Electrical conduction in checkerboard geometries

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We consider electrical conduction in a two-dimensional two-phase material. The geometry is that of conduction perpendicular to fibers of rectangular cross section stacked to form a rectangular lattice in a matrix. An analytic expression for the potential is given in the form of infinite series with coefficients determined by an infinite set of linear equations. As a numerical example we calculate the effective conductivity for the special case of square inclusions in a square matrix. This is compared with a square array of circles in the dilute limit.

I. INTRODUCTION

Consider electrical conduction in a two-dimensional generalized checkerboard, with rectangular inclusions stacked to form a rectangular lattice; cf. Fig. 1. This problem is important in a study of inhomogeneous media, where it represents a system on which various models can be tested. It is also appropriate for conduction perpendicular to the fiber axis in a fiber composite. Further, the regular checkerboard geometry has recently become of considerable interest, since it has been shown[1,2] that in a random composite, statistical composition fluctuations leading to a right-angle corner geometry give rise to hot spots and regions of high-field strengths. Thermal and electrical conduction in planar systems with straight-line and corner phase boundaries is also of interest in microelectronics.

There are some previous studies of related systems. Milton *et al.*[3] obtained the total effective conductivity for the geometry of Fig. 1 but with fibers of a square cross section arranged in a square lattice, and in a numerical optimization procedure. McPhedran[4] and Perrins *et al.*[5] considered transport perpendicular to a square array of cylinders in a matrix. Fogelholm and Grimvall[6] calculated the current distribution in the same geometry, modeled by a resistor network. Söderberg and Grimvall[7] considered the current distribution in a checkerboard corner in the limit when the two phases have very different conductivities. Keller,[8] in the same limit, treated parallelograms in a checkerboard geometry. Guo-Qing and Tao[9] developed a theory for periodic structures of various inclusions in terms of an integral equation using so-called transformation fields. Berdichevskii[10] used conformal invariance to give an exact solution for the regular checkerboard.

Because of the importance of the geometry considered here as a model system, it is desirable to have an analytic expression for the potential. It is the purpose of this paper to derive such an analytic form in terms of a series expansion. We solve numerically for the expansion coefficients to obtain the electric field. Our results agree with those of Milton *et al.*[3] Furthermore, to give a numerical example and a quantitative measure of the geometrical effect of having squares instead of circular inclusions, we calculate the effective conductivity in the dilute limit.

II. AN ANALYTIC SOLUTION

A. Formulation of the problem

An electric field E_a is applied along the x axis (Fig. 1). For symmetry reasons it suffices to consider the unit cell of Fig. 2. No current flows across the sides EF and $E'F'$,

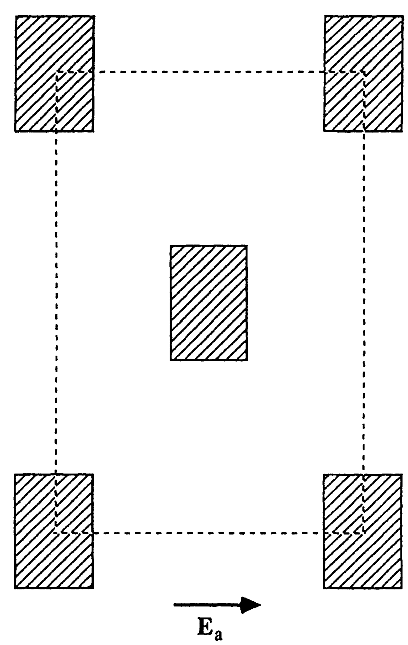

FIG. 1. A generalized checkerboard geometry in an external field $\mathbf{E}_a$.

and no current flows along $E'E$ and $F'F$. We now solve for the potential $u(x,y)$ in the unit cell, using the fact that within a homogeneous part of the cell $\mathbf{E}=-\nabla u(x,y)$, $\nabla\times\mathbf{E}=0$, $\mathbf{j}=\sigma_i\mathbf{E}$, and $\nabla\cdot\mathbf{j}=0$, where $i=1,2$ denote the two phases that have conductivities σ_1 and σ_2. The unit cell is separated into three regions; I (the rectangle $HH'F'F$), II ($GG'H'H$), and III ($EE'G'G$). We introduce the notations

$$\epsilon=c/b,\ \eta=d/a,\ s^2=\sigma_1/\sigma_2\ . \tag{1}$$

A uniform system of phase 2 has $\epsilon=0$, $\eta=1$ and the regular checker board has $\epsilon=1$, $\eta=0$, $a=b$.

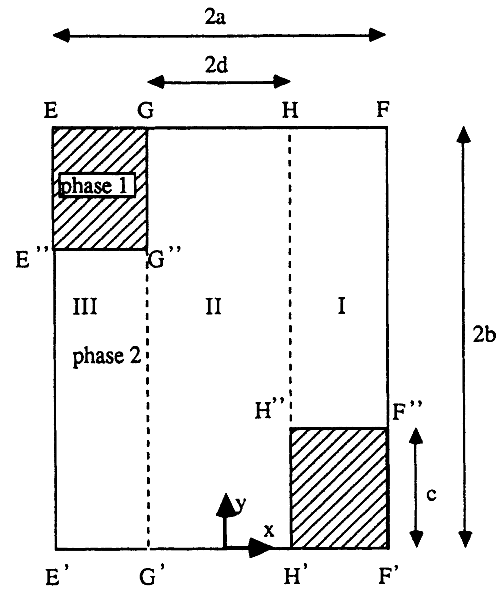

FIG. 2. The unit cell for which calculations are performed.

B. Expansion of the potential

The potential $u(x,y)$ is expanded in eigenfunctions to the operator ∇^2,

$$u_{\mathrm{I}}(x,y)=A_0(1-x/a)+2\sum_{n=1}^{\infty}A_n\{\sinh[\lambda_n a(1-x/a)]/\sinh(\lambda_n a)\}Y_{\mathrm{I},n}(y) \tag{2}$$

for $\eta\le x/a\le 1$,

$$u_{\mathrm{II}}(x,y)=aE_a-A_0[1+(\epsilon/2)(s^2-1)](x/a)+\sum_{m=1}^{\infty}\{C_m\sinh[\delta_m a(x/a)]+D_m\cosh[\delta_m a(x/a)]\}\cos[\delta_m b(y/b)] \tag{3}$$

for $-\eta\le x/a\le\eta$, and

$$u_{\mathrm{III}}(x,y)=2aE_a-\left[A_0^*(1+x/a)+2\sum_{n=1}^{\infty}A_n^*\sinh[\lambda_n a(1+x/a)]/\sinh(\lambda_n a)\right]Y_{\mathrm{III},n}(y) \tag{4}$$

for $-1\le x/a\le -\eta$. Here

$$\delta_m=m\pi/(2b)\ , \tag{5}$$

$$Y_{\mathrm{I},n}(y)=\begin{cases}\cos[\lambda_n b(2-y/b)]/\cos[\lambda_n b(2-\epsilon)], & \epsilon\le y/b\le 2\\ \cos[\lambda_n b(y/b)]/\cos(\lambda_n b\epsilon), & 0\le y/b\le\epsilon\ ,\end{cases} \tag{6}$$

$$Y_{\mathrm{III},n}(y)=\begin{cases}\cos[\lambda_n b(2-y/b)]/\cos(\lambda_n b\epsilon), & 2-\epsilon\le y/b\le 2\\ \cos[\lambda_n b(y/b)]/\cos[\lambda_n b(2-\epsilon)], & 0\le y/b\le 2-\epsilon\ .\end{cases} \tag{7}$$

C. Boundary conditions

We require continuity of the current normal to a boundary and the field parallel to a boundary for all the boundaries between phases 1 and 2 and between regions I, II, and III. These conditions, applied at $E''G''$ and $H''F''$, give the following equation for the characteristic values λ_n ($n=1,2,3,\ldots$):

$$s^2\tan(\lambda_n b\epsilon)+\tan[\lambda_n b(2-\epsilon)]=0\ . \tag{8}$$

It remains to find the coefficients A, A^*, C, and D. The total current flowing into the unit cell is equal to the total outgoing current. Further, symmetry requires that for any x, $u_{\mathrm{I}}(x,0)+u_{\mathrm{III}}(-x,2b)=2aE_a$. These two conditions yield

$$A_0=A_0^*,\quad A_n=A_n^*\ . \tag{9}$$

To satisfy the matching conditions at $x=\pm d$ we make an auxiliary expansion,

$$Y_{I,n}(y)=L_{n,0}/2+\sum_{m=1}^{\infty} L_{n,m}\cos[\delta_m b(y/b)] , \tag{10}$$

and analogous expansions for some other quantities appearing in the potentials $u(x,y)$. The conductivity $\sigma(x,y)$, which has a step at the phase boundary, is also expanded in a Fourier series analogous to (10). Then, after a considerable amount of algebra, the matching conditions for u, $\mathbf{E}$, and $\mathbf{j}$ yield

$$C_{2m}\sinh(\delta_{2m}a\eta)=2(s^2-1)\sum_{n=1}^{\infty} A_n f_1(\lambda_n,\delta_{2m}) , \tag{11}$$

$$D_{2m-1}\cosh(\delta_{2m-1}a\eta)=2(s^2-1)\sum_{n=1}^{\infty} A_n f_1(\lambda_n,\delta_{2m-1}) , \tag{12}$$

$$C_{2m-1}=D_{2m}=0 , \tag{13}$$

where $m=1,2,3,\ldots$ and

$$f_1(\lambda_n,\delta_m)=\lambda_n\tan(\lambda_n b\epsilon)\sinh[\lambda_n a(1-\eta)]\cos(\delta_m b\epsilon)/[b(\delta_m-\lambda_n)\sinh(\lambda_n a)] . \tag{14}$$

The matching also yields

$$2aE_a=A_0[2+\eta\epsilon(s^2-1)]+2(1-s^2)\sum_{n=1}^{\infty} A_n\tan(b\lambda_n\epsilon)\sinh[a\lambda_n(1-\eta)]/[b\lambda_n\sinh(\lambda_n a)] . \tag{15}$$

Further, the requirement (13) and the matching conditions yield

$$2\sum_{n=1}^{\infty} A_n[f_2(\lambda_n,\delta_{2m})\sinh(\delta_{2m}a\eta)+f_1(\lambda_n,\delta_{2m})\cosh(\delta_{2m}a\eta)]+A_0 f_3(\delta_{2m})\sinh(\delta_{2m}a\eta)=0 \tag{16}$$

and

$$2\sum_{n=1}^{\infty} A_n[f_2(\lambda_n,\delta_{2m-1})\cosh(\delta_{2m-1}a\eta)+f_1(\lambda_n,\delta_{2m-1})\sinh(\delta_{2m-1}a\eta)]+A_0 f_3(\delta_{2m-1})\cosh(\delta_{2m-1}a\eta)=0 , \tag{17}$$

where

$$f_2(\lambda_n,\delta_m)=\lambda_n\sin(\delta_m b\epsilon)\cosh[\lambda_n a(1-\eta)]/[b(\delta_m-\lambda_n)\sinh(\lambda_n a)] , \tag{18}$$

$$f_3(\delta_m)=\sin(\delta_m b\epsilon)/(ab\delta_m) . \tag{19}$$

III. GENERAL RESULTS FOR THE EFFECTIVE CONDUCTIVITY

A dual transformation of a material is defined by interchanging phases 1 and 2, without altering the shape of the phase boundaries; cf. Fig. 1. For such systems, Keller[11] has derived the relation

$$\sigma_x(c_1)\sigma_y(1-c_1)=\sigma_1\sigma_2 , \tag{20}$$

where subscripts x and y denote the two coordinate axes. The ordinary checkerboard has the property of being self-dual, i.e., invariant under the transformation (cf. Dykhne[12]), and has $\sigma_x=\sigma_y=\sigma_e$ and $c_1=c_2=0.5$, with

$$\sigma_e=(\sigma_1\sigma_2)^{1/2} . \tag{21}$$

We now seek the effective conductivity σ_e, expressed as averages over the electric field in the composite. The phase boundaries are assumed to be squares, so that the effective conductivity is isotropic. A calculation of the conductivities σ_x and σ_y along the principal axes for a system with rectangular phase boundaries is a straightforward generalization.

Let $\mathbf{E}$ and $\mathbf{j}$ be the local electric field and current, related through $\mathbf{j}=\sigma_i\mathbf{E}$, where $i=1$ or 2. Angular brackets $\langle\cdots\rangle$ and $\langle\cdots\rangle_i$ denote spatial averages over the entire specimen and over all regions of phase i, respectively. Then, with $\mathbf{E}_a$ being the applied field, $\mathbf{j}_a$ the corresponding current, $0<c_1<1$, $c_2=1-c_1$, and $\rho_e=1/\sigma_e$,

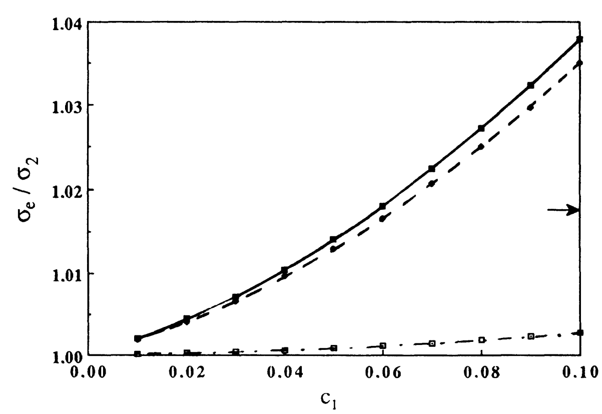

FIG. 3. $\alpha(s,c_1)$ for $\sigma_1/\sigma_2=2$(—·—·—·), 50 (— — —) and 200 (——) in the interval $0.01\le c_1\le 0.1$. The arrow indicates the ratio of the conductances between a square array of squares and a square array of circles, for $\sigma_2/\sigma_1=50$. The value for the square array of circles is from Perrins *et al.* (Ref. 5).

$$\mathbf{E}_a = \langle \mathbf{E} \rangle = c_1 \langle \mathbf{E} \rangle_1 + c_2 \langle \mathbf{E} \rangle_2 = (c_1/\sigma_1)\langle \mathbf{j} \rangle_1 + (c_2/\sigma_2)\langle \mathbf{j} \rangle_2 , \quad (22)$$

$$\mathbf{j}_a = \langle \mathbf{j} \rangle = c_1 \langle \mathbf{j} \rangle_1 + c_2 \langle \mathbf{j} \rangle_2 = c_1\sigma_1 \langle \mathbf{E} \rangle_1 + c_2\sigma_2 \langle \mathbf{E} \rangle_2 , \quad (23)$$

$$\langle \mathbf{j}\cdot\mathbf{E} \rangle = \langle \mathbf{j} \rangle \cdot \langle \mathbf{E} \rangle = \mathbf{j}_a \cdot \mathbf{E}_a = \sigma_e \langle \mathbf{E} \rangle^2 = \rho_e \langle \mathbf{j} \rangle^2 . \quad (24)$$

From (22)–(24) we write the effective conductivity as

$$\sigma_e/\sigma_2 = (R+s^2)/(R+1) , \quad (25)$$

where

$$R \equiv c_2 \langle \mathbf{E} \rangle_2 / c_1 \langle \mathbf{E} \rangle_1 . \quad (26)$$

Using the expressions for the potential $u(x,y)$ derived in Sec. II, and with $\mathbf{E} = -\nabla u(x,y)$, the averages $c_1\langle \mathbf{E} \rangle_1$ and $c_2 \langle \mathbf{E} \rangle_2$ can be obtained analytically. One finds

$$ac_1\langle \mathbf{E} \rangle_1 = A_0\epsilon^2/2 + \sum_{n=1}^{\infty} A_n \sinh(\lambda_n a\epsilon)\tan(\lambda_n b\epsilon)/[b\lambda_n \sinh(\lambda_n a)] , \quad (27)$$

$$ac_2\langle \mathbf{E} \rangle_2 = A_0\{1-[1-(1-\epsilon)s^2\epsilon/2]\} - s^2 \sum_{n=1}^{\infty} A_n \sinh(\lambda_n a\epsilon)\tan(\lambda_n b\epsilon)/[b\lambda_n \sinh(\lambda_n a)] . \quad (28)$$

To get R in Eq. (26), we need A_0 and all A_n. They are obtained from an infinite set of linear equations which have to be solved numerically. We note that the approaches by Perrins *et al.*[5] and Gu-Qing and Tao[9] also lead to similar systems of equations. Here they are solved by a truncation of the system. Using that $\mathbf{E}_a = c_1\langle \mathbf{E} \rangle_1 + c_2 \langle \mathbf{E} \rangle_2$, Eq. (22), we can write

$$ac_1\langle E \rangle_1 = \{aE_a - A_0[1+(s^2-1)\epsilon/2]\}/(1-s^2) , \quad (29)$$

$$ac_2\langle E \rangle_2 = \{s^2 aE_a - A_0[1+(s^2-1)\epsilon/2]\}/(1-s^2) . \quad (30)$$

IV. DILUTE SUSPENSION

As an illustration of our method and a quantitative measure of the difference between squares and circles we consider a dilute suspension. For square inclusions with one side of the squares parallel to the applied field $\mathbf{E}_a$, we have, to lowest order in the surface fraction c_1 of phase 1,

$$\sigma_e/\sigma_2 = [1+(s^2-1)c_1\langle \mathbf{E} \rangle_1/E_a] . \quad (31)$$

For circular inclusions the effective medium result[13] is exact in the dilute limit, and to lowest order in the surface fraction we have

$$\sigma_e/\sigma_2 = 1+2c_1(s^2-1)/(s^2+1) . \quad (32)$$

Next, define a quantity $\alpha(s,c_1)$ by

$$\alpha(s,c_1) \equiv (\sigma_e/\sigma_2)/[1+2c_1(s^2-1)/(s^2+1)] . \quad (33)$$

For circles, when $c_1 \to 0$, $\alpha(s,c_1)=1$ to lowest order in c_1. In Fig. 3 we give $\alpha(s,c_1)$ for various values of s and c_1 for square inclusions. We note that the conductivity for circular inclusions, in the dilute limit, agrees with the lower Hashin-Shtrikman[14] bound and thus is the lowest possible for a given c_1 (when $s>1$).

V. CONCLUSIONS

We conclude by noting that the potential $u(x,y)$ is obtained as follows. First, λ_n are determined from Eq. (8). The expansion coefficients A_0, A_n are found from Eqs. (15), (16), and (17) and A_0^*, A_n^* from (9). C_m and D_m follow from Eqs. (11)–(13). Although the method is exact, one has to solve numerically an infinite set of linear equations, and one is forced to make a truncation. Comparing with similar work by Perrins *et al.*[5] and Guo-Qing and Tao[9] similar sets of equations always seem to arise when treating an array of inclusions. In the dilute limit we find that an array of circles has almost the same conductivity as an array of squares. For high concentrations, when the inclusions get close to touching, their effective properties of course differ considerably.

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