Confined electron and hydrogenic donor states in a spherical quantum dot of GaAs-$Ga_{1-x}Al_xAs$

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According to hydrogenic-effective-mass theory, exact solutions and quantum-level structures are presented for confined electron and hydrogenic donor states in a spherical quantum dot (SQD) of GaAs-$Ga_{1-x}Al_xAs$. Calculated results reveal that the values of the quantum levels of a confined electron in a SQD can be quite different for cases with finite and infinite barrier heights. The quantum-level sequence and degeneracy for an electron in a SQD are similar to those of a superatom of GaAs-$Ga_{1-x}Al_xAs$ but different from those in a Coulomb field. There is stronger confinement and larger binding energy for a hydrogenic donor in a SQD of GaAs-$Ga_{1-x}Al_xAs$ than in the corresponding quantum-well wires and two-dimensional quantum-well structures. The binding energy and its maximum of the ground state of a donor at the center of a quantum well are found to be strongly dependent on the well dimensionality and barrier height.

I. INTRODUCTION

The advances in crystal-growth techniques such as molecular-beam epitaxy (MBE) and metalorganic chemical vapor deposition (MOCVD) have made possible the fabrication of quantum-well structures in which the electrons are confined to move in low dimensions. Presently, intensive work is done on the fabrication of one-dimensional quantum wires[1-4] and zero-dimensional quantum dots[5-10] in a number of laboratories. The optical and electronic properties of these quasi-two-,[11,25] one-,[26-29] and zero-dimensional[30-39] structures have been the subject of both theoretical and experimental investigations. The exciton and impurity states in two-dimensional quantum wells (2D QW's) and superlattices have been calculated by a number of authors.[11-25] For an infinite barrier height of 2D QW's, the binding energy approaches 4 Ry* (Ry* is effective Rydberg, i.e., the binding energy of the ground state in the bulk semiconductor) as the well size is reduced. For a finite barrier height of 2D QW's, however, several calculations have shown that the binding energy goes through a maximum as the well size is reduced instead of continuously increasing as is found in the infinite barrier calculation. The maximum of the binding energy is dependent on the barrier height.

In the last few years there has been increasing interest in the study of the electronic properties of quantum-well wires (QWW's). In such structures, the electrons are confined to movement along the length of the wires while the motion normal to the wires is quantized in the two dimensions. Several calculations[26-29] have been performed for electron and impurity levels of QWW's. Lee and Spector[28] have calculated the binding energies for bound states of a hydrogenic impurity placed on the axis of cylindrical QWW of infinite confining potential. It is found that the binding energy is increased continuously as the well size is reduced. Assuming a finite barrier height for the confining potential, Bryant[26] has also calculated the binding energies and found that they are two or three times greater than the values in comparable 2D QW's. It is also shown that there is a maximum of the binding energy for a fixed barrier height of QWW's, and that the binding energy is larger as the confinement is stronger.

Although perfect quantum-dot structures (boxes or balls) have not yet been realized, Asada *et al.*[30-32] have shown that the linear gain of quantum boxes is much larger than that of bulk crystals at fixed carrier density, and that the laser threshold can be reduced by the structures of boxes. Schmitt-Rink *et al.*[33] have reported a theory of the linear and nonlinear optical properties of semiconductor microcrystallites. They have also shown very desirable optical properties of a perfect quantum dot. Therefore it is interesting to study the electronic structures of quantum dots, such as the quantum levels of electrons and holes and the binding energies of impurity and exciton states,[34-39] which are important for much better understanding of the problems mentioned above.

Because the transverse and longitudinal variables do not separate, the impurity states in 2D QW's and QWW's cannot be solved exactly. Therefore approximation methods should be used. A reasonable trial function is needed to obtain a correct variational state of an impurity in 2D QW's and QWW's, and calculated results are more accurate if the coupling effect between the impurity

and well potentials is considered using a trial function which or a part of which has correctly both donor and well potential effects.[40] However, for a hydrogenic donor at center of spherical quantum dots (SQD's) the exact solutions[41] can be obtained. It is interesting not only from a physical point of view but also from a mathematical point of view to compare the solutions and binding energies with those of 2D QW's and QWW's. In this paper, we report exact solutions and quantum-level structures for confined electron and hydrogenic donor states in SQD's. The dependence of the quantum levels and the binding energies on the dimensionality of quantum wells is also presented.

In Sec. II we present the calculation method for confined electron and hydrogenic donor states in a SQD. Main results are shown in Sec. III. A summary of the results is presented in Sec. IV.

II. CALCULATION METHOD

Our calculation is based on the effective-mass approximation. It has been known to give excellent results for electronic structure of GaAs-$Ga_{1-x}Al_xAs$ 2D QW's and $(AlAs)_n/(GaAs)_n$ superlattices if the well width or n is sufficiently large. The limit is estimated to be about 30 Å ($n\sim 10$).[42] Therefore it should also be valid for the GaAs-$Ga_{1-x}Al_xAs$ quantum dots as the size (diameter for a ball) is sufficiently large. Based on the facts mentioned above, the limit for a ball diameter is also estimated to be the same value and equal to 30 Å. Here we treat the cases where the diameter is larger than the critical size. It is interesting to point out that the maximum quantum confinement of an electron in the GaAs-$Ga_{1-x}Al_xAs$ quantum ball is already obtained before the diameter approaches to the critical value. In addition, polarization and image charge effects can be significant if there is a large dielectric discontinuity between the quantum ball and the surrounding medium.[39] However, this is not the case for the GaAs-$Ga_{1-x}Al_xAs$ quantum system; therefore we ignore such effects.

According to hydrogenic-effective-mass theory, the electron bound states and their binding energies have been found in 2D QW's and QWW's. Normally, the effective-mass equation is reliable weakly bound states, and one might worry that the effective-mass equation is inappropriate when the binding energy is greatly enhanced in SQD's of GaAs-$Ga_{1-x}Al_xAs$. However, the band gap of GaAs is 1.4 eV, while $Ry^*=5.3$ meV. Thus roughly a 100-fold enhancement of the binding energy is necessary before the effective-mass equation becomes inapplicable. This difference is much greater than the enhancement seen in the cases considered here, so that the theory is still reliable for the bound states in SQD's of GaAs-$Ga_{1-x}Al_xAs$.

Let us for definiteness consider a hydrogenic donor at the center of the SQD of radius R_0. The potential due to the discontinuity of the band edges at the GaAs-$Ga_{1-x}Al_xAs$ interface $r=R_0$ is as follows:

$$V(r)=\begin{cases} V_0 & \text{if } r\geq R_0 \\ 0 & \text{if } r<R_0 \end{cases} \tag{1}$$

where r is the electron-donor distance. The barrier height V_0 is obtained from a fixed ratio of the band-gap discontinuity. According to hydrogenic-mass theory, the Hamiltonian for the donor is

$$H=-\nabla^2-\frac{2w}{r}+V(r)\ . \tag{2}$$

It is written in a dimensionless form so that all energies are measured in units of the effective Rydberg Ry^* and all distances are measured in units of effective Bohr radius a^*. w is equal to 1.

In order to solve the Schrödinger-like equation

$$H\Psi(r,\theta,\phi)=E\Psi(r,\theta,\phi) \tag{3}$$

the wave functions of an electron with well-defined values of the orbital (l) and magnetic (m) quantum numbers in a spherically symmetric potential, which is the quantum well and Coulomb potential, are written in the form

$$\Psi_{lm}(r,\theta,\phi)=\Psi^{(l)}(r)Y_{lm}(\theta,\phi)\ , \tag{4}$$

where $Y_{lm}(\theta,\phi)$ and $\Psi^{(l)}(r)$ are the spherical harmonic and radial wave function, respectively. Substituting (4) into (3), we find an equation for the function $\Psi^{(l)}(r)$:

$$r^2\frac{d^2\Psi^{(l)}(r)}{dr^2}+2r\frac{d\Psi^{(l)}(r)}{dr}+\{[E(l)-V(r)]r^2-l(l+1)+2wr\}\Psi^{(l)}(r)=0\ . \tag{5}$$

Using the method of series expansion, we can solve Eq. (5) exactly. It should be noted that the zero and infinity are a regular and an irregular singular point of Eq. (5), respectively. In the region $0<r$, we have a series solution, which has a finite value at $r=0$, as follows:

$$\Psi^{(l)}(r)=Ar^l\sum_{n=0}^{\infty}a_n^{(l)}r^n \tag{6}$$

where

$$a_0^{(l)}=1,\quad a_1^{(l)}=-1/(l+1) \tag{7}$$

and

$$a_n^{(l)}=-[2wa_{n-1}^{(l)}+E(l)a_{n-2}^{(l)}]/n(n+2l+1)\ ,\quad \text{for } n=2,3,4,\ldots\ ; \tag{8}$$

A is a constant. In the region $R_0<r$, we can obtain a normal solution.[43] It approaches zero at $r=\infty$ and is found in the form

$$\Psi^{(l)}(r)=B\exp(-K_lr)r^{\rho_l}\sum_{n=0}^{N}b_n^{(l)}r^{-n} \tag{9}$$

where

$$K_l=[V_0-E(l)]^{1/2}\ , \tag{10}$$

$$\rho_l=-1+w/K_l\ , \tag{11}$$

and

$$b_0^{(l)}=1\ ,$$

$$b_{n+1}^{(l)}=-(\rho_l-n-1)(\rho_l-n+l+1)b_n^{(l)}/2K_l(n+1)\ ,\quad \text{for } n=0,1,2,\ldots\ ; \tag{12}$$

B is a constant. The series appears suitable for numerical

computations for large r.[43] However, they are not suitable for R_0 if it is small. In order to get exact value at small R_0, we find a solution of uniformly convergent Taylor series in the region $R_0 < r \leq R_p$, where R_p is a proper point, e.g., $R_P \geq 2a^*$, for using Eq. (9). For the sake of using the matching conditions at $r = R_p$ to obtain the eigenenergy equation below, it is written as follows:

$$\Psi^{(l)}(r) = C \sum_{n=0}^{\infty} c_n (r - R_p)^n + D \sum_{n=0}^{\infty} d_n (r - R_p)^n , \quad (13)$$

where C and D are constants, c_0 and d_1 are equal to 1, and c_1 and d_0 are equal to 0, respectively. Noting that c_n and d_n are equal to 0 for negative n, the other c_n can be determined by the following recurrence relation:

$$c_n = \{ -2R_p(n-1)^2 c_{n-1} + [-(n-2)(n-1) + l(l+1) - 2wR_p + K_l^2 R_p^2] c_{n-2} + 2(K_l^2 R_p - w) c_{n-3} + K_l^2 c_{n-4} \} / [R_p^2 n(n-1)] \quad (14)$$

and d_n's obey a similar recurrence relation.

Using the matching conditions at the interface $r = R_0$ and R_p, we can obtain the equation of the eigenenergies $E(l)$ as follows:

$$\begin{vmatrix} W_{11} & 0 & W_{13} & W_{14} \\ W_{21} & 0 & W_{23} & W_{24} \\ 0 & W_{32} & 1 & 0 \\ 0 & W_{42} & 0 & 1 \end{vmatrix} = 0 , \quad (15)$$

that is

$$W_{21}(W_{42}W_{14} + W_{32}W_{13}) - W_{11}(W_{42}W_{24} + W_{32}W_{23}) = 0 \quad (16)$$

where

$$W_{11} = \sum_{n=0}^{\infty} a_n^{(l)} R_0^n , \quad W_{13} = \sum_{n=0}^{\infty} c_n (R_0 - R_p)^n ,$$
$$W_{14} = \sum_{n=1}^{\infty} d_n (R_0 - R_p)^n , \quad W_{21} = \sum_{n=0}^{\infty} (l+n) a_n^{(l)} R_0^{n-1} ,$$
$$W_{23} = \sum_{n=0}^{\infty} n c_n (R_0 - R_p)^{n-1} , \quad W_{24} = \sum_{n=1}^{\infty} n d_n (R_0 - R_p)^{n-1} , \quad (17)$$
$$W_{32} = \sum_{n=0}^{N} b_n^{(l)} R_p^{-n} , \quad W_{42} = \sum_{n=0}^{N} (-n + \rho_l - K_l R_p) b_n^{(l)} R_p^{-n-1} .$$

It can be solved numerically. Once the nth eigenenergy $E_n(l)$ is known, the A, B, C, and D [hence $\Psi_n^{(l)}(r)$] are known with use of the normalized condition of $\Psi_n^{(l)}(r)$. This $\Psi_n^{(l)}(r)$ depends on the value of l, the quantum well, Coulomb potential, and energy $E_n(l)$. We should point out that we have neglected the difference of the electron effective masses between GaAs and $Ga_{1-x}Al_xAs$ in the Hamiltonian and the matching conditions. If the effective-mass difference is considered, similar formulas can be obtained.

If there is no Coulomb potential in the Hamiltonian of Eq. (2), i.e., $w = 0$, using the same formulas, we can obtain wave function $\Psi_n^{(l)}(r, w = 0)$, and quantum levels $E_n(l, w = 0)$ of an electron in the quantum well. In fact, Eqs. (6) and (9) become the spherical Bessel function and Hankel function if $w = 0$. The equation of eigenenergies $E(l, w = 0)$ is as follows:

$$k_0 + K_0 \tan(k_0 R_0) = 0 \quad \text{if } l = 0 , \quad (18a)$$

$$ik_l h_l(iK_l R_0) j_{l-1}(k_l R_0) + K_l h_{l-1}(iK_l R_0) j_l(k_l R_0) = 0 \quad \text{if } l \geq 1 , \quad (18b)$$

and

$$k_l = [E(l, w = 0)]^{1/2}, \quad K_l = [V_0 - E(1, w = 0)]^{1/2} \quad (19)$$

where J_l and h_l are the lth-order spherical Bessel function and Hankel functions of the first kind, respectively. Then, the same results are obtained if the wave functions and quantum levels are calculated with use of the Bessel and Hankel functions. Once $E_n(l, w = 1)$ and $E_n(l, w = 0)$ are obtained, the binding energy of the corresponding donor states in the SQD is given by

$$E_{nb}(l) = E_n(l, w = 0) - E_n(l, w = 1) . \quad (20)$$

III. QUANTUM LEVELS AND BINDING ENERGIES

We have performed a numerical calculation for GaAs-$Ga_{1-x}Al_xAs$ SQD's of the R_0 between $0.15a^*$ and $7.0a^*$ with different V_0. In Table I, we have shown quantum levels of an electron in SQD's with different R_0 and V_0. The levels $E_n(l)$ are indicated by two symbols n and l as shown in Sec. II. The n is equal to the number of the root of Eq. (16) or (18) in order of increasing magnitude, i.e., $n = 1, 2, 3 \ldots$ and the $n - 1$, hence, is the radial quantum number as usual. The l is the usual notation, i.e., $s, p, d, \ldots$. Thus we have $1s$, $1p$, $1d$, $2s$, $1f$ levels (states) and so on if the n and l are used as the level notation, and we have $1s$, $2p$, $3d$, $2s$, $4f$ levels, and so on, if the principal quantum number, which is equal to $n + l$, and l is used as the notation. It is interesting to point out that when V_0 approaches infinity

$$E_n(l) = (X_{nl}/R_0)^2 \quad (21)$$

where X_{nl} is the nth root of the lth-order spherical Bessel function. In Table I, it is shown that the different values of $E_n(l)$ are obtained as the R_0 is equal to $1a^*$ and $2.5a^*$, respectively. It is also shown that the values of quantum

TABLE I. Quantum levels of an electron in a SQD of GaAs-$Ga_{1-x}Al_xAs$ with $R_0=1$ and $2.5a^*$ and $V_0=40$, 60, 80, and ∞ Ry*, respectively. The notation with the principal quantum number is shown in parentheses. Effective atomic units are used.

$E(l)$	nl	$1s$ ($1s$)	$1p$ ($2p$)	$1d$ ($3d$)	$2s$ ($2s$)	$1f$ ($4f$)	$2p$ ($3p$)
$R_0=1$	$V_0=\infty$	9.872	20.187	33.212	39.476	48.832	59.676
	80	7.957	16.225	26.593	31.425	38.919	47.016
	60	7.702	15.679	25.642	30.191	37.420	44.789
	40	7.292	14.786	24.045	28.004	34.777	
2.5	∞	1.580	3.230	5.314	6.316	7.814	9.548
	80	1.446	2.958	4.866	5.781	7.151	8.735
	60	1.427	2.919	4.800	5.702	7.054	8.613
	40	1.396	2.854	4.692	5.571	6.892	8.409

levels are different between infinite and finite barrier heights. The differences increase as the R_0 and finite V_0 decrease. There are an infinite number and a finite number of bound states for a SQD with infinite and finite barrier heights, respectively. There is no bound state if $R_0<R_c=0.5\pi/(V_0)^{1/2}$.[44] However, the order of $E_n(l)$ is the same for both infinite and finite barriers, i.e., the unique level sequence $1s$, $1p(2p)$, $1s(3d)$, $2s$, $1f(4f)$, and so on. We should note that the level order is different between both cases of a SQD and Coulomb field, in which the level order of an electron is $1s$, $2s$, $2p$, $3s$, $3p$, $3d$, and so of if the principal and orbital quantum numbers are used as the level notation. It is because of the lack of the deep attractive region in the vicinity of the center of a SQD. For the motion of an electron in a Coulomb field, the quantum levels are only dependent on the principal quantum number n_p and degenerate with respect to both l (orbital quantum number) and m (magnetic quantum number). The total degree of degeneracy of a quantum level with the n_p is equal to n_p^2 (excluding spin degeneracy). For an electron in a SQD, however, the quantum levels are dependent on both n and l and only degenerate with respect to the m. The total degree of degeneracy of a quantum level with n and l is equal to $2l+1$ (excluding spin degeneracy). It is worthwhile to point out that the degeneracy can be lifted in the other kinds of quantum dots. In quantum boxes with circle cross sections, for example, the degeneracy is lifted partly. Now, we can conclude that the quantum-level sequence and degeneracy for an electron in a SQD are quite different from those in a Coulomb field, and that this distinguishing feature of levels might cause new phenomena in this type of GaAs-$Ga_{1-x}Al_xAs$ structure.

In Fig. 1, we have, respectively, plotted the ground and first excited energy levels of an electron in a SQD as a function of R_0 for an infinite barrier height and two finite barrier heights $V_0=40$ and 80 Ry*. It is shown that the differences of energy levels between different barrier heights increase as the R_0 is decreased, and that the difference of the first excited-state energy is larger than that of the ground-state energy for a fixed value of R_0. It is also shown that there are no bound states for a SQD with a finite V_0 if $R_0<R_c$ as mentioned above. In Fig. 2, we have shown the binding energies of the ground and first excited states of a donor in a SQD as a function of the R_0 for three barrier heights $V=80$, 60, and 40 Ry*, respectively. It is readily seen that as R_0 decreases both

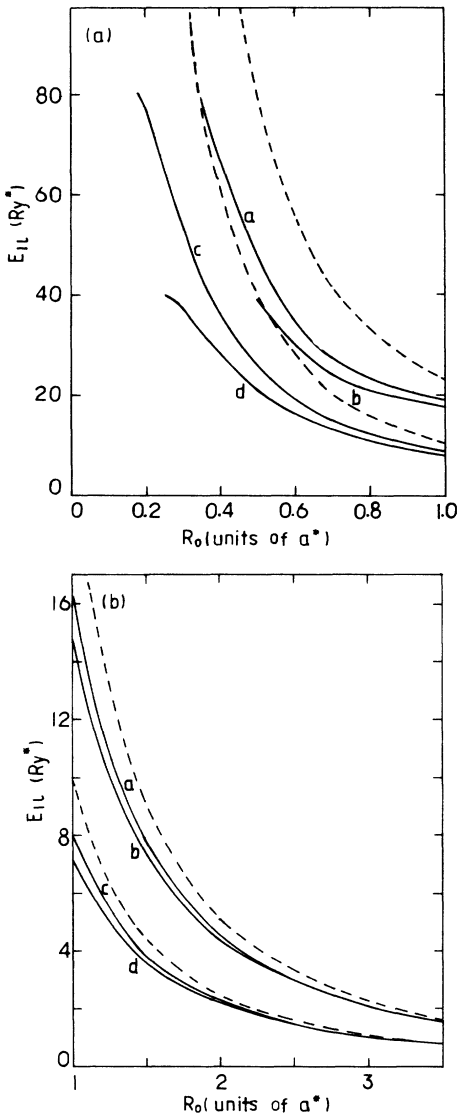

FIG. 1. Ground-state (E_{10}) and first excited-state (E_{11}) energy levels of an electron in a SQD vs the well radius R_0. The top and middle dashed curves represent the levels E_{11} and E_{10} of the well of $V_0=\infty$, respectively. The solid curves a, b, c, and d represents the levels E_{11} and E_{10} of the wells of $V_0=80$ and 40 Ry*, respectively. All energies are expressed in terms of the effective Rydberg (Ry*) and all distances are expressed in terms of the effective Bohr radius (a^*). Same units are used in all of the following figures.

the binding energies increase continuously until their maxima and, then, decrease fast. The values of the binding energies can be much larger than those of QWW's and 2D QW's as R_0 is smaller. It is interesting to point out that the ratio $E_{1B}(0)/E_{1B}(1)$ increases as R_0 increases from some small value. The $E_{1B}(0)$ and $E_{1B}(1)$ are almost independent of V_0 and respectively equal to 1.192 and 0.576 Ry* at $R_0=7.0a^*$. However, the ratio 1.192/0.576 is still much less than 4, which is the limit value of a three-dimensional hydrogenic donor as R_0 approaches infinity.

In Figs. 1 and 2, it is easily seen that as the R_0 decreases the binding energies with respect to different states of a donor in a SQD increase until their maxima, and that the increases of the binding energies are always much less than the increases of the energies of the corresponding states of an electron only confined by the SQD although the binding energies can be much larger than those in the corresponding 2D QW and QWW. It means that confinement effects[37] are dominant in the range of R_0. Further, it is also true for the higher excited states. Therefore we can know what kind of quantum-level sequence we will have if the motion of an electron is confined by both SQD and Coulomb field with a same center. The level sequence is similar to that of a three-dimensional hydrogenic donor if R_0 is much larger and quantum confinement due to the SQD is very weak. However, it is similar to that of the electron in the SQD if the quantum confinement of the SQD is stronger than that of the Coulomb potential. Based on what we have mentioned above, we can understand why the quantum-level structure of GaAs-$Ga_{1-x}Al_xAs$ superatoms[45,46] is similar to that of an electron in a SQD and quite different from those of ordinary atoms, and that the electronic structure of the superatoms is dominated by no-radial-node states of $1s$, $1p(2p)$, $1d(3d)$ and so on.

In Fig. 3, we have plotted the maximum binding energies $E_{B\,\max}$ for the hydrogenic-donor ground and first excited states in a SQD as a function of the barrier height V_0. It is shown that the enhancement of the maximum is greater in a SQD (quasi-zero-dimensional) than in the corresponding QWW (Q1D) and 2D QW (Q2D) as V_0 is increased. This is because of the enhancement of the electron confinement in three dimensions in the SQD. In Fig. 4, we have shown that the maximum binding energy $E_{B\,\max}$ of ground states of a donor at the center of a different kind of quantum well depends on the well dimensionality and barrier height V_0 and presents quasi-

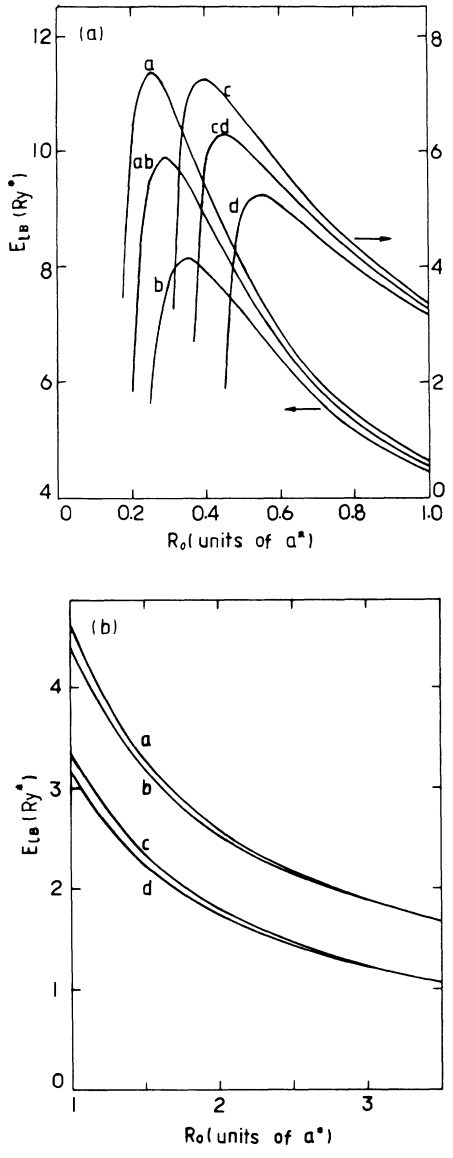

FIG. 2. Binding energies of the ground (E_{0B}) and first excited (E_{1B}) states of a donor in a SQD vs the well radius R_0. The curves *a*, *ab*, and *b* represent E_{0B} of the well of $V_0=80$, 60, and 40 Ry*, and the curves *c*, *cd*, and *d* represent E_{1B} or $V_0=80$, 60, and 40 Ry*, respectively. Arrows indicate the relevant vertical scales.

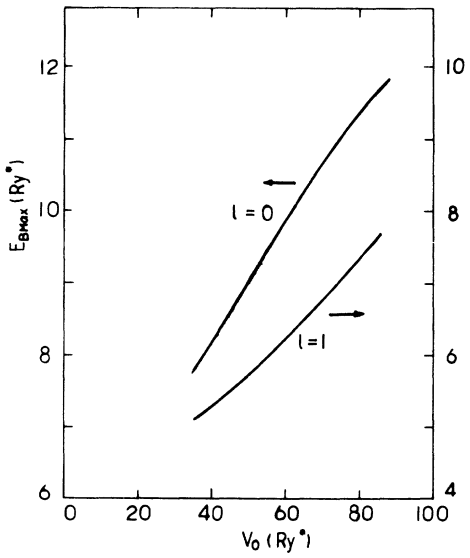

FIG. 3. Maximum binding energies $E_{B\,\max}$ for the hydrogenic-donor ground ($l=0$) and first excited ($l=1$) states in a SQD vs the barrier height V_0. Arrows indicate the relevant vertical scales.

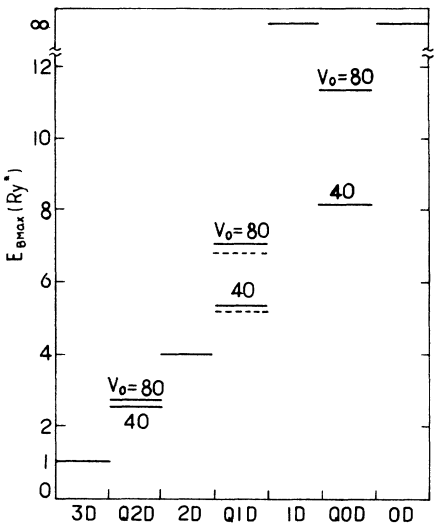

FIG. 4. Maximum binding energy $E_{B\ \max}$ of a donor ground state in a quantum well vs the well dimensionality and barrier height V_0. For the Q1D case, the dashed lines represent the maximum binding energies of QWW's (Ref. 26) and the solid lines represent the mean values of the maxima of the 2D QW's (Ref. 40) and the SQD's.

two-, one-, and zero-dimension features of the hydrogenic donor, respectively. It is interesting to note that the mean values of the maxima of the 2D QW's (Q2D) (Ref. 40) and SQD's (Q0D) are very close to (slightly larger than) the maxima for the QWW's (Q1D's).[26] We should point out that the maximum binding energies of higher excited states can also be used to present the dimension features.

IV. SUMMARY

We have solved the radial equation (5) and obtained the exact solutions of confined electron and donor states in a SQD. The quantum levels and binding energies of a donor in the SQD are calculated numerically. The numerical results reveal that the values of the quantum levels of a confined electron in a SQD with a finite barrier height are different from those with an infinite barrier height. The differences increase as the R_0 decrease. However, the quantum-level order is the same for both infinite and finite barrier heights. It is also shown that the quantum-level sequence and degeneracy for an electron in a SQD are similar to those of a superatom and different from those in a Coulomb field. The quantum-level structure of a donor in a SQD is similar to that of an electron only confined by the SQD as the quantum confinement due to the well potential is stronger than that due to the donor potential. It is useful for understanding the shell model[47] in microclusters.

On the basis of the calculated results, the crossover from three-dimensional to zero-dimensional behavior of the donor states in a SQD is shown when the radius becomes small. The binding energy of a hydrogenic donor state in the well of GaAs-$Ga_{1-x}Al_xAs$ and its maximum are strongly dependent on the well dimensionality and the barrier height and there is a larger confinement and binding energy of a donor state in a SQD than in a QWW and 2D QW. Using calculated results of 2D QW's and QWW's, we have shown that the maxima of the binding energies of hydrogenic donors in 2D QW's, QWW's, and SQD's of GaAs-$Ga_{1-x}Al_xAs$ can be used to present, respectively, quasi-two-, one-, and zero-dimensional features of the hydrogenic donor states. Further, we have found that the maximum of the binding energy of a donor ground state in a GaAs-$Ga_{1-x}Al_xAs$ QWW is about half of the summation of the maximum binding energies in the corresponding 2D QW and SQD.

To close this paper, we should point out that impurities could be located anywhere in a SQD, and that the binding energies will decrease and the level ordering will change as the impurity location shifts to the edge or out of the SQD. Based on the exact solutions obtained, the quantum levels and binding energies of a donor located out of the center of a SQD can be obtained by use of a variation method. This work is in progress. The exact solutions are also useful for the calculation of exciton states in a SQD, which is a kind of quantum dot. It will be interesting to compare the calculated results about quantum levels and binding energies of impurity and exciton states in a SQD with those of other kinds of quantum dots.

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