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Superfluxons in periodically inhomogeneous long Josephson junctions

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Dynamics of a periodic array of fluxons in a dc-driven damped long Josephson junction with an installed periodic lattice of local inhomogeneities are investigated analytically by means of the perturbation theory. In the case when the array and the lattice are commensurable, the array as a whole remains in a pinned state unless the dc bias current density exceeds a certain critical value. It is demonstrated that, in the same time, stable defects in the form of a "hole" or surplus fluxon may propagate along the pinned array. In the long-wave approximation, an evolution equation (an "elliptic sine-Gordon" equation) for local deformations of the array is deduced. That equation supports exact kinklike solutions ("superfluxons") which describe the defects mentioned. In the presence of dissipation and dc bias current (with the density smaller than critical), I-V characteristics of the junction corresponding to the motion of a superfluxon are found. The results obtained are in good agreement with results of recent numerical and physical experiments.

The dominating role of magnetic flux quanta (fluxons) in dynamics of long Josephson junctions (LJJ's) is generally acknowledged today. Of special interest are dynamics of periodic arrays of fluxons in a LJJ with an installed lattice of local inhomogeneities.[1–7] In particular, a dependence of the critical bias current density f_{cr}, i.e., that which tears off an array pinned by the lattice, upon an array's density (proportional to the magnetic field at LJJs' edges) demonstrates sharp peaks at the values of the magnetic field corresponding to the array-lattice commensurability.[7] If the bias current density does not exceed f_{cr}, the commensurable array cannot move as a whole. However, since the array possesses finite rigidness, compression waves may propagate along it. It has been revealed in recent numerical simulations[1] that in the case where the pinned array is almost commensurable, i.e., a sufficiently long commensurable segment of the array contains a defect in the form of a surplus or lacking fluxon, the defect may propagate along the pinned array as a deformation wave under the action of the dc bias current with a density smaller than f_{cr}. The aim of the present paper is to analyze this phenomenon.

The analysis is based on the well-known sine-Gordon (SG) model of a dc-driven damped inhomogeneous LJJ:[8]

$$\phi_{tt}-\phi_{xx}+\sin\phi=-\alpha\phi_t-f+\varepsilon\sum_{n=-\infty}^{+\infty}\delta(x-an)\sin\phi\,. \quad (1)$$

In Eq. (1), ϕ is the normalized magnetic flux, α is a dissipation constant, f is the dc bias current density, a is a spacing of the lattice formed by pointlike inhomogeneities, and ε is a "strength" of a separate inhomogeneity. The cases $\varepsilon>0$ and $\varepsilon<0$ correspond to the so-called microresistor and microshort (microshunt), i.e., a short region in the LJJ where the tunneling of the superconducting pairs across the junction is, respectively, suppressed or enhanced.

Supposing the parameters α, f, and ε small, I will base the analysis upon the perturbation theory.[9] As for the spacing a, no restrictions will be imposed on it. In the zeroth approximation ($\alpha=f=\varepsilon=0$), a quiescent fluxon array is described by the following exact solution to the unperturbed SG equation:

$$\phi(x)=\pi-2\,\mathrm{am}[(x-\xi)/k]\,, \quad (2)$$

where am is the Jacobi elliptic amplitude, the elliptic modulus k ($0<k<1$) is an arbitrary parameter that determines the array's period

$$L=2kK(k) \quad (3)$$

[$K(k)$ is the complete elliptic integral of the first kind], and ξ is an arbitrary constant (the "array's coordinate"). The period L may be interpreted as a spacing between adjacent fluxons in the array. I will assume the following commensurability relation to hold in the zeroth approximation:

$$a=pL\,, \quad (4)$$

with p an arbitrary integer [the more general case of the commensurability, $a=(p/q)L$, can be analyzed too, but it gives rise to much more tedious calculations[7]].

The full Hamiltonian of the dissipationless ($\alpha=0$) model (1) includes the Hamiltonian of the unperturbed SG system,

$$H_0=\int_{-\infty}^{+\infty}dx[\tfrac{1}{2}(\phi_t^2+\phi_x^2)+(1-\cos\phi)]\,, \quad (5)$$

the array-lattice interaction Hamiltonian

$$H_\varepsilon=-\varepsilon\sum_{n=-\infty}^{+\infty}[1-\cos\phi(x=an)]\,, \quad (6)$$

and the item which takes account of the dc drive,

$$H_f=f\int_{-\infty}^{+\infty}dx\,\phi(x)\,. \quad (7)$$

Now let us assume that the array is deformed at a large scale $\lambda\gg L$. To this end, it is sufficient to presume that the quantity ξ in the expression (2) is a slowly varying function $\xi(x,t)(\xi_x\sim L/\lambda)$, while the modulus k remains a constant. By inserting the expression (2) with the variable ξ into terms (5) through (7), it is straightforward to find the full Hamiltonian $H=H_0+H_\varepsilon+H_f$ expressed in terms of $\xi(x,t)$:

$$H=\int_{-\infty}^{+\infty} dx[\tfrac{1}{2}\rho(\xi_t^2+\xi_x^2)-2\varepsilon a^{-1}\mathrm{cn}^2(\xi/k)+G\xi] , \quad (8)$$

where cn is the elliptic cosine, $\rho\equiv 4k^2E(k)/K(k)$ is the array's mass density [$E(k)$ is the complete elliptic integral of the second kind], and $G\equiv\pi fk/K(k)$.

The Hamiltonian (8) gives rise to the equation of motion

$$\xi_{tt}-\xi_{xx}+4\varepsilon a^{-1}\rho^{-1}k^{-1}\mathrm{sn}(\xi/k)\mathrm{cn}(\xi/k)\mathrm{dn}(\xi/k) = -\rho^{-1}G-a\xi_t , \quad (9)$$

where sn and dn are the standard elliptic functions. The last (dissipative) term in Eq. (9) was added on the basis of energy-balance analysis. Let us note that the limit velocity corresponding to the left-hand side of Eq. (9) is the same as the limit velocity (equal to unity in our notation) for Eq. (1), i.e., it coincides with the Swihart velocity of the LJJ considered. In the case $G=a=0$, Eq. (9) can be cast into the dimensionless form

$$\Xi_{TT}-\Xi_{XX}+2(\mathrm{sgn}\varepsilon)\mathrm{sn}(\tfrac{1}{2}\Xi)\mathrm{cn}(\tfrac{1}{2}\Xi)\mathrm{dn}(\tfrac{1}{2}\Xi)=0 , \quad (10)$$

where $\Xi\equiv 2\xi/k$, and X and T are related to x and t in an obvious way. Equation (10) may be naturally called the elliptic sine-Gordon (ESG) equation. It depends upon the continuous parameter k and the sign parameter sgnε. In fact, we have two different ESG equations corresponding to two different signs of ε. At $k^2\ll 1$, Eq. (10) is close to the usual SG equation:

$$\Xi_{TT}-\Xi_{XX}+\sin\Xi=-\tfrac{5}{8}k^2\sin(2\Xi)+O(k^4) . \quad (11)$$

It is well known that the SG equation (11) perturbed by the small term $\sim\sin(2\Xi)$ is not integrable; see, e.g., Ref. 9. This suggests that Eq. (10) cannot be integrable either.

Let us return to Eq. (9). At $G=0$, it has an exact solution describing a quiescent superfluxon:

$$\xi(x)=kF(\sin^{-1}\{(1-k^2)\cosh^2[2xk^{-1}(\varepsilon/a\rho)^{1/2}]+k^2\}^{-1/2}) , \quad -\infty<x<0 , \quad (12a)$$

$$\xi(x)=2kK(k)-\xi(-x), \quad 0<x<+\infty , \quad (12b)$$

for $\varepsilon>0$, and

$$\xi(x)=-kF\{\cos^{-1}[(1-k^2)^{1/2}(\cosh^2\{2xk^{-1}[(1-k^2)|\varepsilon|/a\rho]^{1/2}\}-k^2)^{-1/2}]\}, \quad -\infty<x<0 , \quad (13a)$$

$$\xi(x)=-\xi(-x), \quad 0<x<+\infty , \quad (13b)$$

for $\varepsilon<0$. Here $F(z)$ is the incomplete elliptic integral of the first kind.

The superfluxon solutions (12) and (13) are distinguished by the boundary condition

$$\xi(x=+\infty)-\xi(x=-\infty)=2kK(k)\equiv L \quad (14)$$

[recall that L is the spacing (3) of the fluxon array]. Thus the superfluxon (12) or (13) may indeed be interpreted as a "hole" (a lacking fluxon) in the array. Quite analogously, a superfluxon of the opposite polarity, given by the solution (12) or (13) with the opposite sign, describes a surplus fluxon in the array. A solution describing a moving superfluxon can be obtained (in the case $a=G=0$) from (12) or (13) by the obvious Lorentz transformation.

In relation to the nonintegrability of Eq. (10), recall that nonintegrable equations may support exact one-soliton solutions, but collisions between solitons in nonintegrable systems ought to be inelastic on account of emission of radiation (quasilinear waves).[9–11] A collision between superfluxons of the opposite polarity has recently been simulated numerically within the framework of the ESG equation (10) in Ref. 2. Despite the apparent nonintegrability of this equation (see above), the collision seems practically absolutely elastic in a wide range of parameters: In a system with periodic boundary conditions (a total length of the system was taken to be much greater than the size of a superfluxon), a net radiative energy loss, defined as a drop of the kinetic energy of the colliding superfluxons, was surely less than 1% of the initial kinetic energy after 200 collisions. A reason for this fact remains to be understood.

Let us now proceed to the case $G,a\neq 0$. In the range

$$f^2<f_{\mathrm{cr}}^2\equiv\tfrac{4}{27}(\varepsilon/\pi pk^3)^2[(5k^2-2k^4-2)(1+k^2) +2(1+k^4-k^2)^{3/2}] , \quad (15)$$

where p is the commensurability index defined by Eq. (4), the array as a whole remains pinned.[7] In the same time, a defect of the array in the form of a "hole" or a surplus fluxon, described by the superfluxon solution, moves with a certain velocity V determined by the energy balance between the dissipation and dc drive. For a solitary fluxon in the homogeneous dc-driven damped LJJ, the equilibrium velocity has been found in Ref. 8:

$$V^2(1-V^2)^{-1}=(\pi f/4a)^2 . \quad (16)$$

In the present case, the energy-balance analysis yields the following results:

$$V^2(1-V^2)^{-1}=(\pi^2p/8)k^3[K^2(k)/E(k)] \times(f^2/a^2\varepsilon)\left(\ln\frac{1+k}{1-k}\right)^{-2} \quad (17)$$

for $\varepsilon>0$, and

$$V^2(1-V^2)^{-1}=(\pi^2p/32)k^3[K^2(k)/E(k)] \times(f^2/a^2|\varepsilon|)(\sin^{-1}k)^{-2} \quad (18)$$

for $\varepsilon<0$.

As is known, the x-averaged quantity $\bar{\phi}_t$ is proportional to the voltage across the junction. Using Eqs. (2) and (14), one can find the following general relation:

$$\bar{\phi}_t = 2\pi V/l\,, \tag{19}$$

l being a total length of the system. Thus, a superfluxon carries the same voltage as a usual fluxon moving with the same velocity V. Nonetheless, the I-V characteristics, i.e., a dependence $\bar{\phi}_t(f)$, are different for a superfluxon and for a usual fluxon owing to different dependences $V(f)$ [compare Eqs. (17) and (18) to Eq. (16)].

Let us estimate limits of applicability of the theory developed. According to expressions (12) and (13), a characteristic size λ of the superfluxon can be estimated as follows:

$$\lambda^2 \sim pk|\varepsilon|^{-1}/\ln(1-k^2)^{-1}\,. \tag{20}$$

Equation (9) may be regarded as a Whitham-type equation[12] for envelopes. Its applicability condition $L \ll \lambda$ can be cast, with regard to Eqs. (3) and (20), into the form

$$L^3 \ll a/|\varepsilon|\,. \tag{21}$$

Thus the description developed is applicable to fluxon arrays which are not too rarefied.

In conclusion, it is worthwhile to note that if we analyzed a model of a periodically inhomogeneous LJJ with a harmonic modulation function $\cos[(2\pi/a)x]$ [instead of $\sum_{n=-\infty}^{+\infty}\delta(x-an)$] in Eq. (1), we would have obtained, instead of Eq. (9), a usual SG equation for $\xi(x,t)$, with perturbing terms similar to those in Eq. (9) (cf. an analogous situation in the one-fluxon problem[13]). However, the case of the harmonic modulation can scarcely be realized in an experiment.

Detailed comparison of the results reported here with results of numerical simulations of both the SG model (1) and the ESG model (9), as well as with experimental results, will be given elsewhere.[2] The agreement between theoretical, numerical, and experimental results proves to be fairly good.

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