lambda_l(T)}\right]^2\ . \tag{9}$$

When nonlocal effects come into play, the Pippard or London limits are often introduced. The Pippard limit is characterized by $\lambda\ll\xi(0)$ while the London limit implies $\lambda\gg\xi(0)$. The Pippard limit is determined through the zero frequency limit of Eq. (5) where most of the contribution comes from the $q\to\infty$ region of the integral. The result is[1]

$$\lambda_P(T)=\frac{4}{3\sqrt{3}}\left[\frac{3\pi^2}{v_F}\frac{n}{m}e^2T\sum_{n=1}^{\infty}\frac{\Delta^2(i\omega_n)}{\omega_n^2+\Delta^2(i\omega_n)}\right]^{-1/3}\ , \tag{10}$$

where n is the free electron density and m the electron mass. It is clearly related to the local limit and will not be discussed further here. The London limit which depends mainly on the $q\to 0$ limit of the electromagnetic response function is given by[1]

$$\lambda_L(T)=\left[\tfrac{4}{3}\pi N(0)e^2v_F^2T\sum_{n=1}^{\infty}\frac{\Delta^2(i\omega_n)}{Z(i\omega_n)[\omega_n^2+\Delta^2(i\omega_n)]^{3/2}}\right]^{-1/2}\ , \tag{11}$$

where we see that the renormalization factor $Z(i\omega_n)$ of Eq. (2) now enters explicitly. It was not needed in the local limit case.

Another quantity of interest is the electromagnetic coherence length $\xi(T)$ which describes the nonlocality in the response of a superconductor to an electromagnetic field. It is given by the formula

$$\lim_{q\to\infty}\frac{qK(q,0)}{K(0,0)}=\frac{3\pi}{4\xi(T)}\ , \tag{12}$$

and more explicitly[19]

$$\xi(T)\frac{\frac{v_F}{2}\sum_{n=1}^{\infty}\frac{\Delta^2(i\omega_n)}{Z(i\omega_n)[\omega_n^2+\Delta^2(i\omega_n)]^{3/2}}}{\sum_{n=1}^{\infty}\frac{\Delta^2(i\omega_n)}{\omega_n^2+\Delta^2(i\omega_n)}}\ , \tag{13}$$

and is therefore related to the ratio of local and London limit penetration depth. Explicitly

$$\left[\frac{\lambda_l(T)}{\lambda_L(T)}\right]^2=\frac{\xi(T)}{l}\ , \tag{14}$$

which is just the ratio of coherence length to mean free path $l=v_F\tau$. Local, London, and coherence lengths will be discussed in turn in the following three sections (III–V). To calculate these quantities for any given spectral density $\alpha^2F(\omega)$ and Coulomb pseudopotential μ^*, it is necessary to have numerical solutions of Eqs. (1) and (2) for the Matsubara gaps $\Delta(i\omega_n)$ and renormalization factors $Z(i\omega_n)$. Besides precise numerical solutions of these equations, we will also want to consider approximate analytic solutions which contain a first correction for strong-coupling corrections to the BCS results for the corresponding quantity. To obtain such approximate expressions for the above-mentioned electromagnetic properties, we follow our previous work on thermodynamic properties and use a model for the gap Δ and for the renormalization Z, namely[14]

$$\Delta(i\omega_n = \Delta_0(T), \quad |\omega_n| < \omega_0 \, , \qquad = \Delta_\infty, \quad |\omega_n| > \omega_0 \, , \tag{15}$$

and

$$Z(i\omega_n) = Z_0(T), \quad |\omega_n| < \omega_0 \, , \qquad = 1, \quad |\omega_n| > \omega_0 \, , \tag{16}$$

where ω_0 represents a frequency which is taken to be roughly a few times the maximum phonon energy in the system. We will introduce as well a characteristic phonon energy $\omega_{\ln}$ which is related to $\alpha^2F(\omega)$ through the definition[13]

$$\ln\omega_{\ln} = \left[\frac{2}{\lambda} \int_0^\infty \frac{\ln(\omega)\alpha^2F(\omega)}{\omega} d\omega \right] . \tag{17}$$

This characteristic frequency has already played an important role in the work of Allen and Dynes[13] on critical temperature and of Marsiglio and Carbotte[14] on thermodynamic properties. In the work to be described below and particularly in the Appendix, we will assume that $\omega_{\ln}/\omega_0 \ll 1$ and also that $T_c/\omega_{\ln} \ll 1$. The BCS limit corresponds to the $T_c/\omega_{\ln} \to 0$ limit. With these approximations, we can obtain, after considerable algebra, solutions for $\Delta_0(T)$ and $Z_0(T)$. The essential derivations are outlined in the Appendix. The results will be used in the next three sections.

III. LOCAL PENETRATION DEPTH AND dc JOSEPHSON CURRENT

For both local limit penetration depth and dc Josephson current, the relevant quantity that is to be calculated is [see Eq. (7) and (9)]

$$I_l = \pi T \sum_{n=1}^{\infty} \frac{\Delta^2(i\omega_n)}{\omega_n^2 + \Delta^2(i\omega_n)} \, . \tag{18}$$

We will start with T near T_c. In conformance with the discussion in the Appendix, we can neglect the n dependence of $\Delta(i\omega_n)$, since the sum in (18) converges, and get for the model gap defined in Eq. (15) and (16)

$$I_l = \frac{\pi\Delta_0^2(T)}{8T} \, , \tag{19}$$

where to lowest order in $(T - T_c)$ we get from the Appendix equation (A19)

$$\Delta_0^2(T) = \frac{F'(T)}{G(T)}(T_c - T) \, , \tag{20}$$

and so for $T \to T_c$

$$Y_l(T) \equiv \lambda_l^{-2}(T) = 4\sigma_N I_l = \frac{\pi\sigma_N}{2} \frac{F'(T_c)}{G(T_c)}(1-t) \, , \tag{21}$$

with $t = T/T_c$, the reduced temperature. Reference to Eq. (A17) gives, after some simple algebra,

$$\frac{F'(T_c)}{G(T_c)} = \frac{8\pi^2}{7\zeta(3)} T_c \left\{ 1 + \frac{2(\pi T_c)^2}{\lambda} \left[\left[1 + \frac{6}{7\zeta(3)} \right] a(T_c) - \left[\frac{8}{6} + \frac{2}{7\zeta(3)} \right] b \right] \right\} . \tag{22}$$

On substituting Eqs. (A20) and (A21) into (22), we get

$$T_c \frac{Y_l'(T_c)}{4\sigma_N} = \frac{\pi^3}{7\zeta(3)} T_c \left\{ 1 + 2\pi^2 \left[\frac{T_c}{\omega_{\ln}} \right]^2 \alpha_1 \left[\ln \left[\frac{1.13\omega_{\ln}}{K_B T_c} \right] - \frac{\alpha_2}{\alpha_1}(1.57) \right] \right\} , \tag{23}$$

where α_1 and α_2 are to be treated as arbitrary parameters.

Next, we consider the $T=0$ limit. In this case sums over Matsubara frequencies are replaced by integrals with $\pi T \sum_{n=1}^{\infty} \to \frac{1}{2}\int_0^\infty d\omega$ so that

$$\frac{Y_l(0)}{4\sigma_N} = \int_0^\infty \frac{d\omega}{2} \frac{\Delta^2(\omega)}{\omega^2 + \Delta^2(\omega)} \, . \tag{24}$$

If we ignore the ω dependence of the gap, this integral becomes

$$\frac{Y_l(0)}{4\sigma_N} = \frac{\pi}{4}\Delta_0(T=0) \, ,$$

where $\Delta_0(T=0)$ is given in the Appendix[20–23,14] by Eq. (A34),

$$\frac{2\Delta_0(T=0)}{k_B T_c} = 3.53 \left[1 + a_3 \left[\frac{T_c}{\omega_{\ln}} \right]^2 \ln \left[\frac{\omega_{\ln}}{b_3 T_c} \right] \right] , \tag{25}$$

where a_3 and b_3 are again to be thought of as arbitrary constants. A best fit to data on real materials gives $a_3 = 12.5$ and $b_3 = 2$. Such formula have already appeared in the literature for the gap to critical temperature ratio, for some thermodynamic coefficients as well as other quantities.[20–23,14]

From Eq. (23) and (25) we get

$$\frac{Y_l(0)}{T_c|Y_l'(T_c)|} = 0.376 \left[1 - a_1 \left[\frac{T_c}{\omega_{\ln}} \right]^2 \ln \left[\frac{\omega_{\ln}}{b_1 T_c} \right] \right] , \tag{26}$$

where a_1 and b_1 are related to α_1, α_2, α_3, and b_3, but, as these are treated as arbitrary, we need not exhibit this relationship explicitly. In Fig. 1 (solid dots), we show results for $Y_l(0)/T_c|Y_l'(T_c)|$ in real materials based on tunneling-derived kernels and complete numerical solutions of the Eliashberg equations (1) and (2) and the exact evaluation of formula (18). The materials in increasing

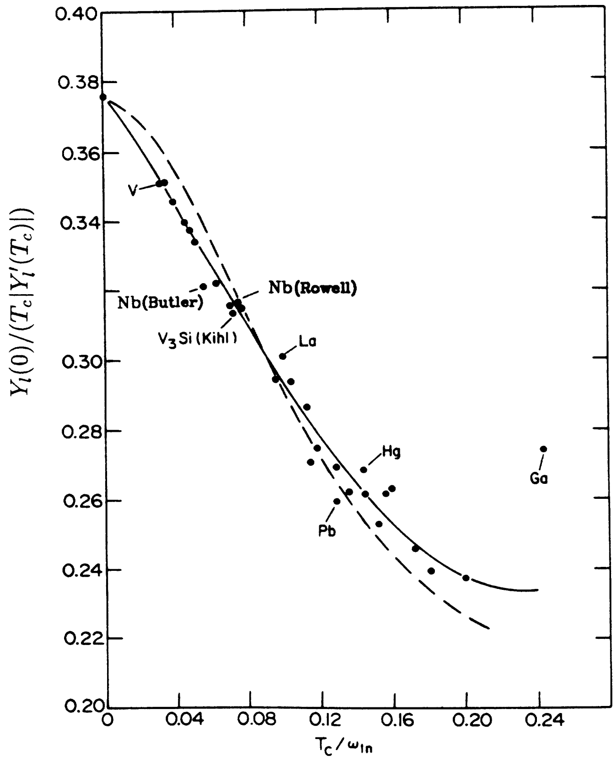

FIG. 1. The ratio $Y_l(0)/T_c|Y_l'(T_c)|$ vs $T_c/\omega_{\ln}$. See text for identification of materials. Note that amorphous Ga has also been included and deviates substantially from the trend. The solid curve corresponds to

$$\frac{Y_l(0)}{T_c|Y_l'(T_c)|}=0.376\left[1-1.5\frac{T_c}{\omega_{\ln}}-7.6\left[\frac{T_c}{\omega_{\ln}}\right]^2\ln\left[\frac{\omega_{\ln}}{4T_c}\right]\right].$$

Note that a linear term has been required for an accurate fit. The dashed curve corresponds to a model Einstein spectra for $\alpha^2F(\omega)$, the model spectra upon which our derivations are based. The trend is quite similar to that of the real materials. Note, however, that the initial decrease from BCS is more quadratic, and hence no linear term would be required. Thus, it appears that the effect of the realistic shapes used has been to produce a linear correction below the BCS value.

order of $T_c/\omega_{\ln}$ are Al, V, Ta, Sn, Tl, $Tl_{0.9}Bi_{0.1}$, In, Nb (Butler), Nb (Arnold), $V_3Si(1)$, V_3Si (Kihl), Nb (Rowell), Mo, $Pb_{0.4}Tl_{0.6}$, La, V_3Ga, $Nb_3Al(2)$, $Nb_3Ge(2)$, $Pb_{0.6}Tl_{0.4}$, Pb, $Nb_3Al(3)$, $Pb_{0.8}Tl_{0.2}$, Hg, Nb_3Sn, $Pb_{0.9}Bi_{0.1}$, $Nb_3Al(1)$, $Nb_3Ge(1)$, $Pb_{0.8}Bi_{0.2}$, $Pb_{0.7}Bi_{0.3}$, and $Pb_{0.65}Bi_{0.35}$. For more details of these spectra, including their origin, the reader is referred to the previous work of Marsiglio and Carbotte.[14] The solid line in Fig. 1 is a visual best fit through the exact numerical data of the form

$$\frac{Y_l(0)}{T_c|Y_l'(T_c)|}=0.376\left[1-1.5\left[\frac{T_c}{\omega_{\ln}}\right]-7.6\left[\frac{T_c}{\omega_{\ln}}\right]^2\ln\left[\frac{\omega_{\ln}}{4T_c}\right]\right], \tag{27}$$

where we have added, purely on a phenomenological basis, a linear term to the derived form (26). Such a term is needed to get a really good fit. It can be thought of as necessary to compensate for the approximations made during the derivation of (26). This derivation should be more accurate for a δ function spectrum. Such results are shown as the dashed line in Fig. 1. We note that near $T_c/\omega_{\ln}\rightarrow 0$ this curve is more quadratic and does not indicate the need for a linear term which can then be interpreted to be resulting from the frequency spread that occurs in real spectra. Phonon frequencies are normally spread over a significant frequency range. To end, we note that amorphous Ga falls way off the main trend curve. This is not surprising since such a spectrum exhibits considerable weight at low energies and hence our approximation $\nu/T_c \gg 1$ does not apply for all important frequencies (ν).

IV. LONDON LIMIT PENETRATION DEPTH

In analogy with the procedure followed in the preceding section, we start by introducing a quantity $I_L(T)$ which does not depend on external normal-state parameters, but depends only on the solutions of the Eliashberg equations (1) and (2). That is, we write

$$I_L(T)=\frac{\lambda_L^{-2}(T)}{\frac{4}{3}N(0)e^2v_F^2}=\pi T\sum_{n=1}^{\infty}\frac{\Delta^2(i\omega_n)}{Z(i\omega_n)[\omega_n^2+\Delta^2(i\omega_n)]^{3/2}}. \tag{28}$$

For T near T_c, the sum in (28) is sufficiently convergent that we can replace $Z(i\omega_n)$ by its $n=1$ value which is $(1+\lambda)$. This also holds for $\Delta(i\omega_n)$ and we get

$$I_L(T)=-\frac{7\zeta(3)\Delta_0^2(T)}{8(1+\lambda)(\pi T)^2}, \tag{29}$$

from which we conclude that

$$\frac{T_cY_l'(T_c)}{\frac{4}{3}N(0)e^2v_F}=\frac{1}{1+\lambda}\left\{1+\frac{2(\pi T_c)^2}{\lambda}\left[\left[1+\frac{6}{7\zeta(3)}\right]a(T_c)-\left[\frac{4}{3}+\frac{1}{7\zeta(3)}\right]b\right]\right\}, \tag{30}$$

where use was made of formula (22) and by definition $Y_L(T)\equiv\lambda_L^{-2}(T)$. A similar formula was derived by Masharov[24] using very different methods.

At zero temperature, we get, from Eq. (28) on converting the sums to integrals, the result

$$I_L(0)=\int_0^{\infty}\frac{d\omega}{2}\frac{\Delta^2(\omega)}{Z(\omega)[\omega^2+\Delta^2(\omega)]^{3/2}}. \tag{31}$$

Ignoring the implicit ω dependence in both Z and Δ and replacing them by their $\omega=0$ value leads to a known integral and

$$I_L(0)=\frac{1}{2Z_S(0)} . \tag{32}$$

The quantity $Z_S(0)$ is calculated, for our model solution (15) and (16), in the Appendix. Equation (A28) with $\omega=0$ gives

$$Z_S(0)=1+\lambda+\Delta_0^2(T=0)\int_0^\infty 2\nu\,d\nu\,\alpha^2F(\nu)\frac{1}{\nu 4}\left[1-\ln\left[\frac{2\nu}{\Delta_0(T=0)}\right]\right] , \tag{33}$$

which can be worked out to read

$$Z_s(0)=(1+\lambda)\left[1-\left[\frac{2\Delta_0}{T_c}\right]^2\frac{1}{(2\pi)^2}\frac{(\pi T_c)^2}{1+\lambda}[a(T_c)-b]\right] . \tag{34}$$

Using Eq. (30), (32), and (34), we get

$$\frac{Y_L(0)}{T_c|Y_L'(T_c)|}=0.50\left\{1-2\frac{(\pi T_c)^2}{\lambda}\left[\left[1+\frac{6}{7\zeta(3)}\right]a(T_c)-1.57b\right]\right\} . \tag{35}$$

When $a(T_c)$ and b, given by Eq. (A20) and (A21), respectively, are finally substituted into (35), we get

$$\frac{Y_L(0)}{T_c|Y_L'(T_c)}=0.50\left[1-a_2\left[\frac{T_c}{\omega_{\ln}}\right]^2\ln\left[\frac{\omega_{\ln}}{b_2T_c}\right]\right] , \tag{36}$$

where a_2 and b_2 are arbitrary constants to be fit to data

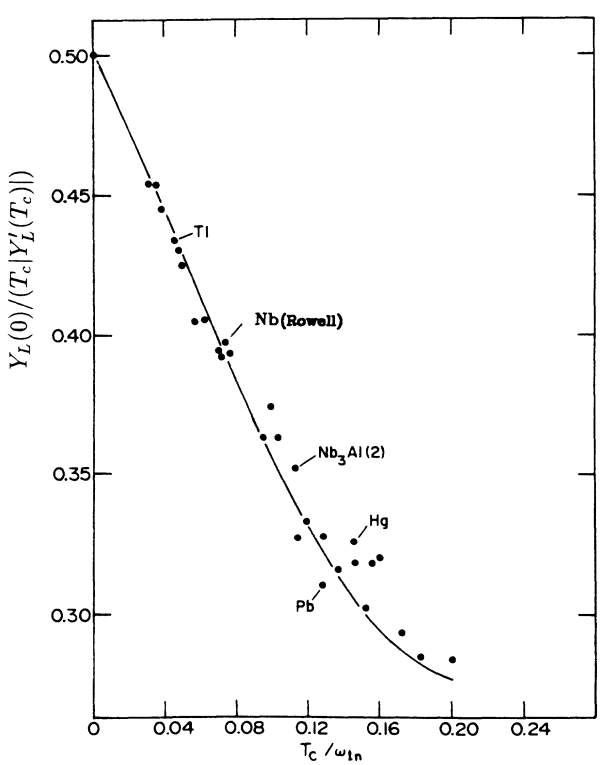

FIG. 2. The ratio $Y_L(0)/T_c|Y_L'(T_c)|$ vs $T_c/\omega_{\ln}$. See Sec. III for identification of materials. The solid curve corresponds to

$$\frac{Y_L(0)}{T_c|Y_l'(T_c)|}=0.5\left[1-2\frac{T_c}{\omega_{\ln}}-11\left[\frac{T_c}{\omega_{\ln}}\right]^2\ln\left[\frac{\omega_{\ln}}{4.5T_c}\right]\right] .$$

for real materials obtained from the complete numerical solutions of the Eliashberg equations (1) and (2) with exact prescription (11) for $\lambda_L(T)$.

Results of exact numerical solutions for the many real materials (solid dots) identified in the preceding section are shown in Fig. 2. A reasonable fit can be obtained to this data with a form (36) provided an additional linear term is introduced phenomenologically as in Sec. III. The solid line of Fig. 2 corresponds to the form

$$\frac{Y_L(0)}{T_c|Y_L'(T_c)}=0.50\left[1-2.0\left[\frac{T_c}{\omega_{\ln}}\right]-11.0\left[\frac{T_c}{\omega_{\ln}}\right]^2\ln\left[\frac{\omega_{\ln}}{4.5T_c}\right]\right] . \tag{37}$$

which is our final result for strong-coupling corrections. If one is interested in deviations from Eq. (37) for a particular material, a full numerical solution is required.

Note finally that the Pippard limit penetration depth was not explicitly commented on in this section because it is simple matter to prove that it does not lead to new results. This is easily seen if we introduce $Z_P(T)\equiv\lambda_p^{-3}(T)$. It follows that

$$\frac{Z_P(0)}{T_c|Z_P'(T_c)|}=\frac{Y_l(0)}{T_c|Y_l'(T_c)|} , \tag{38}$$

a quantity we have already plotted in Fig. 1 in the preceding section.

V. ELECTROMAGNETIC COHERENCE LENGTH

As pointed out in Sec. II [see Eq. (14) in particular], the electromagnetic coherence length $\xi(T)$ is related to local and London limit penetration depth. So, no new algebra is required to discuss $\xi(T)$. Before giving results, however, we wish to discuss the effect of normal impurity scattering on the gap and renormalization factor which in turn enter the formulas for the electromagnetic properties.

So far, no impurities have been included explicitly in the Eliashberg equations (1) and (2). To include them requires adding to the right hand side of Eq. (1) a terms of the form[25]

$$\pi t_+ \frac{\Delta(i\omega_n)}{[\omega_n^2+\Delta^2(i\omega_n)]^{1/2}}\ , \qquad (39a)$$

and in Eq. (2)

$$\pi t_+ \frac{1}{[\omega_n^2+\Delta^2(i\omega_n)]^{1/2}}\ , \qquad (39b)$$

with $t_+=1/2\pi\tau_N$ with τ_N the impurity lifetime already introduced. It can easily be seen, on substitution of the modified Eq. (2) into Eq. (1), that for an isotropic superconductor the new impurity terms will cancel right out of the combined equation for the Matsubara gaps. This cancellation is in accordance with Anderson's[26] theorem which states that T_c is unaffected by normal impurities in an isotropic superconductor. Of course, when anisotropy is included, the theorem no longer holds and the effect of normal impurities is to wash out the anisotropy and in the dirty limit we recover isotropy.

While the impurity contribution drops out of the gaps $\Delta(i\omega_n)$, it remains in $Z(i\omega_n)$ itself. Noting from Eq. (7) that only the $\Delta(i\omega_n)$ are required in $\lambda_l(T)$, we see that adding t_+ explicitly to our equations changes nothing. This is not surprising since the local limit is derived under the assumption that $l \ll \xi(0)$ and so the dirty limit is already built into it. It is then only consistent that adding an additional πt_+ term in the Eliashberg equations themselves makes no difference. The situation is different for the London limit. In this case, Eq. (11) applies and we see that $Z(i\omega_n)$ enters explicitly so that the πt_+ contribution in a modified Eq. (2) will come in explicitly and the London limit will depend on impurity scattering time τ_n. In fact, in the dirty limit with $\pi t_+ \to \infty$ the new term (39) in the Z channel will dominate, and it is appropriate to replace $Z(i\omega_n)$ in Eq. (11) by

$$Z(i\omega_n)\cong \pi t_+/[\omega_n^2+\Delta^2(i\omega_n)]^{1/2}$$

which gives

$$\lambda_L(T)=\left[\tfrac{4}{3}\pi\frac{N(0)e^2}{\pi t_+}v_F^2 \times T\sum_{n=1}^{\infty}\frac{\Delta^2(i\omega_n)}{[\omega_n^2+\Delta^2(i\omega_n)]^{1/2}}\right]^{-1/2}. \qquad (40)$$

The constant in the definition of $\lambda_L^{-2}(T)$ is simply $4\pi\sigma_N$ with σ_N given by (8) and so the London limit result in the dirty limit reduces to the local limit result [Eq. (7)].

Since, as noted in Eq. (14), there is a relationship between coherence length and the ratio of local to London penetration depth, $\xi(T)$ will depend explicitly on the πt_+ factor in the modified Eliashberg equations. In fact, from Eq. (13) in the dirty limit, we see immediately that

$$\xi(T)\cong\frac{v_F}{2\pi t_+}=v_F\tau=l\ . \qquad (41)$$

The electromagnetic coherence length becomes the mean free path in this limit.

Results for the ratio $\xi(0)/\xi(T_c)$ for real materials are shown as solid dots in Fig. 3. These results were obtained from exact numerical calculations based on Eq. (1) and (2), without an explicit πt_+ term added on, and the prescription (13), so they apply to the clean limit case. The solid curve represents the best visual fit that we could achieve and is

$$\frac{\xi(0)}{\xi(T_c)}=1.33\left[1-0.83\left[\frac{T_c}{\omega_{\ln}}\right]-0.75\left[\frac{T_c}{\omega_{\ln}}\right]^2\ln\left[\frac{\omega_{\ln}}{40T_c}\right]\right], \qquad (42)$$

where the numerical coefficients in this form were fit to the data of Fig. 3 and were not derived from our previous fits to Y_l and Y_L even though an exact relationship is

$$\frac{Y_L(0)}{T_c|Y_L'(T_c)|}=\frac{\xi(0)}{\xi(T_c)}\frac{Y_l(0)}{T_c|Y_l'(T_c)|}\ . \qquad (43)$$

This was done to increase the accuracy of our fit. As a consequence, we stress that Eq. (42), (37), and (26) violate (43) in the first order correction.

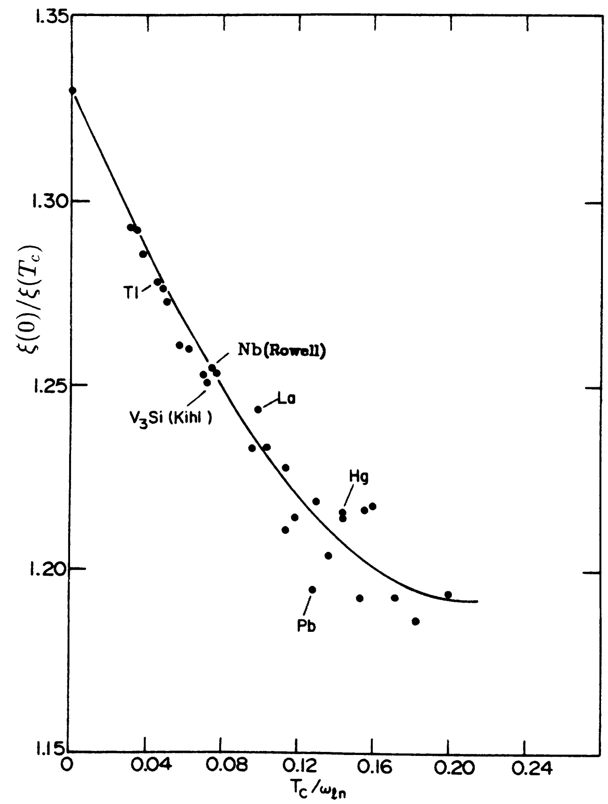

FIG. 3. The ratio $\xi(0)/\xi(T_c)$ vs $T_c/\omega_{\ln}$. See Sec. III for identification of materials. The curve corresponds to

$$\frac{\xi(0)}{\xi(T_c)}=1.33\left[1-0.83\frac{T_c}{\omega_{\ln}}-0.75\left[\frac{T_c}{\omega_{\ln}}\right]^2\ln\left[\frac{\omega_{\ln}}{40T_c}\right]\right].$$

VI. CONCLUSIONS

We have calculated the local and London limit penetration depth as well as the electromagnetic coherence length for a large number of conventional materials. The calculations involve the complete numerical solutions of the Eliashberg equations based on known electron-phonon spectral densities and Coulomb pseudopotential. These normal-state parameters, which determine the kernels in the gap equations, are known from tunneling inversion. The data so generated are used to fit parameters in semiphenomenological forms derived from the exact expressions for the electromagnetic properties through a series of approximations. In all cases, the final forms obtained depend only on the single characteristic strong-coupling parameter of the material $T_c/\omega_{\ln}$ where $\omega_{\ln}$ is the characteristic phonon energy first defined by Allen and Dynes. Deviations from the trend curves are found in some cases. These can only be described through full numerical calculations.

ACKNOWLEDGMENTS

This work was supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC). One of us, J. P. Carbotte, was also supported by the Canadian Institute for Advance Research (CIAR).

APPENDIX: DERIVATION OF STRONG-COUPLING CORRECTIONS TO THE GAP

In this Appendix, we outline the derivation of strong-coupling corrections to the gap. Near $T=T_c$ we can expand Eqs. (1) and (2) in powers of the gap since it is small:

$$Z_S(\omega_n)\Delta(\omega_n)=\pi T\sum_{m=-N_0+1}^{N_0}\lambda(i\omega_n-\omega_m)\frac{\Delta_0}{|\omega_m|}\left[1-\frac{1}{2}\frac{\Delta_0^2}{\omega_m^2}+\frac{3}{8}\frac{\Delta_0^4}{\omega_m^4}\right], \tag{A1}$$

$$Z_s(\omega_n)=Z_N(\omega_n)+\frac{\pi T}{\omega_n}\int_0^\infty 2\nu\, d\nu\, \alpha^2F(\nu)\sum_{m=-N_0+1}^{N_0}\frac{\mathrm{sgn}\omega_m}{\nu^2+(\omega_m-\omega_n)^2}\left[-\frac{1}{2}\frac{\Delta_0^2}{\omega_m^2}+\frac{3}{8}\frac{\Delta_0^4}{\omega_m^4}\right]. \tag{A2}$$

Here the subscript $S(N)$ means superconducting (normal). Also, $N_0=\omega_0/(2\pi T)+\frac{1}{2}$ enumerates the Matsubara frequencies in the sums. However, the convergence is sufficiently rapid that N_0 can be replaced by infinity. The summations are folded to the domain $(0,\infty)$. In the Z channel [Eq. (A2)], this procedure results in sums like

$$\sum_{m=1}^{\infty}\frac{1}{\omega_m^{2i-1}}\frac{1}{(\nu^2+\omega_m^2+\omega_n^2)^2}\left[1+\frac{4\omega_m^2\omega_n^2}{(\nu^2+\omega_m^2+\omega_n^2)^2}+\cdots\right]. \tag{A3}$$

Noting that $\omega_n=\pi t(2n-1)$, and only small n is required, one sees that terms of $O(T_c/\nu)^6$ have been neglected, consistent with our assumption, $T_c\ll\nu$. The required sums are

$$U_i\equiv 4\pi T\sum_{m=1}^{\infty}\frac{1}{\omega_m^{2i-1}}\frac{1}{(\omega_m^2+a_n^2)^2} \tag{A4}$$

and

$$V_i\equiv 4\pi T\sum_{m=1}^{\infty}\frac{1}{\omega_m^{2i-1}}\frac{4\omega_n^2\omega_m^2}{(\omega_m^2+a_n^2)^4}. \tag{A5}$$

Here, $a_n^2\equiv\nu^2+\omega_n^2$, and $i=1,2,3,\ldots$. In accordance with the remarks made above, only $i=1$, 2 are required in Eq. (A4) and all i can be neglected in Eq. (A5). These are readily evaluated in terms of digamma functions:

$$U_1\equiv\frac{2}{a_n^4}\left[\ln\frac{1.13a_n}{k_BT}-\frac{1}{2}\right], \tag{A6}$$

$$U_2\equiv\frac{7}{2}\frac{\zeta(3)}{(\pi T)^2}\frac{1}{a_n^4}. \tag{A7}$$

$\zeta(3)$ is the Riemann zeta function [$\zeta(3)\approx 1.202\ldots$]. It is easy to show $V_1=O(T/\nu)^4$. Equation (A2) becomes

$$Z_S(\omega_n)-Z_N(\omega_n)=-\Delta_0^2(T)\int_0^\infty 2\nu\, d\nu\, \alpha^2F(\nu)\frac{1}{(\nu^2+\omega_n^2)^2}\left[\ln\frac{1.13(\nu^2+\omega_n^2)^{1/2}}{k_BT}-\frac{1}{2}\right]$$
$$+\frac{21}{16}\Delta_0^4(T)\frac{\zeta(3)}{(\pi T)^2}\int_0^\infty 2\nu\, d\nu\, \alpha^{2F}(\nu)\frac{1}{(\nu^2+\omega_n^2)^2}. \tag{A8}$$

The n dependence in Eq. (A7) is dropped in the argument of the logarithm, to facilitate the calculation. An identical expansion can be performed for $Z_N(n)$, with the result

$$Z_N(\omega_n)=1+\lambda-\frac{1}{3}(\pi T)^2[(2n-1)^2-1]\int_0^\infty 2\nu\, d\nu\, \alpha^2F(\nu)\frac{1}{\nu^4}. \tag{A9}$$

For $n=1$, there is no strong-coupling correction in (A9); $Z_n = 1+\lambda$, which is the $\lambda^{\theta\theta}$ result. For a higher value of n, a strong-coupling correction results. We prefer to use the former value, since it is exact for $n=1$. This is seen most readily from the exact equation for $Z_N(\omega_n)$:

$$Z_N(\omega_n)=1+\frac{\pi T}{\omega_n}\left[\lambda(0)+2\sum_{m=1}^{n-1}\lambda(m)\right] . \quad \text{(A10)}$$

Similar remarks apply to the Δ channel. Equation (A1) can be reduced to

$$Z_S(n)\Delta_0=\Delta_0\int_0^\infty 2\nu\, d\nu\, \alpha^2F(\nu)[P_1+Q_1-\tfrac{1}{2}\Delta_0^2(P_2+Q_2)+\tfrac{3}{8}\Delta_0^4(P_3+Q_3)] , \quad \text{(A11)}$$

where

$$P_i \equiv \sum_{m=1}^{\infty}\frac{2\pi T}{\omega_m^{2i-1}}\frac{1}{\omega_m^2+a_n^2} , \quad \text{(A12)}$$

and

$$Q_i \equiv \sum_{m=1}^{\infty}\frac{2\pi T}{\omega_m^{2i-3}}\frac{4\omega_n^2}{(\omega_m^2+a_n^2)^3} . \quad \text{(A13)}$$

These sums are

$$P_1=\frac{1}{a_n^2}\ln\frac{1.13a_n}{k_BT}-\frac{(\pi T)^2}{6a_n^4} , \quad \text{(A14a)}$$

$$P_2=\frac{7}{4}\frac{\zeta(3)}{(\pi T)^2}\frac{1}{a_n^2}-\frac{1}{a_n^4}\ln\frac{1.13a_n}{k_BT} , \quad \text{(A14b)}$$

$$P_3=\frac{31}{16}\frac{\zeta(5)}{(\pi T)^4}\frac{1}{a_n^2}-\frac{7}{4}\frac{\zeta(3)}{(\pi T)^2}\frac{1}{a_n^4} , \quad \text{(A14c)}$$

$$Q_1=\frac{\omega_n^2}{a_n^4} , \quad \text{(A14d)}$$

$$Q_2=Q_3=O\left[\frac{1}{a_n^6}\right] . \quad \text{(A14e)}$$

We evaluate Eq. (A11) for small n (specifically, $n=1$) and define

$$a(T)\equiv\int_0^\infty 2\nu\, d\nu\, \alpha^2F(\nu)\frac{1}{\nu^4}\ln\frac{1.13\nu}{k_BT} \quad \text{(A15a)}$$

and

$$b(T)\equiv\int_0^\infty 2\nu\, d\nu\, \alpha^2F(\nu)\frac{1}{\nu^4} . \quad \text{(A15b)}$$

The gap equation (A11) becomes [using $Z_S(n=1)$]

$$1=F(T)+\Delta_0^2G(T)+\Delta_0^4J(T) , \quad \text{(A16)}$$

where

$$F(T)=\frac{\lambda}{1+\lambda}\ln\frac{1.13\omega_{\ln}}{k_BT}-\frac{(\pi T)^2}{1+\lambda}[a(T)-\tfrac{4}{3}b] , \quad \text{(A17a)}$$

$$G(T)=\frac{-\lambda}{1+\lambda}\frac{7}{8}\frac{\zeta(3)}{(\pi T)^2}+\frac{3}{2}\frac{a(T)}{1+\lambda}+(\tfrac{7}{8}\zeta(3)-\tfrac{1}{2})\frac{b}{1+\lambda} , \quad \text{(A17b)}$$

$$J(T)=\frac{\lambda}{1+\lambda}\frac{93}{128}\frac{\zeta(5)}{(\pi T)^4}+\frac{93}{128}\frac{\zeta(5)}{(\pi T)^2}\frac{b}{1+\lambda}-\frac{63}{32}\frac{\zeta(3)}{(\pi T)^2}\frac{b}{1+\lambda} . \quad \text{(A17c)}$$

The T_c equation is given by $1=F(T_c)$, with the result

$$T_c=1.13\omega_{\ln}\exp\left[-\left[\frac{1+\lambda}{\lambda}\right]\right]\times\left[1-\frac{(\pi T_c)^2}{\lambda}[a(T_c)-\tfrac{4}{3}b]\right] . \quad \text{(A18)}$$

This is not an accurate T_c equation, but will prove useful later. The gap parameter (near T_c) is obtained from Eq. (A16):

$$\Delta_0^2(T)=-\frac{F'}{G}(T-T_c)\left[1+\left[\frac{1}{2}\frac{F''}{F'}-\frac{G'}{G}+\frac{F'J}{G^2}\right](T-T_c)\right] , \quad \text{(A19)}$$

where it is understood that the derivatives are with respect to temperature, and the functions are all evaluated at T_c. To simplify the formulas (A17) for F, G, and J, we write

$$a(T_c)=\frac{\alpha_1}{\omega_{\ln}^2}\int_0^\infty 2\nu\, d\nu\, \alpha^2F(\nu)\frac{1}{\nu^2}\ln\frac{1.13\nu}{k_BT_c}=\frac{\alpha_1}{\omega_{\ln}^2}\lambda\ln\frac{1.13\omega_n}{k_BT_c} . \quad \text{(A20)}$$

The first equality follows for a given spectrum from the mean value theorem of calculus; α_1 can be chosen to compensate for the averaging. The second equality follows from the definition of $\omega_{\ln}$ [Eq. (2.3)]. Similarly,

$$b=\frac{\alpha_2}{\omega_{\ln}^2}\lambda . \quad \text{(A21)}$$

At zero temperature the Eliashberg equations are modified according to the prescription

$$2\pi T\sum_n f(\omega_n)\rightarrow\int_{\infty}^{+\infty}d\omega\, f(\omega) . \quad \text{(A22)}$$

The equation for $Z_S(\omega)$ is folded to the domain $[0,\infty]$, with result that it can be written

$$Z_S(\omega)=1+4\int_0^\infty \nu\, d\nu\, \alpha^2F(\nu)(A_1+A_2) , \quad \text{(A23)}$$

where

$$A_1=\int_0^\infty d\omega'\frac{\omega'^2}{(\omega'^2+\Delta_0^2)^{1/2}}\frac{1}{(\omega'^2+a_0^2)^2} \quad \text{(A24)}$$

and

$$A_2=\int_0^\infty d\omega'\frac{4\omega^2\omega'^2}{(\omega'^2+\Delta_0^2)^{1/2}}\frac{1}{(\omega'^2+a_0^2)^4} . \quad \text{(A25)}$$

Here again $a_0^2\equiv\omega^2+\nu^2$. The results are

$$A_1=\frac{1}{2a_0^2}-\frac{\Delta_0^4}{2a_0^4}\left[\ln\frac{2a_0}{\Delta_0}-1\right] \quad (A26)$$

and

$$A_2=\frac{1}{3}\frac{\omega^2}{a_0^4} . \quad (A27)$$

Other contributions are of higher order. The net result is

$$Z_S(\omega)=Z_N(\omega)+\Delta_0^2\int_0^\infty \frac{2\nu\, d\nu\, \alpha^2F(\nu)}{(\nu^2+\omega^2)^2} \times\left[1-\ln\frac{2(\nu^2+\omega^2)^{1/2}}{\Delta_0}\right] . \quad (A28)$$

$Z_N(\omega)$ can be evaluated exactly:

$$Z_N(\omega)=1+\frac{1}{\omega}\int_0^\infty 2\,d\nu\, \alpha^2F(\nu)\tan^{-1}\left[\frac{\omega}{\nu}\right] . \quad (A29)$$

Recall that we have used a constant model for $\Delta(\omega)$ and $Z_S(\omega)$. Hence it should not matter at what frequency we evaluate Eq. (A29), as long as it is small. However, there is a dependence, and we will choose $\omega=0$. Then $Z_N(0)=1+\lambda$ and there is no strong-coupling correction. One can argue that a more appropriate frequency at which to evaluate $A_N(\omega)$[and subsequently $\Delta(\omega)$] is Δ_0, the gap edge, since this defines the gap edge. However, it must be kept in mind that this calculation is on the imaginary axis, so that corrections proportional to ω^2 actually have the opposite sign from the correction on the real axis [since $\omega^2=-(i\omega)^2$]. Hence it is more accurate to evaluate at $\omega=0$.

The gap channel is treated similarly, with the result

$$Z_S(0)\Delta_0\int_0^\infty 2\nu\, d\nu\, \alpha^2F(\nu)(A_3+A_4) , \quad (A30)$$

where

$$A_3=\int_0^\infty \frac{d\omega'}{(\omega'^2+\Delta_0^2)^{1/2}}\frac{1}{\omega'^2+a_0^2} \quad (A31)$$

and

$$A_4=\int_0^\infty \frac{d\omega'}{(\omega'^2+\Delta_0^2)^{1/2}}\frac{4\omega^2\omega'^2}{(\omega'^2+a_0^2)^3} . \quad (A32)$$

Evaluation of these integrals is straightforward. Combining with Eq. (A28) evaluate at $\omega=0$, we obtain

$$\Delta_0=2\omega_{\ln}\exp\left[-\left[\frac{1+\lambda}{\lambda}\right]\right]\left[1+\frac{3}{2}\frac{\Delta_0^2}{\lambda}(a-\tfrac{5}{6}b)\right] . \quad (A33)$$

Here, a and b are the same functions (within BCS) as in the preceding section (evaluated at $T=T_c$). Combining Eq. (A18) with (A33), we obtain

$$\frac{2\Delta_0}{K_BT_c}3.53\left[1+a_3\left[\frac{T_c}{\omega_{\ln}}\right]^2\ln\left[\frac{\omega_{\ln}}{b_3T_c}\right]\right] , \quad (A34)$$

where

$$a_3=[\tfrac{3}{8}(3.53)^2+\pi^2]\alpha_1 \quad (A35)$$

and

$$b_3=\frac{1}{1.13}\exp\left[\left[\frac{\frac{5}{16}(3.53)^2+\frac{4}{3}\pi^2}{\frac{3}{8}(3.53)^2+\pi^2}\right]\frac{\alpha_2}{\alpha_1}\right] . \quad (A36)$$

The values first fit by Mitrovic *et al.*[20] are $a_3=12.5$ and $b_3=2$. The values we would obtain by setting $\alpha_i\approx 1$ are $a_3=14.5$ and $b_3=2.9$, which are not much different from the best fit.

*Present address: University of California at San Diego, La Jolla, California 92093.
†Present address: MPB Technologies, 1725 North Service Road, Trans-Canada Highway, Dorval, Quebec, Canada H9P 1J1.

[1]Sang Boo Nam, Phys. Rev. **156**, 470 (1967).
[2]Sang Boo Nam, Phys. Rev. **156**, 487 (1967).
[3]J. C. Swihart and W. Shaw, in *Superconductivity,* edited by F. Chilton (North-Holland, Amsterdam, 1971), p. 678.
[4]W. Shaw and J. C. Swihart, Phys. Rev. Lett. **20**, 1000 (1968).
[5]R. Blaschke and R. Blocksdarf, Z. Phys. B **49**, 99 (1982).
[6]C. S. Lim J. D. Leslie, H. J. T. Smith, P. Vashishta, and J. P. Carbotte, Phys. Rev. B **2**, 1651 (1970).
[7]P. Vashishta and J. P. Carbotte, Phys. Rev. B **7**, 1874 (1973).
[8]H. R. Kerchner and D. M. Ginsberg, Phys. Rev. B **8**, 3190 (1973).
[9]D. M. Ginsberg, R. E. Harris, and R. C. Dynes, Phys. Rev. B **14**, 990 (1976).
[10]J. P. Carbotte and P. Vashishta, Can. J. Phys. **49**, 1493 (1971).
[11]J. Q. Blezius and J. P. Carbotte, Phys. Rev. B **33**, 3509 (1986).
[12]W. L. McMillan and J. M. Rowell, in *Superconductivity,* edited by R. D. Parks (Marcel Dekker, New York, 1969), Vol. 1, p. 561.
[13]P. B. Allen and R. C. Dynes, Phys. Rev. B **12**, 905 (1975).
[14]F. Marsiglio and J. P. Carbotte, Phys. Rev. B **33**, 6141 (1986).
[15]D. Rainer and G. Bergmann, J. Low Temp. Phys. **14**, 501 (1974).
[16]J. M. Daams and J. P. Carbotte, J. Low Temp. Phys. **43**, 263 (1981).
[17]J. Bardeen, L. N. Cooper, and J. R. Schreiffer, Phys. Rev. **108**, 1175 (1957).
[18]V. Ambegaokar and A. Baratoff, Phys Rev. Lett. **10**, 486 (1963).
[19]T. R. Lemberger, D. M. Ginsberg, and G. Rickayzen, Phys. Rev. B **18**, 6057 (1978).
[20]B. Mitrovic, H. G. Zarate, and J. P. Carbotte, Phys. Rev. B **29**, 184 (1984).
[21]B. T. Geilikman, V. Z. Kresin, and N. F. Masharov, J. Low Temp. Phys. **18**, 241 (1975).
[22]B. T. Geilikman and V. Z. Kresin, Fiz. Tverd. Tela (Leningrad) **1**, 3294 (1954) [Sov. Phys.—Solid State **7**, 2659 (1966)].
[23]V. Z. Kresin and V. P. Parkhomenko, Fiz. Tverd. Tela (Leningrad) **16**, 3363 (1974) [Sov. Phys.—Solid State **16**, 2180 (1975)].
[24]N. F. Masharov, Fiz. Tverd. Tela (Leningrad) **16**, 2342 (1974) [Sov. Phys.—Solid State **16**, 1524 (1975)].
[25]P. R. Allen and B. Mitrovic, in *Solid State Physics,* edited by H. Ehrenreich, F. Seitz, and D. Turnbull (Academic, New York, 1982), Vol. 37, p. 1.
[26]P. W. Anderson, J. Phys. Chem. Solids **11**, 26 (1959).