

Transport in networks with a power-law distribution of conductances: The ladder and the Sierpinski gasket

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(Received 7 April 1989; revised manuscript received 10 October 1989)

We study the transport properties of two types of resistor networks: the ladder and the Sierpinski gasket (SG), where *all* their bonds obey a power-law distribution of conductances: $P(\sigma) \sim \sigma^{-\alpha}$, $\alpha < 1$, $\sigma \leq 1$. We argue that for the ladder there exists a critical value of α , $\alpha_c = \frac{1}{2}$, that separates normal ($\alpha \leq \alpha_c$) and anomalous transport ($\alpha > \alpha_c$), whereas for the SG the transport is normal for all α . Extensive numerical simulations, based on triangle-star transformations, support our predictions. The difference between the two structures is discussed.

Various physical systems can be mapped onto the problem of transport on networks with a power-law distribution of conductances.¹⁻⁷ It was shown that when the bond conductivities are distributed according to

$$P(\sigma) \sim \sigma^{-\alpha}, \quad \alpha < 1, \quad \sigma \leq 1, \quad (1)$$

the transport exponents $\tilde{\xi}$ and d_w depend on α . Recently, Havlin *et al.*⁸ have studied transport on a quasi-one-dimensional (quasi-1D) system—a strip consisting of n connected parallel chains of length L , where the conductances of the horizontal bonds follow the distribution (1). The vertical bonds are taken to be perfect conductors. They found that there exists a critical value of α , $\alpha_c = 1 - 1/n$, so that the resistance exponent $\tilde{\xi}$ is

$$\tilde{\xi} = \begin{cases} 1, & \text{for } \alpha \leq \alpha_c \text{ (normal transport),} \\ \frac{1}{n(1-\alpha)}, & \text{for } \alpha > \alpha_c \text{ (anomalous transport).} \end{cases} \quad (2)$$

This result is in agreement with Straley's earlier prediction.² According to Straley,

$$\tilde{\xi} = \max \left[\tilde{\xi}_0, \frac{1}{\nu(1-\alpha)} \right], \quad (3)$$

where $\tilde{\xi}_0$ is the normal resistance exponent, and ν is the correlation length exponent. One can show that for n connected parallel chains $\nu = n$. If the probability of a horizontal bond to be conducting is $p = 1 - \epsilon$, then the probability of a connected chain of L links is

$$P_L = [1 - (1-p)^n]^L \equiv e^{-L/\xi}, \quad (4)$$

where ξ is the correlation length. From Eq. (4) follows

$$\xi^{-1} = -\ln(1 - \epsilon^n) \approx \epsilon^n, \quad (5)$$

which is the universal relation for ξ , except that ν is replaced by n . Hence Eq. (3) (with $\nu = n$) and Eq. (2) are identical, indicating a continuous transition at α_c .

The values in Eq. (2) represent rigorous lower bounds if the transverse bonds are not perfect conductors, but have the same power-law distribution. The diffusion exponent d_w is obtained from (2) via the relation⁷ $d_w = d_f + \tilde{\xi}$.

In this work we argue that (2) is also correct when *all* bonds, vertical and horizontal, have the distribution (1). We present numerical simulations for $n = 2$ (hereafter referred to as a "ladder") to support this claim.

Next, we study the transport exponents for the same distribution on the Sierpinski gasket (SG),^{7,9,10} which is "marginal" between one dimension (1D) and higher dimensionalities. The SG is considered as quasi-1D due to its finite order of ramification.⁷ Yet, it is an interesting object, since it was proposed as a model for the backbone of percolation.⁹ We argue that for the SG $\alpha_c = 1$, i.e., the transport is normal for all α . Extensive numerical simulations substantiate this conjecture.

Our numerical simulations, for both the ladder and the SG, are based on the use of triangle-star transformations, in calculating the conductance of the system. These transformations were recently used for several other problems,¹¹⁻¹³ and were found to be very efficient.

We discuss the reason for the different result between the ladder and the SG, which is related to the concentration of singly connected bonds in each structure. This leads to different probability of low conducting configurations, which dominate the transport properties.

Consider first the case of strips ($n \times \infty$) where the vertical bonds are perfect conductors. It was shown⁸ that the dominant contribution to the system resistance comes from a configuration in which all horizontal bonds in one column of n bonds have low conductivities. Since the probability of this low-conducting configuration decreases as n increases, a sensitive dependence on n is obtained [Eq. (2)].

When the vertical perfect conductors are replaced by conductors with the same distribution (1), the total conductance is still determined primarily by the low horizontal conductors in one column. Other low-conducting

configurations, including also low-conducting vertical bonds, involve more than n low-conducting bonds and therefore are less probable. Thus, we expect that for the case of power-law distribution of both vertical and horizontal bonds, the transport exponents in Eq. (2) remain unchanged.

We test this conjecture numerically. For simplicity we perform the simulations for the case $n=2$. The total resistance or conductance is calculated in a technique using triangle-star transformations, as shown in Fig. 1. The two bonds on the first column of the chain are held in the same potential. Thus, together with the first vertical bond they form a triangle, which is transformed into a star according to the formulas

$$\begin{aligned}
 R_1 &= \frac{r_2 r_3}{r_1 + r_2 + r_3}, \\
 R_2 &= \frac{r_1 r_3}{r_1 + r_2 + r_3}, \\
 R_3 &= \frac{r_1 r_2}{r_1 + r_2 + r_3},
 \end{aligned}
 \tag{6}$$

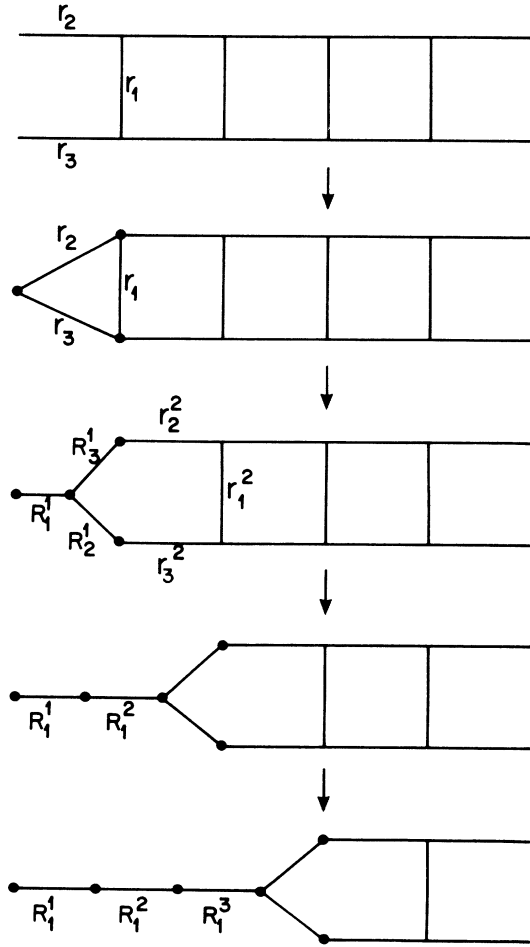


FIG. 1. The first steps of the transformation from a ladder to a linear chain. The resistance is calculated according to Eqs. (6)–(8).

where r_i and R_i are the resistances in the triangle and the star, respectively.

In this way, a new triangle is formed, built of R_2 , R_3 , and of the bonds in the *next* column r_1 , r_2 , and r_3 . This procedure is repeated iteratively, using the following formulas:

$$\begin{aligned}
 R_1^{i+1} &= \frac{(R_3^i + r_2^{i+1})(R_2^i + r_3^{i+1})}{R_2^i + R_3^i + r_1^{i+1} + r_2^{i+1} + r_3^{i+1}}, \\
 R_2^{i+1} &= \frac{r_1^{i+1}(R_2^i + r_3^{i+1})}{R_2^i + R_3^i + r_1^{i+1} + r_2^{i+1} + r_3^{i+1}}, \\
 R_3^{i+1} &= \frac{r_1^{i+1}(R_3^i + r_2^{i+1})}{R_2^i + R_3^i + r_1^{i+1} + r_2^{i+1} + r_3^{i+1}},
 \end{aligned}
 \tag{7}$$

where the superscripts denote the column index.

After the whole ladder is converted into a simple one-dimensional chain, the total resistance is given by the sum

$$R = \sum_{i=1}^L R_1^i.
 \tag{8}$$

A statistical ensemble of ladders has been generated, with

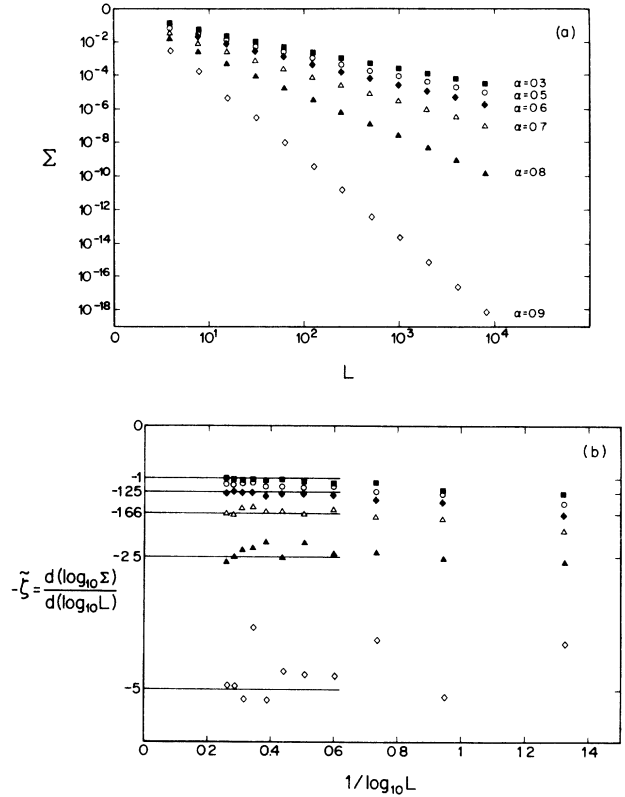


FIG. 2. *The ladder.* (a) Results for the averaged conductance Σ as a function of the length L , for various values of α . (b) Plot of the local slope $-\tilde{\xi} = d(\log_{10} \Sigma) / d(\log_{10} L)$ as a function of $1/\log_{10} L$, for the same values as in (a). The solid lines in the asymptotic region $L \rightarrow \infty$ represent the values of $\tilde{\xi}$ as predicted by Eq. (2). Above $\alpha_c = 0.5$, anomalous transport is obtained.

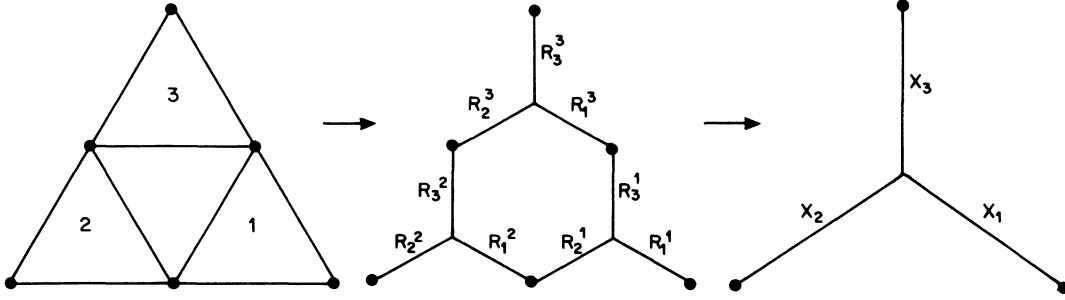


FIG. 3. The transformation of the SG into an effective network, using triangle-star transformations.

bonds having resistances

$$\rho = \eta^{-1/(1-\alpha)}, \quad (9)$$

where η is a pseudorandom number, chosen from a uniform distribution ($0 \leq \eta \leq 1$). Equation (9) corresponds to the conductivity distribution (1). The total resistance was calculated according to Eqs. (6)–(8), then converted into conductance. The results for the averaged conductance Σ are shown in Fig. 2(a) for various values of α . According to the definition of $\tilde{\zeta}$, $\Sigma \sim L^{-\tilde{\zeta}}$, the slopes in Fig. 2(a) represent $\tilde{\zeta}$ for each α . In Fig. 2(b) we plot the local slope $d(\log_{10}\Sigma)/d(\log_{10}L)$ as a function of $1/\log_{10}L$. The correct values of the $\tilde{\zeta}$'s are obtained from the asymptotic region $L \rightarrow \infty$. The values of $\tilde{\zeta}$ agree with those predicted by Eq. (2), for normal and anomalous transport ($\alpha_c = \frac{1}{2}$).

A related result has recently been derived by Schnörer and Blumen¹⁴ for transient transport in a hopping model with a power-law distribution of transition rates.

Next, we consider the Sierpinski gasket. For this structure the low-conducting configurations are connected with quadruply connected bonds, since the minimum number of bonds one has to cut in order to stop the flowing current is four. Hence, comparing the SG with a strip of $n=4$, a rigorous lower bound for α_c is obtained when substituting $n=4$ in Eq. (2), i.e., $\alpha_c \geq \frac{3}{4}$. However, since the number of these bonds scales as $\log L$, we expect that these quadruply connected bonds will not dominate the transport behavior. Thus we argue that for the SG $\alpha_c = 1$, and both $\tilde{\zeta}$ and d_w should accept the same values as for uniform bond conductivities, i.e.,

$$\tilde{\zeta} = \frac{\ln \frac{5}{3}}{\ln 2}, \quad d_w = \frac{\ln 5}{\ln 2}, \quad (10)$$

for all values of the distribution exponent α in (1).

We test this conjecture using extensive numerical simulations. The end-to-end conductance is calculated in a technique we recently used elsewhere.¹³ We transform the gasket into an effective equivalent network, passing through triangle-star transformations (see Fig. 3). The first step is similar to that of Eq. (6) above. After getting a gasket of stars only, we replace every three connected stars by an equivalent star of three resistors, using the following formulas:

$$\begin{aligned} X_1 &= R_1^1 + \frac{(R_1^3 + R_3^1)(R_2^1 + R_2^2)}{R_1^3 + R_2^1 + R_2^2 + R_3^1 + R_3^2 + R_3^3}, \\ X_2 &= R_2^2 + \frac{(R_2^1 + R_2^2)(R_3^2 + R_3^3)}{R_1^3 + R_2^1 + R_2^2 + R_3^1 + R_3^2 + R_3^3}, \\ X_3 &= R_3^3 + \frac{(R_1^3 + R_3^1)(R_3^2 + R_3^3)}{R_1^3 + R_2^1 + R_2^2 + R_3^1 + R_3^2 + R_3^3}, \end{aligned} \quad (11)$$

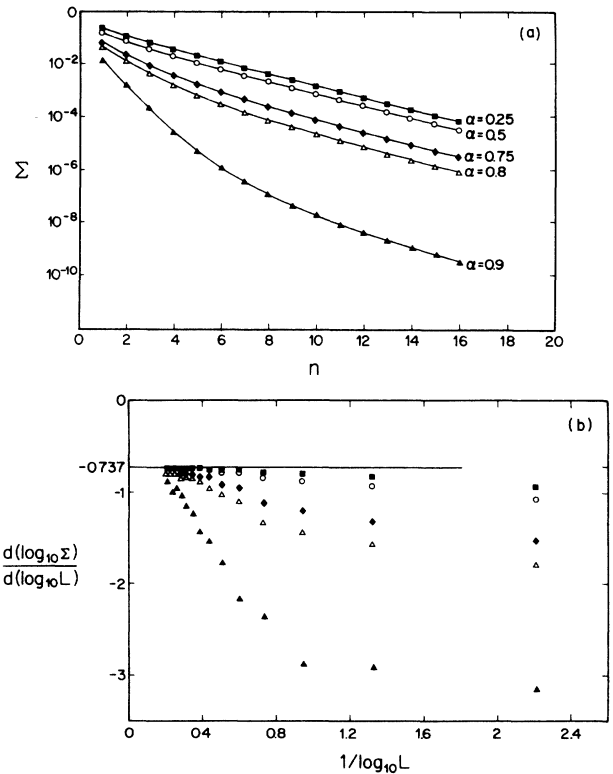


FIG. 4. The Sierpinski gasket. (a) Results for the averaged conductance Σ as a function of n , the order of the gasket, for various values of α . (b) Plot of the local slope $-\tilde{\zeta} = d(\log_{10}\Sigma)/d(\log_{10}L)$ as a function of $1/\log_{10}L$, for the same values as in (a), where $L = 2^n$. The solid line represents the value of the normal resistance exponent as given by Eq. (10). One can see that in the asymptotic region $L \rightarrow \infty$, all slopes converge to this value, which is an exact lower bound for $\tilde{\zeta}$. (For $\alpha > 0.9$, the same qualitative results were obtained, but much larger gaskets are required to gain full convergence.)

where superscripts denote the three original triangles. This last procedure is repeated iteratively until covering the entire gasket. The single star obtained as a final result is transformed back into an effective triangle, using the inverse of the transformation (6). Then the end-to-end resistance (or conductance) is simply obtained.

The simulations were performed for finite gaskets of order n (length $L = 2^n$, containing 3^{n+1} bonds), where n takes the values $n = 2, 3, \dots, 16$. The largest gasket we studied contained $3^{17} = 129,140,163$ bonds. The resistance of each bond was determined according to Eq. (9). The larger the gasket, the fewer ensembles needed to achieve convergence, because of the self-averaging in the larger systems. Due to the huge gaskets we considered, the total computational effort in terms of CPU time was 250 h on IBM 3081.

The results are shown in Fig. 4(a). It can be seen that as n gets larger, all slopes converge to the same value of ζ , which is the value given by Eq. (10), as shown in Fig. 4(b). Thus we conclude that transport on an infinite gasket with bond conductivity distribution (1) is equivalent to transport on an *ordered* gasket, where all bonds have the same conductivity.

This result also agrees with Straley's relation (3). It is known^{9-10,13} that for the SG $1/\nu = 0$ or $\nu = \infty$. Hence the resistance exponent in (3) remains normal for all $\alpha < 1$.

In summary, the ladder and the SG, both possessing 1D features, were shown to behave differently when all their bonds follow the distribution (1). The ladder has an effective critical α , $\alpha_c = \frac{1}{2}$, whereas the SG has not. The basic difference between these two structures is the probability of low-conducting configurations. These configurations, which dominate the transport properties, are related to the number of singly connected bonds. The number of doubly connected bonds in the same column of the ladder scales as L , the length, while the quadruply connected bonds or any other multiply connected bonds in the SG scale as $\log L$. This fact causes a region of anomalous transport for the ladder, but not for the gasket. It is interesting to compare these results to more complicated random systems, such as percolation, where the number of singly connected bonds scale as a power of L . This may explain why the transport on percolation is affected by the distribution (1) for large values of α ($\alpha > \alpha_c$),^{5,6} while for the SG, where the number of singly connected bonds scale only as $\log L$, the transport is normal for all α .

ACKNOWLEDGMENTS

We acknowledge J. Sadigurschi for helping in the computer programming.

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