Fully frustrated fractal

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An exact technique is developed for the thermodynamic functions of an Ising system on a Sierpinsky gasket fractal, which in the antiferromagnetic case is fully frustrated. An analytic expression is obtained for the zero-temperature entropy per spin $S(0)$ in the fully frustrated fractal, resulting in $S(0)=0.493006\ldots$ (about 50% *larger* than the corresponding result for the frustrated triangular lattice). Spin-spin correlations are also provided by the method and in the frustrated fractal always exhibit extremely rapid exponential decay.

Frustration is the crucial feature in spin glasses, where it occurs together with disorder. Perhaps the most important frustrated system is the Ising antiferromagnet on the triangular lattic both because of its exact solution demonstrating a finite zero-point specific entropy[1] and because disorder can be added to it in order to further the understanding of spin glasses.[2-4]

In particular it has been suggested that whereas the full frustration of the triangular lattice antiferromagnet makes it paramagnetic down to $T=0$, random removal of some spins can reduce the huge ground-state degeneracy so that the system may order into a spin-glass phase, and Monte Carlo calculations support this idea.[4]

There is a need for secure results on nonuniform fully frustrated systems, particularly ones with bonds removed (holes) whether randomly or in some organized fashion, as in certain fractals. An obvious candidate, which we treat exactly here, is the triangular Sierpinsky gasket fractal, which is essentially a triangular lattice with holes on all scales. The scaling of the thermal parameter has been given previously,[5-8] from which it is known that the ferromagnetic case is not ordered at any finite temperature. Subsequent work[9] has emphasized the apparent long-range order in the Ising ferromagnet on finite Sierpinsky gaskets and has provided a scaling of Boltzmann probabilities.

This paper and independent work by Grillon and Brady Moreira,[10] investigates the thermodynamic behavior of the fully frustrated antiferromagnetic Ising system on Sierpinsky gasket fractals. The conclusion is that the ground-state degeneracy is *higher* in the fractal than in the triangular lattice, which is opposite to the suggestion referred to above for random dilution. Grillon and Brady Moreira have obtained this result by numerically iterating renormalization-group equations for the scaling of thermal parameter and free energy. This paper gives an analytic method for obtaining the partition function, etc., and also correlation functions, and hence derives an exact analytic result for the ground-state entropy, etc. so the fully frustrated fractal can be regarded as solved to the same extent as the triangular Ising antiferromagnet.[1,11]

The method is similar to a transfer-matrix technique, in the sense that a partition function "tensor" is built up by contracting corresponding tensors for subunits of the fractal. This generates a simple and exact scaling procedure, which applies to a wide variety of fractals of various fractal dimensions. The results for thermodynamic quantities and correlation functions can be cast in various forms as is convenient for discussion of scaling properties (e.g., hyperscaling in the critical regime of the ferromagnetic fractal) or noncritical properties (the fully frustrated fractal is actually never critical). It is convenient to give the zero-point entropy of the fully frustrated fractal as a series, whose convergence is so rapid that stopping after three terms produces an error of only $O(10^{-15})$. A further significant point is that the length scaling transformations produced by applying the method to the fully frustrated fractal are not chaotic, unlike those occurring in frustrated hierarchical Ising models with competing interactions[12]

The body of this paper is as follows. First the general method is developed [Eqs. (1)–(10)]. Then its application in a scaling context is very briefly illustrated for the ferromagnetic Ising model on the fractal. The remainder of the analysis [Eqs. (12)–(16)] is for the fully frustrated fractal (antiferromagnetic case), resulting in the rapidly convergent series (15) [see also (16)] for the zero-temperature entropy per spin. Selected results for spin-spin correlations are also briefly presented, including their extremely rapid exponential decay in the zero-temperature frustrated fractal. A concluding discussion then follows.

The method exploits the hierarchical character of the fractal. The fractal is obtained as the n-infinite limit of a sequence of generations, two successive members $(n+1,n)$ of which are related as shown in Fig. 1.

$Z_n^{\alpha\beta\gamma}$ is defined to be the "constrained" partition function of the zero field, spin-$\frac{1}{2}$ Ising model on the nth generation fractal, but with the three vertex spins in arbitrarily specified states α,β,γ respectively (with $\alpha=\pm1$ denoting spin up or down, etc.). Then, with the "summation convention" that repeated indices (α',β',γ') are summed over ±1, we have from Fig. 1

$$Z_{n+1}^{\alpha,\beta,\gamma}=Z_n^{\alpha\beta'\gamma'}Z_n^{\beta\gamma'\alpha'}Z_n^{\gamma\alpha'\beta'} . \tag{1}$$

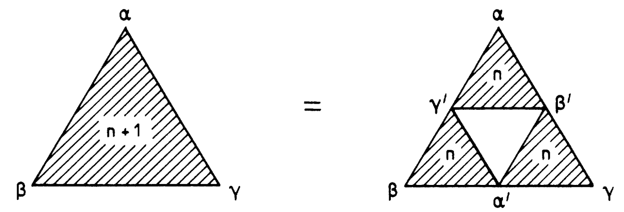

FIG. 1. Recursive construction of Sierpinsky gasket fractal, and labeling of vertex spin states corresponding to Eq. (1) for the "constrained" partition function.

$Z_n^{\alpha\beta\gamma}$ for $n=0$ is just the Boltzmann weight for the specified configurations (α,β,γ) of a triangle of just three spins:

$$Z_0^{\alpha\beta\gamma}=\exp[(\alpha\beta+\beta\gamma+\gamma\alpha)K] , \quad (2)$$

where K is exchange interaction divided by $k_B T$. The equations imply that $Z_n^{\alpha\beta\gamma}$ is independent of the order of α, β, γ, as is physically obvious. (1) can be used to show by induction that

$$Z_n^{\alpha\beta\gamma}=A_n[1+B_n^{-1}(\alpha\beta+\beta\gamma+\gamma\alpha)] , \quad (3)$$

where

$$A_{n+1}=8A_n^3(1+B_n^{-3}) , \quad (4)$$

$$B_{n+1}=B_n^2-B_n+1 , \quad (5)$$

and [from (2)],

$$A_0=c^3(1+t^3) , \quad (6)$$

$$B_0=t-1+1/t , \quad (7)$$

where $c\equiv\cosh K$, $t\equiv\tanh K$. Then, since the usual (unconstrained) partion function is given by

$$Z_n\equiv\sum_{\alpha\beta\gamma}Z_n^{\alpha\beta\gamma}=8A_n , \quad (8)$$

[using (3)], exact solution, or iteration, of (4) and (5) using initial conditions (6) and (7) provides the partition function.

The recurrence equations (4) and (5) are actually the renormalization-group transformation equations for partition function and thermal parameter under a scaling of the fractal by dilation factor 2. The corresponding transformation of the number of sites N_n is

$$N_{n+1}=N_n+3^{n+1}, \quad N_0=3 . \quad (9)$$

It is convenient to convert (4), using (8), into an equation for the transformation of the free energy per spin, $F_n\equiv(\ln Z_n)/N_n$:

$$F_{n+1}=3F_nN_n/N_{n+1}+(1/N_{n+1})\ln[(1+B_n^{-3})/8] . \quad (10)$$

This equation is of the form expected from standard scaling approaches to free energies[13] and so should also be derivable by extending the recent work of Grillon and Brady Moreira.[10] Equation (10), together with (5), (9) provides the means to obtain the free energy per spin $F\equiv\lim_{n\to\infty}F_n$ and other thermal properties, particularly the entropy $S(T)$ per spin

$$[S(T)=F-K\partial F/\partial K] ,$$

of the gasket.

We are principally interested in the ground-state degeneracy and associated zero-temperature entropy $S(0)$ for the antiferromagnetic case. So we concentrate hereafter on low-temperature properties.

It is helpful to first briefly discuss the ferromagnetic case ($K>0$). Here $B_n\geq 1$ for all n and so the scaling Eq. (5) for the thermal parameter is monatonic increasing with fixed points at $B^*=1,\infty$. This is to be contrasted with the antiferromagnetic case where it will be seen that the negative initial K changes sign under a single scaling and the only accessible fixed point is $B^*=\infty$. Equation (5) was first derived and discussed for the ferromagnetic case in Ref. 5, where it was shown to yield $T=0$ as an unstable fixed point (so $T_c=0$), and to lead to the following pathological behavior of the correlation length ξ

$$\xi\sim\exp(\tfrac{1}{4}e^{4K}\ln 2) , \quad (11)$$

This form arises because, linearizing around the unstable fixed point with the use of

$$\epsilon\equiv B-1\sim\exp(-4K) ,$$

ϵ is actually marginal and so the "scaling variable" is not a power of ϵ but instead $\exp[(-\ln 2)/\epsilon]$. This raises the question whether hyperscaling may be valid in this anomalous situation in which the correlation length exponent ν is infinite. The answer provided by exploiting (10) with (5) and (9), is that hyperscaling is satisfied, in the sense that near the zero-temperature fixed point the singular part of the total free energy N_nF_n scales like ξ^{-d_f} where $d_f=\ln 3/\ln 2$ is the fractal dimension of the gasket. In addition we find that the ground-state energy per spin is -2 K, which is two thirds of the value for the ferromagnetic triangular lattice.[1]

For the rest of this letter we discuss the more important antiferromagnetic case, where K is negative. Then, at low temperatures

$$B_n=c_n+d_n\delta^2+O(\delta^4) , \quad (12)$$

where $\delta\equiv 2\exp(-2|K|)$, and the coefficients c_n, d_n are of order unity. In particular, from (7), $c_0=-3$, $d_0=-1$. The form (12) then follows from (5). c_n is the zero-temperature limit of B_n, and so also satisfies the same equation (5).

It can be seen that the zero-temperature initial value $c_0=-3$ is not a fixed point of (5), so the system is not critical at zero temperature. Further, the negative c_0 scales into a succession of larger and larger positive c_n's, tending asymptotically to the paramagnetic fixed point of the $K>0$ system. This means that the zero-temperature antiferromagnetic fractal is like a ferromagnetic fractal at finite temperature, hence explaining the macroscopic zero-temperature entropy of the antiferromagnet.

To evaluate the low-temperature free energy (per spin)

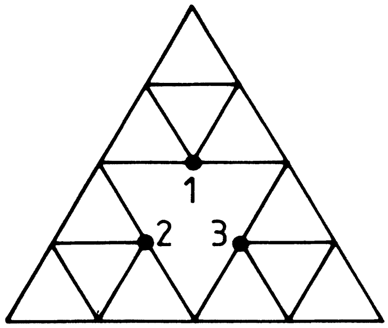

FIG. 2. For Ising antiferromagnetic interactions, the lattice shown has more degenerate ground states than would occur if sites 1, 2, and 3, were linked, as on a triangular lattice.

and hence the zero-temperature entropy $S(0)$ for the antiferromagnetic fractal, it is sufficient to use (10) in the form

$$F_{n+1}=\frac{3N_n}{N_{n+1}}F_n+\frac{1}{N_{n+1}}\ln\left[\frac{1}{8}\left[1+\frac{1}{c_n{}^3}\right]\right]+O(\delta^2)\ , \tag{13}$$

where [using (8) and (6) with $t=-(1-\delta)$],

$$F_0=\tfrac{1}{3}\ln 6+\tfrac{1}{3}|K|+O(\delta^2)\ . \tag{14}$$

The resulting (exact) expression for the zero-temperature entropy per spin of the fully frustrated antiferromagnetic fractal is

$$S(0)=\tfrac{2}{3}\ln 6+\frac{2}{3^2}\sum_{p=0}^{\infty}\frac{1}{3^p}\ln\left[\frac{1}{8}\left[1+\frac{1}{c_p^3}\right]\right]\ . \tag{15}$$

Then, from (5), $c_0=-3$ goes into $c_1=13$, $c_2=157$, etc. so the series in (15) converges extremely rapidly, and the error in stopping after three terms is only $O(10^{-15})$. The result is

$$S(0)=0.493\,006\,107\ldots\ . \tag{16}$$

This agrees, within the accuracy of their numerical calculation, with the value obtained by Grillon and Brady Moreira. As they have noted, it is a higher value per spin than that occurring in the Ising antiferromagnet on the triangular lattice[1] where $S(0)=0.3383\ldots$. We conclude that the holes of the fractal have not relieved frustration but rather have increased the degeneracy of the ground state. This increase can be easily understood by considering the Ising antiferromagnetic on just the $n=2$ member of the sequence of generators, shown in Fig. 2. It is trivial to see that this system has more degenerate ground states than would occur if sites 1, 2, and 3, were linked, as they are on a triangular lattice. So holes will not, in general, relieve frustration (in the usual sense of reducing the ground-state entropy). However, specific irregular removal of bonds or sites (with all incident bonds) from small samples of the triangular lattice can certainly reduce $S(0)$: so, it is possible that while on the infinite triangular lattice regular Sierpinsky-like holes will increase $S(0)$, random dilution of bonds or sites may decrease it, as suggested by the Monte Carlo work.[4]

Comparison of (16) with the entropy per spin at infinite temperature ($\ln 2=0.6931$) emphasises how high the ground-state entropy is, and the huge degeneracy [$\exp(0.49\ldots N)$] of the ground state in the N-spin system. It is also interesting to note that (13) and (14) lead to a ground-state energy per spin of $-\frac{2}{3}|K|$ for the frustrated fractal, which is one third of the ferromagnetic value; the same factor of one third is found in the triangular lattice.[1]

The method used here also allows the calculation of all correlations. For example, the pair correlation function of vertex spins at generation n, i.e., at separation 2^n, is just B_n^{-1}. The scaling equation (5) for B_n provides this and all zero-field correlation functions. Its use shows that for the antiferromagnetic Ising system on the Sierpinsky gasket at zero temperature and field the vertex spin-correlation function is antiferromagnetic for $n=0$ and ferromagnetic for all higher n, and that all correlations fall off exponentially, with correlation length $\xi=\lim_{n\to\infty}(2^n/\ln c_n)=0.79$ (in units of the smallest spacing on the fractal). This extremely small correlation length, for a system at $T=0$, is a quantitative microscopic measure of the highly disordered state of the frustrated system.

In conclusion, an exact analytic treatment has been given of the "ground-state" entropy of a fully frustrated fractal, which is like a triangular Ising antiferromagnet with regularly disposed holes. The holes enhance the degeneracy. Though our emphasis was on low temperatures, the techniques used provide the thermodynamic functions and correlation functions at any temperature, and can be readily generalized to nonzero field and to higher spin, and to the case of Potts models. The basic method [cf Fig. 1 and (1)] is like a generalization of the usual transfer-matrix technique for a chain with cyclic boundary conditions. But is is more suited to scaling considerations. It can be extended without difficulty to such system as the Berker lattice, or the tetrahedral Sierpinsky gasket etc., with an equation like (1) again applying but with more vertex labels.

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