Transmission through a one-dimensional Fibonacci sequence of δ-function potentials

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We study transmission and reflection of a plane wave (with wave number $k>0$) through a one-dimensional array of N δ-function potentials with equal strengths v located on the Fibonacci numbers 1,1,2,3,5,8,... in the limit $N\to\infty$. Our results can be summarized as follows: (i) For $k\in\pi Z[(1+\sqrt{5})/2]$ (a countable dense set on the positive part of the k axis), the system is a perfect reflector; namely, the reflection coefficient equals unity. (Physically, the system is an insulator.) (ii) For $k=\frac{1}{2}(2N+1)\pi$ $(N=0,1,2,\ldots)$ and $3\cos\psi-1>0$ with $\psi=\arctan(v/k)$, the system may conduct. (The reflection coefficient is strictly smaller than unity.) (iii) For $k=\frac{1}{2}(2N+1)\pi$ $(N=0,1,2,\ldots)$ and $3\cos\psi-1<0$, the system is an insulator. (iv) For any k which is a rational noninteger multiple of π, the system conducts for small values of v/k and becomes an insulator for large values of v/k. Results (ii) and (iii) are physically remarkable since they imply for fixed $k=\frac{1}{2}(2N+1)\pi$ $(N=0,1,2,\ldots)$ a phase transition between a conductor and an insulator as the strength v varies continuously near $k\sqrt{8}$. Result (iv) means that at least one phase transition of this kind occurs at any k which is a rational noninteger multiple of π, once v/k becomes large enough.

I. INTRODUCTION

Experimental advances in submicrometer physics which made possible the fabrication of nearly ideal one-dimensional wires[1] naturally lead to increasing interest in their physical properties, especially those related to transport phenomena. The quantum-mechanical relation between the electrical conductance at zero temperature and the transmission probability[2] indicates that some measurable physical quantities can be accurately explained on the microscopic level.

One-dimensional lattices of infinite extent are, of course, extensively studied in the literature in connection with Bloch theory[3] (if they are periodic), Anderson localization[4] (if they are completely disordered), and quasicrystals[5] (if they are of commensurate-incommensurate structure). On the other hand, the theory of scattering from a semi-infinite one-dimensional array of potentials is less familiar.[6,7] A useful technique in this context is the transfer-matrix algorithm from which the conductance is evaluated with the help of the trace map.[5,8,9] Another closely related technique is one that expresses the transmission and reflection amplitudes through $N+1$ scatterers in terms of the amplitudes for N scatterers[9] and to let $N\to\infty$ (the so-called "thermodynamic limit"). Here we adopt this technique from the point of view that it can be regarded as a combination of Möbius transformation and multiplication by a phase. We find this algorithm particularly useful since it enables us to study two aspects of importance from a physical point of view: (a) The dependence of the transmission on the strength of the potential for fixed energy, and (b) the occurrence of phase transition from a conductor to an insulator. We can study here the complex transmission amplitude and not just its squared absolute value (the conductance). The importance of this quantity is that it is directly related to the density of states $N(E)$ through the relation[10]

$$N(E)-N_0(E)=\frac{1}{\pi}\frac{d\{\arg[t(E)]\}}{dE},$$

where $N_0(E)$ is the free-particle density of states.

If the system of scatterers is arranged in an arithmetic progression (a perfectly ordered crystal), the limit $N\to\infty$ can be easily obtained and a band structure of the transmission can be deduced; namely, the transmission as a function of the energy is zero on some segments and greater than zero on other segments. On the other hand, if the position of scatterers is completely random, a closed-form expression for the transmission cannot be found in general, but an ensemble average of the transmission over many samples can sometimes be carried out and the results show the transmission decays exponentially with N (namely with length) with some characteristic localization length.

The intermediate case, where the scatterers are located on an arbitrary sequence, is interesting in itself. For example, one-dimensional quasicrystals are characterized by a system of scatterers located on the sequence

$$x_N=N+\frac{1}{\tau}\left\lfloor\frac{N}{\tau}+\beta\right\rfloor$$

where $\tau=(1+\sqrt{5})/2$ is the golden ratio, $\lfloor\cdot\rfloor$ denotes the integer value (the "floor function"), and β is an arbitrary real number. We plan to report on our study of scattering from one-dimensional quasicrystals in a future paper, but here we concentrate on scattering from an infinite system of δ-function potentials located on the Fibonacci numbers. It will become evident that the mathematical concepts and the calculation techniques developed here can be applied elsewhere.

In Sec. II we will present our study, while in Sec. III we will set the present work in context with other related works. Some formal mathematical arguments (which are clearly very relevant to quasiperiodic systems) will be relegated to the Appendix.

II. RESULTS BASED ON THE RECURSION AND MÖBIUS TRANSFORMATION TECHNIQUES

Consider a one-dimensional array of N δ-function potentials

$$V_N(x)=v\sum_{n=1}^{N}\delta(x-x_n)\ , \tag{1}$$

where $v>0$ and x_n are the Fibonacci numbers $x_{m+1}=x_{m-1}+x_m$ for $m\geq 1$, with $x_1=1$ and $x_2=2$. (We start with $x_1=1$ since we will mainly employ the sequence of differences $y_m=x_{m+1}-x_m$ which is the Fibonacci sequence 1,1,2,3,5,8,13, ...) A plane wave at momentum k, e^{-ikx}, coming from the right will have reflection and transmission amplitudes r_N and t_N, respectively. For $N=1$, these amplitudes are given by

$$r=\frac{v}{2ik-v},\quad t=\frac{2ik}{2ik-v}\ , \tag{2}$$

which satisfy unitarity

$$|r|^2+|t|^2=1,\quad tr^*+t^*r=0 \tag{3}$$

and continuity at the point $x=x_1$,

$$t=1+r\ . \tag{4}$$

The unitarity relation (3) is valid, of course, for any N. For $N>1$ scattering centers, the reflection and transmission amplitudes can be determined from a recursion relation as follows: We define

$$a_n=\frac{1}{t_n},\quad b_n=\frac{r_n}{t_n},\quad \underline{A}=\begin{bmatrix}1/t & -r/t\\ r/t & (t^2-r^2)/t\end{bmatrix},$$
$$\underline{\Delta}_n=\begin{bmatrix}e^{-iky_n} & 0\\ 0 & e^{iky_n}\end{bmatrix}, \tag{5}$$

with $\det(\underline{A})=\det(\underline{\Delta}_n)=1$. Then we have

$$\begin{bmatrix}a_{n+1}\\ b_{n+1}\end{bmatrix}=\underline{A}\,\underline{\Delta}_n\begin{bmatrix}a_n\\ b_n\end{bmatrix}. \tag{6}$$

The conductance (at zero temperature) of this system is given by the limit of $|t_N|^2=1/|a_N|^2$ as $N\to\infty$. Equivalently we may inspect the limit of $|r_N|^2=|b_N/a_N|^2$ and use unitarity. If $|t_N|\to 0$ (equivalently $|r_N|\to 1$) as $N\to\infty$, we say that the system is an insulator. If $|t_N|$ does not tend to 0 as $N\to\infty$ the system may conduct. Our aim is to find out for what values of the momentum k and the strength v the system is an insulator or may conduct.

It may happen, of course, that the sequence of complex numbers r_N will not converge while the sequence of absolute values $|r_N|$ (which is what is relevant from a physical point of view) will converge. This case will be studied later on. Our first goal will be to find out under what conditions the limit of the sequence $\{r_N\}$ exists, and what this limit is. As a quotient, it will not be affected if the matrix $\underline{A}$ is multiplied by -1 and the matrix $\underline{\Delta}_n$ is multiplied by e^{iky_n}. Denoting $\lambda_n=e^{2iky_n}$ and using $r=t-1$, we replace Eq. (6) by

$$\begin{bmatrix}a_{n+1}\\ b_{n+1}\end{bmatrix}=\begin{bmatrix}-1/t & r/t\\ -r/t & -2+1/t\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & \lambda_n\end{bmatrix}\begin{bmatrix}a_n\\ b_n\end{bmatrix}. \tag{7}$$

A very useful relation relating r_{n+1} to r_n in a fractional linear form can be deduced from (7), namely

$$r_{n+1}=\frac{r+(2t-1)\lambda_n r_n}{1-r\lambda_n r_n}\ . \tag{8}$$

Equation (8) tells us that the reflection of $n+1$ barriers is obtained from that of n barriers in two steps, namely, multiplication by a phase followed by a Möbius transformation. This result is, of course, general for any one-dimensional system of scatterers.

Thus, from a physical point of view, the difference between the ordered (periodic) case and the other two (quasiperiodic and disordered) cases is that here λ_n is not a constant, but depends on n. It is then intuitively clear that the behavior of the sequence $\{r_n\}$ is directly related to that of $\{\lambda_n\}$. This is indeed the case. Lemma 1 in the Appendix asserts that if the sequence $\{r_n\}$ converges to a number ρ, then $\lambda_n\to 1$ as $n\to\infty$. In other words, a necessary condition for the convergence of the sequence of complex reflection amplitudes is the convergence of the sequence of phases to 1. [Notice that since $|r|\leq 1$ and $\det(\underline{A})=\det(\underline{\Delta}_n)=1$, we have $\rho\leq 1$.]

The converse statement is not evident. Inspection of Eq. (7) shows that if $\lambda_n=1$ identically, then the reflection is perfect, namely $r_n\to -1$ irrespective of the potential strength v. This is an example of a periodic system in which the interference is destructive. The question which we raise here is what happens if λ_n is not equal to 1 but tends to 1? The answer (proved in Lemma 2 in the Appendix) is that only if $\lambda_n\to 1$ *exponentially* as $n\to\infty$, then $r_n\to -1$.

Now we shall investigate under what conditions (on k) the sequence $\{\lambda_n=e^{2iky_n}\}$ tends exponentially to 1 as $n\to\infty$. This will give us part of the energy values for which the system is an insulator irrespective of the potential strength v. The answer, proved in Lemma 3 in the Appendix, is that $\lambda_n\to 1$ exponentially if, and only if k/π is an algebraic integer in $\mathbb{Q}[\sqrt{5}]$; namely, k can be expressed as

$$k=\pi[a+b(1+\sqrt{5})/2]\quad\text{with } a,b \text{ integers}\ . \tag{9}$$

The set of numbers k with the representation (9) is countable and dense on the real axis. Physically, $k>0$, of course, and our result means that at least for that set of momenta the system reflects perfectly.

We now turn to the study of the transmission as a function of both k and v. Physically, the important question to be asked is whether a phase transition between a conductor and insulator occurs for some values of v and for values of k which are not necessarily equal to the set of values given in Eq. (9). To this end we inspect the properties of Eq. (8) from the point of view of Möbius transformations. First, it is proved (Lemma 4 in the Appendix) that the Möbius transformation

$$w=T(z)=\frac{r+(2t-1)z}{1-rz} \tag{10}$$

[obtained from (8) simply by setting $z=\lambda_n r_n$] maps the unit disk onto itself such that the unit circle is mapped onto itself. [From unitarity we know that $|r_N|\leq 1$, so that the transformation (10) should map the unit disc onto itself.] Another important property of $T(z)$ is that it leaves invariant the set U of all circles with the center on the real axis passing through the point $z=-1$ including the vertical line $-1+iy$ ($y\in\mathbb{R}$). This is proved in Lemma 5. It is also possible to show that the transformation T moves the points (clockwise or counterclockwise, depending on the parameters) on each circle belonging to U. The point -1 is never reached, and hence the points on such a circle near -1 are very dense. Thus, the transformation T does not conserve the Lebesgue measure for arcs on a circle in U.

These considerations can help us to inspect the dependence of the conductance on N (namely on length), which is of great importance from a physical point of view. The iteration procedure (8) is a successive operation of T followed by a multiplication by a phase λ_n, which moves the point on a canonical circle. Thus, an orbit of a point traces an arc of a circle from U and then an arc of a canonical circle, and so on. When λ_n tends exponentially to 1 as $n\rightarrow\infty$, the motion on the canonical circle is suppressed and the convergence to -1 along a circle in U dominates. Physically we are interested only in orbits inside the unit circle.

Thus, we have generally studied the convergence of the sequence $\{r_n\}$ and turn now to study the sequence $\{|r_n|\}$. As state before, we want to find out for which values of k and v we have $|r_n|\rightarrow 1$ as $n\rightarrow\infty$. (We have already proved, of course, that it happens for a countable dense set $k\in\pi\mathbb{Z}[(1+\sqrt{5})/2]$ and for any $v\neq 0$) although for almost every k the sequence $\{|r_n|\}$ is dense in the segment [0,1].

Let us start to study the sequence $\{|r_n|\}$ for some specific values of $k\notin\pi\mathbb{Z}[(1+\sqrt{5})/2]$. Specifically, we start with $k=\frac{1}{2}(2N+1)\pi$ ($N=0,1,2,\ldots$). Then the sequence $\{\lambda_n\}$ has a period of 3; $\lambda_1=\lambda_2=-1$, $\lambda_3=1$, $\lambda_{n+3}=\lambda_n$. The matrix of the Möbius transformation (10) is $\begin{pmatrix}2t-1 & r\\ -r & 1\end{pmatrix}$, which can, of course, be multiplied by an arbitrary scalar. We multiply this matrix by 2, use the parametrization [using Eqs. (2) and (4)]

$$\begin{gathered}\psi=\arctan(v/k),\quad \phi=e^{-i\psi}\,,\\ t=\tfrac{1}{2}(1+\phi),\quad r=\tfrac{1}{2}(-1+\phi)\,,\end{gathered} \tag{11}$$

and define

$$\begin{gathered}\underline{A}=2\begin{bmatrix}2t-1 & r\\ -r & 1\end{bmatrix}=\begin{bmatrix}2\phi & \phi-1\\ 1-\phi & 2\end{bmatrix},\\ \underline{\Delta}=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}.\end{gathered} \tag{12}$$

For $k=\frac{1}{2}(2N+1)\pi$ ($N=0,1,2,2,\ldots$), the matrix transforming r_{3n} to r_{3n+3} is then independent of n and is given explicitly by

$$\underline{S}=\underline{A}\;\underline{A}\;\underline{\Delta}\;\underline{A}\;\underline{\Delta}=\begin{bmatrix}8\phi^2-6\phi+2 & -3\phi^2+8\phi-5\\ -5\phi^2+8\phi-3 & 2\phi^2-6\phi+8\end{bmatrix}. \tag{13}$$

The eigenvalues of $\underline{S}$, s_1 and s_2, are given (using $\eta=\phi+\phi^*=2\cos\psi$) explicitly:

$$\left.\begin{matrix}s_1\\ s_2\end{matrix}\right\}=\phi\{5\eta-6\pm[(4\eta-8)(6\eta-4)]^{1/2}\}\,. \tag{14}$$

It is easy to check that $4\eta-8<0$ [otherwise, equality implies $r=1$, contrary to Eq. (2)] and that $6\eta-4>0$ if, and only if $|\cos\psi|<\frac{1}{3}$. Let us then regard $\underline{S}$ as a Möbius transformation and find its fixed points $S(y)=y$. We define $\mu^2=\phi$ so that $\eta=(\mu-\mu^*)^2+2$ and

$$\begin{gathered}\left.\begin{matrix}s_1\\ s_2\end{matrix}\right\}=\phi\{5(\mu-\mu^*)^2\\ +4\pm 2(\mu-\mu^*)[6(\mu-\mu^*)^2+8]^{1/2}\}\,.\\ \left.\begin{matrix}y_1\\ y_2\end{matrix}\right\}=\frac{3(\mu+\mu^*)\pm 2[6(\mu-\mu^*)^2+8]^{1/2}}{-5\mu+3\mu^*}\,,\end{gathered} \tag{15}$$

such that the direction y_i corresponds to the eigenvalue s_i ($i=1,2$). Note also that s_i are defined up to an arbitrary constant different from zero. We now study several possibilities.

Case (a) $6\eta-4>0$. In this case $s_1\neq s_2$ and after some algebraic manipulations we get

$$r_{3n}=\frac{3-5\phi^*}{3-5\phi}\,\frac{1-(s_2/s_1)^n}{y_2-y_1(s_2/s_1)^n}\,. \tag{16}$$

From (14) it is clear that $|s_1|=|s_2|$ and hence we study two subcases of (a).

Subcase (a1) s_2/s_1 is not a root of unity. In this case the sequence $(s_2/s_1)^n$ is dense on the unit circle. Hence, the sequence $\{r_{3n}\}$ has infinitely many limit points contained in the image of the unit circle under the Möbius transformation

$$w=M(z)=\phi\frac{3-5\phi^*}{3-5\phi}\,\frac{1-z}{y_2-y_1 z}\,. \tag{17}$$

The sequence $\{r_{3n}\}$ is contained in the unit circle (by unitarity) and therefore, the pertinent set of limit points is dense on a circle contained in the unit circle. Since $M(1)=0$, this circle is not the unit circle, and in fact it can intersect the unit circle in, at most, one point. The set of limit points for the sequence $\{r_n\}$ lies on three such circles, which, in general, are different from each other. To summarize, in subcase (a1) we find that the sequence $|r_n|$ does not converge to 1. The system may conduct.

Subcase (a2) s_2/s_1 is a root of unity. In that case the sequence $(s_2/s_1)^n$ is periodic and equals 1 infinitely often. Hence, the sequence $\{r_n\}$ is periodic and equals 0 for infinitely many n's. Hence in subcase (a2) we also find that the sequence $\{|r_n|\}$ does not converge to 1. The system may conduct.

Let us now find out which roots of unit may be obtained as the ratio s_2/s_1. Up to complex conjugation, we have

$$\frac{s_2}{s_1}=\frac{5\eta-6\pm[(4\eta-8)(6\eta-4)]^{1/2}}{5\eta-6\pm[(4\eta-8)(6\eta-4)]^{1/2}}\,, \tag{18}$$

which is a root of unity if and only if the angle

$$\alpha=\arctan\left[\frac{[(8-4\eta)(6\eta-4)]^{1/2}}{5\eta-6}\right]$$

is a rational multiple of π. Recalling Eq. (11) and the definition of η after Eq. (13), we inspect the argument of the arctan function as the phase ϕ moves on the unit circle from $\psi=0$ to $\psi=\arccos\frac{1}{3}$. (Similar behavior will be found between $\psi=0$ and $\psi=-\arccos\frac{1}{3}$.) This expression vanishes at $\psi=0$ and grows continuously (apparently monotonically, but that is irrelevant here) to ∞ as ψ approaches $\arccos\frac{3}{5}$ from below. After changing sign it grows (apparently monotonically) from $-\infty$ to 0 at $\psi=\arccos\frac{1}{3}$. Therefore, s_2/s_1 can be any root of unity and, hence, the sequence $\{r_n\}$ can have period $3L$ where L is an arbitrary integer. We have thus exhausted case (a), namely $6\eta-4>0$.

Case (b) $6\eta-4<0$. Equation (16) still holds, but this time $|s_1|\neq|s_2|$, and we assume, say, $|s_1|>|s_2|$. In that case we find $|r_{3n}|\to|y_1|$. However, we claim that $|y_1|=|y_2|=1$, so that the result will be similar if $|s_2|>|s_1|$. Indeed, from Eq. (15), this time with a negative argument of the square root $6(\mu-\mu^*)^2+8=6\eta-4<0$, we see that $y_{1/2}$ are multiples of $1/(-5\mu+3\mu^*)$ by two complex-conjugate numbers and hence $|y_1|=|y_2|$. In addition, $y_1y_2=\phi(3-5\phi^*)/(3-5\phi)$, a complex number with a unit modulus, namely $|y_1y_2|=1$, and therefore $|y_1|=|y_2=1$. Hence the sequence $\{r_n\}$ has at most three limit points, all of them on the unit circle. In fact, the sequence has exactly three limit points. Indeed, if the Möbius transformation $A\Delta$ leaves y_1 untouched, so does the transformation $A\Delta A\Delta$ and hence so does A. But A has only one fixed point, $y=-1$. The other possibility—$A\Delta y_1\neq y_1$ and $A\Delta A\Delta y_1=A\Delta y_1$—is ruled out since, in that case, two different points y_1 and $A\Delta y_1$ are transformed by $A\Delta$ to the same point contrary to the single-value property of the Möbius transformation. We have thus exhausted case (b).

Case (c) $6\eta-4=0$. From (16) and (17) we see that there is one eigenvalue s and one eigendirection (fixed point) of the Möbius transformation $\underline{S}$ [Eq. (13)], $y=(-1+2\sqrt{2i}\,)/3$. It is easy to check by substitution that in that case the matrix $\underline{S}$ is not a scalar matrix so it is similar to the matrix

$$\underline{B}=\begin{bmatrix} s & 1\\ 0 & s\end{bmatrix}\,. \tag{19}$$

Denoting by $u_1=\binom{1}{y}$ the only eigenvector of this matrix and defining a vector u_2 by

$$\underline{B}\,\underline{u}_2=\underline{u}_1+s\underline{u}_2\,,$$

it follows that, since u_1 and u_2 are nearly independent, we may express the initial vector in the basic recursion relation (7) as $\binom{a_0}{b_0}=\alpha_1\underline{u}_1+\alpha_2\underline{u}_2$. Hence, at step $3n$ we have

$$\begin{aligned}\begin{bmatrix} a_{3n}\\ b_{3n}\end{bmatrix}&=\alpha_1\underline{B}^n\underline{u}_1+\alpha_2\underline{B}^n\underline{u}_2\\ &=\alpha_1 s^n\underline{u}_1+\alpha_2(s^n\underline{u}_2+ns^{n-1}\underline{u}_1)\\ &=s^{n-1}[(\alpha_1 s+n\alpha_2)\underline{u}_1+\alpha_2 s\underline{u}_2]\,.\end{aligned} \tag{20}$$

Therefore, whether $\alpha_2=0$ or $\alpha_2\neq 0$ the slopes of these vectors tend to the slope of $\underline{u}_1$, namely y. Hence, the sequence $\{r_{3n}\}$ has one limit point on the unit circle and the sequence $\{r_n\}$ has three limit points on the unit circle. This means, of course, $|r_n|\to 1$ as $n\to\infty$.

We have thus completed the case $k=\frac{1}{2}(2m+1)\pi$. To reemphasize the important physics, we have just established that for $k=\frac{1}{2}(2N+1)\pi$ $(N=0,1,2,\dots)$ and $3\cos\psi-1>0$ with $\psi=\arctan(v/k)$ the system may conduct (namely, the reflection coefficient is strictly smaller than unity), while for $3\cos\psi-1<0$ the system is an insulator. These results are physically remarkable since they imply for fixed $k=\frac{1}{2}(2N+1)\pi$ $(N=0,1,2,\dots)$ a phase transition between a conductor and an insulator as the strength v varies continuously near $k\sqrt{8}$.

We now study the case of k being a rational multiple of π, $k=(p/m)\pi$, with p and m relatively prime $[(p,m)=1]$ and $m>1$. The sequence $\lambda_n=e^{2iky_n}$ is, of course, periodic modulo (m). Furthermore, expressing the Fibonacci recursion relation as a transformation

$$\begin{bmatrix} y_{n+1}\\ y_{n+2}\end{bmatrix}=\begin{bmatrix} 0 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix} y_n\\ y_{n+1}\end{bmatrix}\,, \tag{21}$$

and the fact that the matrix $\left(\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right)$ is invertible modulo m, the periodicity starts at the beginning of the sequence. Let us assume that the period is M. We have to study a matrix of the form

$$\underline{D}=\underline{C}\,\underline{\Delta}_M\underline{C}\,\underline{\Delta}_{M-1}\cdots\underline{C}\,\underline{\Delta}_1\,, \tag{22}$$

where

$$\begin{aligned}\underline{C}&=\begin{bmatrix} 1/t & -r/t\\ r/t & 2-1/t\end{bmatrix}\\ &=\begin{bmatrix} 1-r/t & -r/t\\ r/t & 1+1/t\end{bmatrix}\\ &=\mathbb{1}+iq\begin{bmatrix} -1 & -1\\ 1 & 1\end{bmatrix},\quad r/t=iq,\quad q\in\mathbb{R}\,,\end{aligned} \tag{23}$$

and the matrices $\underline{\Delta}_n$ are defined in Eq. (5). Both matrices $\underline{C}$ and $\underline{\Delta}_n$ have the form $\left(\begin{smallmatrix}a&b\\b^*&a^*\end{smallmatrix}\right)$ for some $a,b\in\mathbb{C}$. It is easy to see that the set of all such matrices form a subring in the ring $M_2(\mathbb{C})$ of all 2×2 matrices over $\mathbb{C}$. Furthermore, $\underline{C}$ and $\underline{\Delta}_n$ have a unit determinant and so does $\underline{D}$ [defined in Eq. (22) above]. We then write $\underline{D}$ as

$$\underline{D}=\begin{bmatrix} d_1 & d_2 \\ d_2^* & d_1^* \end{bmatrix}, \tag{24}$$

and study its eigenvalues

$$\left.\begin{matrix} s_1 \\ s_2 \end{matrix}\right\}=\tfrac{1}{2}\{d_1+d_1^*\pm[(d_1+d_1^*)^2-4]^{1/2}\} . \tag{25}$$

Hence if $|d_1+d_1^*|\geq 2$ both eigenvalues are real and distinct (unless $|d_1+d_1^*|=2$), while if $|d_1+d_1^*|<2$ they are not real and $s_2=s_1^*$. In particular, $|s_1|=|s_2|$ for $|d_1+d_1^*|\leq 2$. To find the eigendirections we have to find the fixed points of the transformation $\underline{D}$, namely,

$$w=\frac{d_1^*w+d_2^*}{d_2w+d_1},$$

$$\left.\begin{matrix} w_1 \\ w_2 \end{matrix}\right\}=\frac{1}{2d_2}\{d_1^*-d_1\pm[(d_1+d_1^*)^2-4]^{1/2}\} . \tag{26}$$

The product of the eigendirections is $w_1w_2=-d_2^*/d_2$ and hence $|w_1w_2|=1$. Also, since $d_1^*-d_1$ is purely imaginary, we have $|w_1|=|w_2|=1$ if, and only if $|d_1+d_1^*|\geq 2$. We now argue that

$$\underline{\Delta}_M\underline{\Delta}_{M-1}\cdots\underline{\Delta}_1=\beta\mathbb{1},\quad \beta=\pm 1 . \tag{27}$$

Indeed, it is sufficient to show that

$$y_1+y_2+\cdots+y_M\equiv 0 \pmod m . \tag{28}$$

We define a matrix $\underline{\sigma}=\binom{0\ 1}{1\ 1}$ and note from the definition of M that

$$\underline{\sigma}\left[\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}+\begin{bmatrix} y_2 \\ y_3 \end{bmatrix}+\begin{bmatrix} y_3 \\ y_4 \end{bmatrix}+\cdots+\begin{bmatrix} y_M \\ y_{M+1} \end{bmatrix}+\begin{bmatrix} y_{M+1} \\ y_{M+2} \end{bmatrix}\right]$$
$$\equiv\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}+\begin{bmatrix} y_2 \\ y_3 \end{bmatrix}+\begin{bmatrix} y_3 \\ y_4 \end{bmatrix}+\cdots+\begin{bmatrix} y_M \\ y_{M+1} \end{bmatrix} \pmod m , \tag{29}$$

and, since $\underline{\sigma}$ is invertible (modulo m), then

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}+\begin{bmatrix} y_2 \\ y_3 \end{bmatrix}+\begin{bmatrix} y_3 \\ y_4 \end{bmatrix}+\cdots+\begin{bmatrix} y_M \\ y_{M+1} \end{bmatrix}\equiv 0 \pmod m ,$$

which proves (28) and hence also proves (27). We now concentrate on $\mathrm{Tr}(\underline{D})$ and find

$$\begin{aligned}\mathrm{Tr}(\underline{D})&=\mathrm{Tr}(\underline{C}\,\underline{\Delta}_M\underline{C}\,\underline{\Delta}_{M-1}\cdots\underline{C}\,\underline{\Delta}_1)\\ &=\mathrm{Tr}[(\mathbb{1}+iq\underline{T})\underline{\Delta}_M(\mathbb{1}+iq\underline{T})\underline{\Delta}_{M-1}\cdots(\mathbb{1}+iq\underline{T})\underline{\Delta}_1]\\ &=\sum_{j=0}^{M}(iq)^j\sum_{l_1+l_2+\cdots+l_M=j;\,l_n=0;1}\mathrm{Tr}(\underline{T}^{l_M}\underline{\Delta}_M\underline{T}^{l_{M-1}}\underline{\Delta}_{M-1}\cdots\underline{T}^{l_1}\underline{\Delta}_1) ,\end{aligned} \tag{30}$$

with $\underline{T}=\binom{-1\ \ -1}{1\ \ \ \ 1}$. If we regard the left-hand side of Eq. (30) as a polynomial in q, then its free term is 2β. We will see now that the coefficient of q is 0. Indeed, the coefficient of q is

$$i\sum_{l_1+l_2+\cdots+l_M=1;\,l_n=0;1}\mathrm{Tr}(\underline{T}^{l_M}\underline{\Delta}_M\underline{T}^{l_{M-1}}\underline{\Delta}_{M-1}\cdots\underline{T}^{l_1}\underline{\Delta}_1) .$$

This is a sum of M terms, each of which is a trace of a product containing all the matrices $\underline{\Delta}_n$ and one matrix $\underline{T}$. Using the commutativity $\mathrm{Tr}(\underline{A}\,\underline{B})=\mathrm{Tr}(\underline{B}\,\underline{A})$ to put $\underline{T}$ to the left, and the form (29) for the product of the matrices $\underline{\Delta}_n$ (which is independent on the order), we find that each term is indeed 0. We can now show that the coefficient of q^2 is $-a\beta$, where a is real, $a>0$. A typical term in the expansion (30) for $j=2$ is

$$i^2\mathrm{Tr}(\underline{\Delta}_M\cdots\underline{\Delta}_{j_2}\underline{T}\,\underline{\Delta}_{j_2-1}\cdots\underline{\Delta}_{j_1}\underline{T}\,\underline{\Delta}_{j_1-1}\cdots\underline{\Delta}_1)$$
$$=-\mathrm{Tr}(\underline{T}\,\underline{\Delta}_{j_2-1}\cdots\underline{\Delta}_{j_1}\underline{T}\,\underline{\Delta}_{j_1-1}\cdots\underline{\Delta}_1\underline{\Delta}_M\cdots\underline{\Delta}_{j_2})=-\mathrm{Tr}(\underline{T}\,\underline{B}\,\underline{T}\beta\underline{B}^{-1}) ,$$

where $\underline{B}$ is a matrix of the form $\underline{B}=\binom{h\ \ 0}{0\ \ h^*}$ with $|h|=1$. Hence, after some algebraic manipulation we find

$$-\mathrm{Tr}(\underline{T}\,\underline{B}\,\underline{T}\beta\underline{B}^{-1})=-\beta[2-h^2-(h^*)^2] .$$

Clearly, $[2-h^2-(h^*)^2]\geq 0$ for all h on the unit circle, with equality only for $h=\pm 1$. It is also evident that for $m>1$ at least some part of the matrices $\underline{B}$ are not equal to $\pm\mathbb{1}$. Hence the coefficient of q^2 is $-a\beta$, where a is real, $a>0$, and

$$\mathrm{Tr}(\underline{D})=2\beta-aq^2\beta+\sum_{j=3}^{M}c_jq^j \tag{31}$$

for some coefficients c_j. Hence, for $q\neq 0$ sufficiently close to 0, $|\mathrm{Tr}(\underline{D})|<2$ and the eigenvalues of $\underline{D}$ are complex conjugates. In this case, exactly as in the case $k=(N+\frac{1}{2})\pi$ we get the result that the sequence $\{|r_n|\}$ does not tend to 1 as $n\to\infty$, namely that the system may conduct. Small q means small r, namely a small value of v/k. On the other hand, since the polynomial in q, $\mathrm{Tr}(\underline{D})$ [Eq. (30)] is not a constant, it will lead to $|\mathrm{Tr}(\underline{D})|\geq 2$ for large enough q, and in that case, $|r_n|\to 1$. This is also similar to the case $k=(N+\frac{1}{2})\pi$ except that here we cannot say that there is only one value of v/k at which the transitions from conductor to insulator (or vice versa) occur.

Again, we reemphasize the important result from a physical point of view. For any k which is a rational noninteger multiple of π, the system conducts for a small value of v/k and becomes an insulator at a large value of v/k. This means that at least one phase transition of this kind occurs at any k which is a rational noninteger multiple of π, once v/k becomes large enough.

III. DISCUSSION

The present study should be matched with those in Refs. 5, 8, and 9. From the point of view of the theory of quasicrystals, our results are less useful since we did not apply our algorithm to a Fibonacci chain. Apparently, the studies of Kollar and Suto[9] and of Holzer[9] are related to the present one when applied to a Fibonacci chain. For example, we can show how to get one of their results.

We consider the one-dimensional array

$$x_N = N + \frac{1}{\tau}\left\lfloor \frac{N}{\tau} \right\rfloor ,$$

where $\tau=(1+\sqrt{5})/2$ is the golden ratio, and study the scattering problem for the discrete set of wave numbers, $k=m\tau\pi$ and m a positive integer. In this case $ky_n=k(x_{n+1}-x_n)$ is either $m\tau\pi$ or $m\tau\pi+m\pi$. These two cases are, in fact, identical since as a quotient the result will not be affected if the matrix $\underline{A}$ [Eq. (5)] is multiplied by a scalar matrix ($-\underline{1}$ in this particular case). Denote $q=-v/2k$, $\alpha=m\tau\pi$, $\mu=e^{i\alpha}$, and using $r=t-1$, the matrix $\underline{D}_n=\underline{A}\,\underline{\Lambda}_n$ [see Eq. (5) for the definition of $\underline{\Lambda}_n$] can be replaced by the constant matrix

$$\underline{D}=\begin{bmatrix}1-iq & -iq\\ iq & 1+iq\end{bmatrix}\begin{bmatrix}\mu^* & 0\\ 0 & \mu\end{bmatrix} = \begin{bmatrix}(1-iq)\mu^* & -iq\mu\\ iq\mu^* & (1+iq)\mu\end{bmatrix} . \tag{32}$$

We have already shown that $|r_N|\to 1$ if and only if $|\mathrm{tr}(\underline{D})|\geq 2$. But $\mathrm{tr}(\underline{D})=\mu+\mu^*+iq(\mu-\mu^*)=2(\cos\alpha-q\sin\alpha)$. Hence we obtain

$$|r_N|\to 1 \text{ if and only if } q\leq q_0\leq 0,\quad q_0=\begin{cases}-(1-\cos\alpha)/\sin\alpha & \text{if } \sin\alpha>0\\ (1+\cos\alpha)/\sin\alpha & \text{if } \sin\alpha<0 .\end{cases} \tag{33}$$

It is worth mentioning here that if $|r_N|\to 1$ as $N\to\infty$, then the sequence $\{r_N\}$ converges, while if the sequence $\{|r_N|\}$ does not tend to unity, then the set of accumulation points of the sequence $\{r_N\}$ is located on a circle which passes through the origin. This is result (ii) obtained by Kollar and Suto.[9]

In conclusion, we have developed the mathematical concepts and tools for the study of scattering from a system of scatterers located on an arbitrary sequence of (positive) real numbers. From this point of view, the present work is quite general since the phase between two scatterers must not assume only two values as in the Fibonacci chain discussed above. We applied it to study transmission through an infinite system of δ-function potentials located on the Fibonacci numbers, but have shown that it can also be applied to Fibonacci chains.

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APPENDIX

Lemma 1. If the sequence r_n converges to a number ρ then $\lambda_n\to 1$. [Notice that since $|r|\leq 1$ and $\det(\underline{A})=\det(\underline{\Lambda}_n)=1$, then $\rho\leq 1$.]

Proof. It is sufficient to demonstrate the convergence $\lambda_n\to\lambda$ for some λ since, then using the fact that y_n is the Fibonacci sequence, we will have $\lambda_{n+2}=\lambda_{n+1}\lambda_n$, so that in the limit $\lambda=\lambda^2$ and, since $\lambda\neq 0$, then $\lambda=1$. From Eq. (8) we get

$$\lambda_n r_n[rr_{n+1}-(1-2t)]=r_{n+1}-r . \tag{A1}$$

From Eq. (A1) we see that $\rho\neq 0$, since otherwise the left-hand side will tend to zero and the right-hand side will tend to $-r$, a contradiction. We also claim that $\rho\neq(1-2t)/r$, since otherwise the left-hand side of Eq. (A1) tends to zero, which implies $\rho=r$ but clearly $(1-2t)/r\neq r$. Therefore, the square-bracketed factor in Eq. (A1) tends to a nonzero limit. Hence, dividing Eq. (A1) by $r_n[rr_{n+1}-(1-2t)]$ indicates that the sequence λ_n converges. This proves the lemma.

Lemma 2. If $\lambda_n\to 1$ exponentially, then $r_n\to -1$. (Although this can be intuitively expected from Eq. (8) by setting $\lambda_n=1$ and assuming that the iterations end at the fixed point -1, the rigorous proof needs some precaution.)

Proof. Denote $s_n=-1/(1+r_n)$. From Eq. 8) we get, after some algebraic manipulation and use of Eq. (4), the recursion relation for s_n as

$$s_{n+1}=s_n+\frac{r}{t}-(1-\lambda_n^{-1})\frac{s_n(1-s_n)}{s_n(1-\lambda_n^{-1})+1} . \tag{A2}$$

Lemma 2 is equivalent to $s_n\to\infty$ (as a complex number). If this does not hold, then there exists a constant $C>0$ for which the number of terms s_n with $|s_n|<C$ is infinite. We will now show that in that case there exists an integer n_0 such that

$$|s_n|\leq C+2n|r/t| \quad \text{for all } n\geq n_0 . \tag{A3}$$

Indeed, we take some $0<q<1$ and n_0 for which the following estimates hold:

$$|s_{n_0}|\le C,\quad |1-\lambda_n|\le q^n,\quad (C+2n|r/t|)q^n\le \tfrac{1}{2},\quad 2(1+C+2n|r/t|)^2q^n\le |r/t| \quad \text{for } n\ge n_0 \;. \tag{A4}$$

From this choice the inequality (A4) is, of course, satisfied for $n=n_0$. If it is true for some $n\ge n_0$, then from Eq. (A2) we get

$$\begin{aligned}|s_{n+1}| &\le |s_n|+|r/t|+|1-\lambda_n^{-1}|\,|s_n|(1+|s_n|)\frac{1}{|s_n(1-\lambda_n^{-1})+1|}\\ &\le C+2n|r/t|+|r/t|+q^n(1+|s_n|)^2\frac{1}{1-|s_n||1-\lambda_n^{-1}|}\\ &\le C+(2n+1)|r/t|+(1+C+2n|r/t|)^2q^n\frac{1}{1-(C+2n|r/t|)q^n}\\ &\le C+(2n+1)|r/t|+2(1+C+2n|r/t|)^2q^n\\ &\le C+(2n+1)|r/t|+|r/t|\le C+2(n+1)|r/t| \;,\end{aligned}$$

which proves the inequality (A3).

Using the recursion relation (A2) from n_0 to n,

$$s_n=s_{n_0}+(n-n_0)\frac{r}{t}-\sum_{k=n_0}^{n-1}(1-\lambda_k^{-1})\frac{s_k(1-s_k)}{s_k(1-\lambda_k^{-1})+1} \;,$$

and employing the inequality (A3), we obtain

$$|s_n-s_{n_0}-(n-n_0)(r/t)|\le\sum_{k=n_0}^{n-1}2(1+C+2k|r/t|)^2q^k\le 2(1+C+2n|r/t|)^2\sum_{k=n_0}^{\infty}k^2q^k=M<\infty \;. \tag{A5}$$

Since $s_{n_0}+(n-n_0)(r/t)\to\infty$ (as a complex number), then also $s_n\to\infty$. This contradicts the negative assumption $|s_n|<C$ for an infinite number of terms. Recalling that $s_n=1/(1+r_n)$ we conclude that $r_n\to -1$ from inside the unit circle ($|r_n|\le 1$ by unitarity) and this proves Lemma 2.

Lemma 3. $\lambda_n\to 1$ exponentially if and only if k/π is an algebraic integer in $\mathbb{Q}[\sqrt{5}]$, namely, k is written as

$$k=\pi[a+b(1+\sqrt{5})/2] \quad (a,b \text{ integers}) \;.$$

Proof. For a real number u, let us define the quantity $\|u\|=\min(u-\lfloor u\rfloor,\ \lfloor u\rfloor+1-u)$ as its distance from the nearest integer. If we set $q=k/\pi$, then

$$\begin{aligned}|\lambda_n-1| &= |e^{2\pi i q y_n}-1|\\ &= |e^{2\pi i\|qy_n\|}-1|\\ &= |1+2\pi i\|qy_n\|+o(\|qy_n\|)-1|\\ &= 2\pi\|qy_n\|+o(\|qy_n\|)\\ &= O(\|qy_n\|) \;.\end{aligned}$$

Therefore, it remains to be seen when the sequence $\{qy_n\}$ tends exponentially to 0 (mod 1). If we use Binet's formula and represent the Fibonacci numbers as

$$y_n=\frac{\sqrt{5}}{5}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right] \tag{A6}$$

then the second term tends exponentially to zero anyhow. Thus, with $q_1=(\sqrt{5}/5)q$ it remains for one to check for which real numbers q_1 the sequence $q_1[(1+\sqrt{5}/2)]^n$ tends exponentially to 0 (mod 1). We will now prove that the sequence $q_1[(1+\sqrt{5}/2)]^n$ converges to 0 (mod 1) if and only if the convergence is exponential if and only if q_1 is of the form

$$q_1=\frac{\sqrt{5}}{5}\left[a+b\frac{1+\sqrt{5}}{2}\right] \quad (a,b \text{ integers}) \;. \tag{A7}$$

Let $\tau=(1+\sqrt{5})/2$ and let $K=\mathbb{Q}[\tau]$. We take the following basis for K over $\mathbb{Q}$:

$$\omega_1=1,\quad \omega_2=\tau \;. \tag{A8}$$

We find the dual basis $\omega_1^*=(5-\sqrt{5})/10$, $\omega_2^*=\sqrt{5}/5$ constructed to satisfy the equalities

$$\mathrm{Tr}_{K/\mathbb{Q}}(\omega_i\omega_j^*)=\delta_{ij},\quad 1\le i,j\le 2 \;. \tag{A9}$$

Indeed, using $\mathrm{Tr}_{K/\mathbb{Q}}(a+b\sqrt{5})=(a+b\sqrt{5})+(a-b\sqrt{5})=2a$ $(a,b\in\mathbb{Q})$, we have

$$\mathrm{Tr}_{K/\mathbb{Q}}(\omega_1\omega_1^*)=\mathrm{Tr}_{K/\mathbb{Q}}\left[\frac{5-\sqrt{5}}{10}\right]=2\tfrac{5}{10}=1 \;,$$

$$\mathrm{Tr}_{K/\mathbb{Q}}(\omega_1\omega_2^*)=\mathrm{Tr}_{K/\mathbb{Q}}\left[\frac{\sqrt{5}}{5}\right]=2\times 0=0 \;,$$

$$\mathrm{Tr}_{K/\mathbb{Q}}(\omega_2\omega_1^*)=\mathrm{Tr}_{K/\mathbb{Q}}\left[\frac{4\sqrt{5}}{20}\right]=2\times 0=0 \;,$$

$$\mathrm{Tr}_{K/\mathbb{Q}}(\omega_2\omega_2^*)=\mathrm{Tr}_{K/\mathbb{Q}}\left[\frac{5+\sqrt{5}}{10}\right]=2\tfrac{5}{10}=1 \;.$$

We now employ results from Ref. 11, according to which the sequence $q_1\tau^n$ tends to 0 (mod 1) if and only if q_1 is of the form

$$q_1=\tau^{-m}(p_1\omega_1^*+p_2\omega_2^*)$$
$$=\left[\frac{\sqrt{5}-1}{2}\right]^m\left[p_1\frac{5-\sqrt{5}}{10}+p_2\frac{\sqrt{5}}{5}\right]$$
$$=\frac{\sqrt{5}}{5}\left[\frac{\sqrt{5}-1}{2}\right]^m\left[p_2+p_1\frac{\sqrt{5}-1}{2}\right], \qquad (A10)$$

for some integers p_1,p_2 and $m\geq 0$. Hence $q_1=(\sqrt{5}/5)\eta$, where $\eta\in\mathbb{Z}[\tau]$. On the other hand, since $(1,(\sqrt{5}-1)/2)$ is a basis of $\mathbb{Z}[\tau]$ over $\mathbb{Z}$, then for suitable p_1 and p_2 on the right-hand side of Eq. (A10) and for $m=0$ it is possible to represent any element of $(\sqrt{5}/5)\mathbb{Z}[\tau]$. Hence there is a convergence to zero if and only if q_1 has the form (A7). It remains to be seen that if q_1 has the form (A7) then the convergence of $q_1[(1+\sqrt{5}/2)]^n$ to 0 (mod 1) is exponential. Indeed, using the representation (A6) for the Fibonacci numbers we get, for every $a,b\in\mathbb{Z}$,

$$\|q_1\tau^n\|=\left|\left|\frac{\sqrt{5}}{5}(a+b\tau)\tau^n\right|\right|$$
$$\leq|a|\left|\left|\frac{\sqrt{5}}{5}\tau^n\right|\right|+|b|\left|\left|\frac{\sqrt{5}}{5}\tau^{n+1}\right|\right|$$
$$=|a|\frac{\sqrt{5}}{5}\tau^{-n}+|b|\frac{\sqrt{5}}{5}\tau^{-n-1},$$

which tends exponentially to zero. This ends the proof of Lemma 3.

Lemma 4. The Möbius transformation (8), $w=T(z)=[r+(2t-1)z]/(1-rz)$, maps the unit disk to itself such that the unit circle is mapped onto itself.

Proof. Let us use the parametrization (11). Then if $|z|=1$, we have

$$|w|=\frac{|-1+\phi+2\phi z|}{|2+z-\phi z|},$$

and by simple algebraic manipulations,

$$|-1+\phi+2\phi z|^2=|2+z-\phi z|^2 .$$

Hence, the unit circle is mapped onto itself. The point $z=0$ is mapped on $w=r$ and the points $z=-1$ is a fixed point of the transformation. Using considerations of connectedness and the one-to-one property of the Möbius transformations, it is clear (without resorting to unitarity) that the interior of the unit disk is mapped onto itself. This proves Lemma 4.

Lemma 5. Let U be the set of all circles with its center on the real axis passing through the point $z=-1$ including the vertical line $-1+iy$ $(y\in\mathbb{R})$. Then any element of S is invariant under T. (Evidently U contains the unit circle.)

Proof. Let z be a point on a circle passing through the point -1 whose center is at a real point c. Then

$$z=c+(1+c)|\mu| \quad (|\mu|=1) .$$

To prove that $T(z)$ remains on the same circle we have to show that

$$|T(z)-c|=|1+c| .$$

Using the representation $t=\frac{1}{2}(1+\phi)$, $r=\frac{1}{2}(-1+\phi)$ we get, after some algebraic manipulation,

$$T(z)-c=[-(1+c)^2+(1+c)^2\phi-c(1+c)\mu+(2+c)(1+c)\phi\mu]/[2+c+(1+c)\mu-c\phi-(1+c)\phi\mu] ,$$

and by inspecting the absolute value squared of the numerator divided by $|1+c|^2$ and the denominator, we find that they are indeed equal. Since the union of all the circles in U is the complex plane minus the line $-1+iy$ $(y\in\mathbb{R})$, it follows from the one-to-one properties of T that this line is also invariant under T. This proves Lemma 5.

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