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Compressibility and superfluidity in the fractional-statistics liquid

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We study the low-temperature properties of a fractional-statistics liquid. Using a hydrodynamic approach based on an extended mean-field approximation, we show that the excitation is gapless, in agreement with a random-phase calculation of Fetter, Hanna, and Laughlin [Phys. Rev. B **39**, 9679 (1989)]. A compressible quantum fluid with only a phonon as low-lying excitation is shown to be a superfluid with a "Meissner effect." We develop a two-fluid model and calculate some properties of the superfluid phase at finite temperatures. These properties are consistent with high-temperature superconductivity. We also discuss experimental signals of time reversal and parity breaking in the superfluid state.

I. INTRODUCTION

The statistical mechanics[1] of a collection of particles satisfying fractional statistics[2–5] poses a challenging problem in theoretical physics. Recently, this problem has taken on great physical interest with the suggestion of Kalmeyer and Laughlin[6] that the elementary excitations in high-temperature superconductors may have fractional statistics. We have shown[7,8] that in a nonlinear σ-model description of the high-temperature superconductivity, fractional statistics, together with the separation of spin and charge, emerge quite naturally. We found that the statistics parameter $e^{i\theta}$ (to be defined later) could have the values $e^{i\pi/2n}$, for n=integer. This conclusion agrees with that of Ref. 6. In particular, for $n=1$, the excitations are semions. Together with Wilczek, we have also shown how fractional statistics could arise in a lattice formulation based on the notion of chiral spin states.[9]

In a profound and original article,[10] Laughlin outlined a general picture for the superfluidity of the fractional-statistics fluid. Microscopically, semions, being half way between fermions and bosons, should be more inclined than fermions to pair. Two semions make a boson and the condensation of such bosons may lead to superconductivity. We have developed an effective Landau-Ginzburg theory[11] based on this picture and have used the theory to calculate possible experimental consequences.

The Lagrangian describing a collection of fractional-statistics particles is rather simple:

$$L=\sum_{i=1}^{N}\tfrac{1}{2}m\left(\frac{d\mathbf{x}_i}{dt}\right)^2+\hbar\frac{\theta}{\pi}\sum_{i\neq j}^{N}\frac{d}{dt}\varphi_{ij}\ . \tag{1.1}$$

Here $\mathbf{x}_i$ is a two-dimensional vector specifying the location of the ith particle and φ_{ij} is the azimuthal angle between particles i and j (measured relative to some fixed reference axis). We may remind the reader that fractional statistics occurs only in $(2+1)$-dimensional space-time. The second term in (1) is a total time derivative and does not contribute classically. Quantum mechanically, however, a phase is induced in the wave function. In particular, if particle i goes halfway around particle j (thus effectively interchanging the two particles after a translation) a phase $e^{i\theta}$ is induced. For $\theta=0\,\mathrm{mod}2\pi$ we have bosons; for $\theta=\pi\,\mathrm{mod}2\pi$, we have fermions, while for an arbitrary value of θ, we have anyons. For physical applications to high-temperature superconductivity, we are particularly interested[6–11] in semions with $\theta=\pi/2$. While most of our discussion holds for arbitrary θ as will be clear from context, for ease of language we will often speak of semions.

From (1) we find that the canonical momentum

$$\mathbf{p}_i=m\dot{\mathbf{x}}_i-\hbar\mathbf{a}_i(x_i)\ , \tag{1.2}$$

where

$$a_{i\alpha}(x_i)=\frac{\theta}{\pi}\sum_{j\neq i}\frac{\epsilon_{\alpha\beta}(x_i-x_j)^\beta}{|x_i-x_j|^2}\ . \tag{1.3}$$

Here the spatial indices $\alpha,\beta=1,2$. Repeated α,β indices are summed. The Hamiltonian is then

$$\begin{aligned}H&=\sum_{i=1}^{N}\tfrac{1}{2}m\left(\frac{dx_i}{dt}\right)^2\\&=\sum_{i=1}^{N}\frac{1}{2m}[\mathbf{p}_i+\hbar\mathbf{a}_i(\mathbf{x}_i)]^2\ .\end{aligned} \tag{1.4}$$

As is well-known by now, the dynamics can be described by bosons interacting via a statistical gauge potential $\mathbf{a}$ (not to be confused with the electromagnetic gauge potential, of course). More precisely, the particles may be regarded as carrying statistical charge 1 and flux $\phi=\theta/\pi$. There is then a Dirac-Aharonov-Bohm phase interaction between the particles. The N-body wave function $\psi(x_1,\ldots,x_N)$ satisfies the Bose condition:

$$\psi(\cdots x_i\cdots x_j\cdots)=\psi(\cdots x_j\cdots x_i\cdots)\ .$$

Alternatively, we can describe the particles as fermions interacting with $\mathbf{a}$ but with the shift $\theta\to\theta-\pi$. In this case the N-body wave function satisfies the Fermi condition:

$$\psi(\cdots x_i \cdots x_j \cdots) = (-1)\psi(\cdots x_j \cdots x_i \cdots) .$$

We can also perform a singular gauge transformation and have a free Hamiltonian

$$H' = \sum_{i=1}^{N} \frac{\mathbf{p}_i^2}{2m} . \tag{1.5}$$

In this gauge, the wave function is constrained by the rather nasty condition that upon interchange of two functional-statistical particles the wave function acquires the phase $e^{i\theta}$.

The difficulty of the problem may be appreciated by noting that in the singular gauge the N-body wave function[12] cannot be constructed out of the one-body wave function as is the case for bosons and fermions. Strictly speaking, even the notion of the wave function is not defined. For instance, the two-body wave functions $\psi(x_1,x_2)$ are reached from some reference positions (x_1^0,x_2^0). The wave function $\psi(x_1,x_2)$ has a phase $e^{i(\theta/\pi)\varphi}$ relative to $\psi(x_1^0,x_2^0)$, where φ is the angle measuring the number of times the particles have wound around each other in going from (x_1^0,x_2^0) to (x_1,x_2). In the N-body case, the phase depends on the positions of the other particles.

As is well known by now, the system described here violates time reversal T and parity P. An exciting possibility is that T and P are violated in high-temperature superconductors,[13] leading to rather dramatic experimental consequences.[11] In superconductors the excitations carry a unit electric charge. In writing down (1), we have ignored the Coulomb interaction term. Its effect is negligible in the high-density limit, when the kinetic energy, which is of order $\hbar^2 n/m$, is much larger than the Coulomb energy $e^2/n^{-1/2}$, in other words, for $n \gg (me^2/\hbar^2)^2$.

The problem is then to understand the statistical mechanics of this system. How does it differ from the free bose gas and the free fermion gas? What are its elementary excitations? Is it a superfluid?

It should be noted that at zero temperature this problem has only two dimensionful parameters: $\hbar/m$ with dimension length squared divided by time and the density n with dimension inverse length squared. Notice that in our convention the spatial component of the gauge potential has dimension inverse length (so that the covariant derivative is $\partial_\alpha - ia_\alpha$).

At this point, let us warn the reader again of a potential confusion in this subject. The statistical gauge potential $\mathbf{a}$ is not to be confused with the electromagnetic gauge potential, which we will denote by $\mathbf{A}$. As it is rather cumbersome to speak of statistical flux, statistical magnetic field, and so on, we will simply use terms such as flux and magnetic field, trusting the reader to discern from the context whether we mean the statistical magnetic field or the standard magnetic field. When we wish to emphasize the distinction we will restore the qualifier "statistical" or put words like magnetic in quotation marks. We could have followed the practice of another field and invented terms like selectric and smagnetic but it is probably clearer not to.

II. MEAN-FIELD APPROXIMATION

The statistical mechanics had been worked out in the low-density (or high-temperature) limit since the problem reduces essentially to a two-body problem.[1–14] Even so, the physics contains some surprising features.

In the high-density limit, Arovas *et al.*[1] suggested that a mean-field approximation in which the statistical flux carried by the particles be averaged over space. The particles are then treated as moving in a uniform magnetic field $b \equiv \epsilon_{\alpha\beta}\partial_\alpha a_\beta = 2n\theta$. The idea is then to reduce the problem to a single-body problem. At first sight, this appears questionable since the gauge potential $\mathbf{a}_i$ in (1.4) felt by each particle is in fact a pure (but topologically nontrivial) gauge and thus associated with zero b: $\epsilon_{\alpha\beta}\partial_\alpha a_{i\beta} = 0$. Consider, however, the scattering of one particle off another. Due to P (and T) violation, the particle tends to scatter preferentially to one side.[13] The net effect is that the particle moves more or less in a circular orbit and acts as if it is experiencing a uniform magnetic field. (Fig. 1). More precisely, as a given particle is transported around a large loop of area A and, enclosing nA particles, its wave function picks up a phase $e^{i(nA)2\theta}$ Thus, the effective "magnetic" field

$$b = 2n\theta . \tag{2.1}$$

This heuristic picture suggests that the mean-field approximation may be sensible, at least as a starting point for further work. Indeed, Laughlin[15] has used this approximation to derive reasonable results for the fractional-statistics liquid. Furthermore, Girvin and MacDonald,[16] and others[17–19] have shown that this mean magnetic field approximation when applied to the fractional Hall effect gives results in agreement with Laughlin's microscopic theory.

In Laughlin's work,[15] he treated the particles in (1.4) as fermions and built up the ground state by filling the Landau levels in accordance with the Pauli exclusion principle. Recall from the theory of the fractional Hall effect that the filling factor ν is equal to

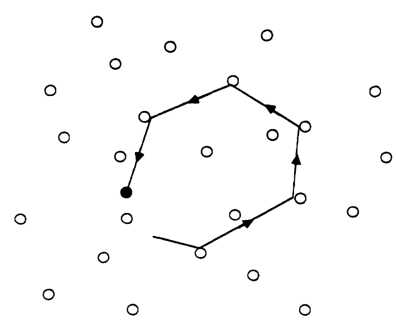

FIG. 1. A fractional-statistics particle moving in a background of many other fractional-statistical particles experiences an effective "magnetic" field.

$$\nu = \frac{N}{BA/2\pi} . \tag{2.2}$$

Here we plug in for B the value for b given above and so $\nu = \pi/\theta$ and the semion case corresponds to filling the two lowest Landau levels.[15]

Recently, Fetter, Hanna, and Laughlin[20] went beyond this work by employing a random-phase approximation. They showed that the semion liquid is a superfluid. While this conclusion is plausible, the physics underlying this random-phase calculation is not totally clear to us. The important physics is inextricably intertwined with the technical complexity of the calculation.

Our goal here is to lend support to the conclusion of Fetter *et al.* by developing an alternative and hopefully more physical picture. Our picture overlaps considerably with the picture developed by Laughlin and his collaborators. Our approach is in the tradition of Landau's and Feynman's treatment of superfluidity;[21–23] it is physical rather than computational.

III. FRUSTRATION AND PAIRING

The difficulty in solving the fractional-statistics-particle liquid may be expressed by saying that the system is frustrated. The wave function has to change sign whenever one particle goes around another. One way of relieving this frustration in the semion case is for two semions to join so as to form a boson.

We would like to sketch some heuristic arguments suggesting that the semions would tend to pair to form bosons. It will be clear that these arguments are incomplete and may only be suggestive.

First, we can attempt to go beyond the $[(N-1)+1]$-body treatment of Ref. 15 by considering an $[(N-2)+2]$-body problem in which two semions move in the mean magnetic field $b=2n\theta=n\pi$ generated by the other $(N-2)$ semions. The two-body Hamiltonian can be decomposed into a center-of-mass Hamiltonian and the relative motion Hamiltonian. Solving the relative motion Hamiltonian, we find that the lowest energy of relative motion is just $\frac{1}{2}\hbar\omega_c=\hbar^2 b/2m$ after we take into account the reduced mass and so on. (Here ω_c denotes the cyclotron frequency.) This is the same as the lowest one-body energy, i.e., the lowest energy of a particle moving around in the field b.

The key is then to argue that the center of mass does not see a magnetic field. When the center of mass moves around a large loop of area A and enclosing N_A particles, it acquires a phase $e^{iN_A(2\theta)2}=e^{i2\pi N_A}=1$ for $\theta=\pi/2$ (Fig. 2). Hence, the center of mass sees a zero magnetic field rather than a magnetic field that is twice as large. If so, then the two semions have energy $E_{2\ \mathrm{body}}+E_{\mathrm{c.m.}}=\frac{1}{2}\hbar\omega_c+(0)$, which is less than $E_{1\ \mathrm{body}}+E_{1\ \mathrm{body}}=\frac{1}{2}\hbar\omega_c+\frac{1}{2}\hbar\omega_c$, thus suggesting that it may be energetically favorable for the semion to pair. This argument is essentially a more precise expression of the words given in the first paragraph of this section.

Our second argument is based on the observation that in contrast to the difficulty of solving the problem of fractional-statistics particles in empty space, we can readily write down the wave function of fractional-statistics particles in an external (electromagnetic) magnetic field B. We simply borrow from the theory of the fractional Hall effect, Laughlin's famous wave function, and change the odd integer he put in the exponential by a fraction:

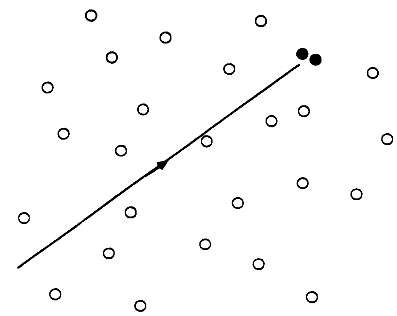

FIG. 2. A pair of fractional-statistical particles moving in a background of other fractional-statistical particles.

$$\psi(z_1,\ldots,z_N)=\prod_{i<j}(z_i-z_j)^{\theta/\pi}\exp\left[-\frac{B}{4}\sum_l |z_l|^2\right] . \tag{3.1}$$

Here $z_i=x_i+iy_i$ is the position of the ith particle in complex coordinates. (This is manifestly a solution, since ψ may be expanded as a superposition of one-body wave function of the form $z_i^n e^{-(B/4)|z_i|^2}$ for any given i.) A heuristic way of saying this is that the phase frustration required of the fractional-statistics particles is just compensated by the phase change imposed by the magnetic field. As is well known, the magnetic field B is constrained to be related to the density n by [cf. (2.2)]

$$\frac{2\pi n}{B}=\frac{\pi}{\theta}=\nu . \tag{3.2}$$

(This is of the crux of the matter for the theory of fractional Hall effect and essentially implies that the "Hall" gas is incompressible.)

However, we do not want to solve the problem of the fractional-statistics-particle gas in an external magnetic field B. We want to solve it for $B=0$. Suppose we try to reach $B=0$ by decreasing B. For $B-\delta B$, the density would want to decrease correspondingly to $n-\delta n$ almost everywhere. To conserve the total number of fractional-statistics particles, the system necessarily has to nucleate regions with higher n (see Fig. 3). These nucleated lumps are just the quasielectron in Laughlin's theory of the fractional Hall effect in which the quantum numbers of these quasielectrons are worked out. For the semion gas, these nucleated lumps contain two semions. This picture suggests strongly that the semion gas tends to pair into bosons.

In the Appendix we give yet another heuristic argument. Taken together, these admittedly heuristic arguments all indicate that pairing of half fermions may be a central feature of the half-fermion liquid. However, the

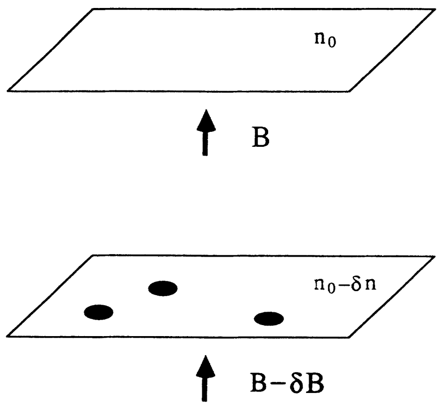

FIG. 3. As the magnetic field is lowered, fractional-statistical particles nucleate into lumps. Each lump contains two fractional-statistical particles if $\theta=\pi/2$.

size of these pairs tends to be of order $b^{-1/2}\sim(n\theta)^{-1/2}$ (for general θ) and is thus comparable to the interparticle separation for semions and larger than the interparticle separation for θ small. This suggests that at least for θ small, a mean-field approximation may be quite reliable in the same way that the mean-field approximation works well in the BCS theory because of the large size of the Cooper pairs. On the other hand, it may be quite misleading to view the fractional-statistics liquid roughly as consisting of point bosons.

IV. A HYDRODYNAMIC TREATMENT

In this section we will develop an effective theory of the fractional-statistics liquid valid at long wavelength and low frequency. For the rest of this paper we will concentrate on the case $\theta=\pi/q$, for q an integer. We will use the mean "magnetic" field approximation, but in contrast to Laughlin *et al.*,[15,20] we treat the particles in (1.4) as bosons. It may be worthwhile to emphasize here that having a Bose gas does not necessarily mean that we have superfluidity or superconductivity, as is sometimes said rather glibly. In his classic paper Landau[21] pointed out that superfluidity requires that the energy dispersion $\omega(k)$ of the elementary excitation has the "roton" form, ie., that the ratio $\omega(k)/k$ is bounded below. In particular, for small k, $\omega(k)$ has to be linear in k, and thus the free Bose gas is certainly not a superfluid. As Bogoliubov[24] showed, a hard-core repulsion changes the energy dispersion from $\omega(k)\propto k^2$ to $\omega(k)\propto k$. Furthermore, we must show that the single-particle ground state the bosons can condense into is not degenerate.

Here we have a system of bosons with a rather unfamiliar magnetic interaction. In the mean magnetic field approximation, the bosons are to put into the states of lowest Landau level

$$\psi_n \sim z^n e^{-(1/4)|z|^2} . \tag{4.1}$$

(We have set the magnetic length to unity here.) Given that these states in the lowest Landau level ψ_n are all degenerate with the energy $\frac{1}{2}\hbar\omega_c$, how do we put the N bosons into these states?

To obtain a hint on this question, we recall that the quantum degeneracy corresponds classically to the freedom of centering the Larmor orbit around any point we choose. In other words, by taking linear combinations of ψ_n, we can form the wave function

$$\psi_\alpha(z)\sim e^{-(1/4)(|z|^2-2\alpha^* z)} , \tag{4.2}$$

where the complex number α specifies the center of the orbit corresponding to ψ_α. We consider an arrangement shown in Fig. 4. We consider a periodic arrangement of orbits so that space is filled. Since the radius of the orbit is of order of the magnetic length

$$1\sim b^{-1/2}\sim(n\theta)^{-1/2} , \tag{4.3}$$

in the classical picture the orbits are essentially touching. Since the filling factor is $\nu=q$, we put q bosons into the quantum state corresponding to each orbit (Fig. 4). In this case the state has a uniform density of anyons, which is consistent with our assumption that the mean "magnetic" field b is constant in space. Notice that the states ψ_{α_i} and ψ_{α_j} are not orthogonal but the overlap is small because of the exponential.

This state is stable as can be seen from the following argument. Suppose we take out a boson from one of the orbits and put it into some other orbit (see Fig. 5). In the region where there is an excess of bosons, the mean magnetic field b is somewhat higher, leading to a higher $\frac{1}{2}\hbar\omega_c$. The opposite holds where there is a deficit of bosons. Thus, it is energetically favorable to have the uniform arrangement shown in Fig. 4. The particle density tends to be spatially uniform.

There is a more direct way to understand the aforementioned result. We know that the anyon wave function obtains a nonzero phase after exchanging two anyons. This nonzero phase implies that the fractional-statistics-particle system is frustrated and the fractional-statistics particles have nonzero kinetic energies even in the ground state. The average kinetic energy per particle is expected to be proportional to the fractional-statistics-particle density. In particular, for fermions the average

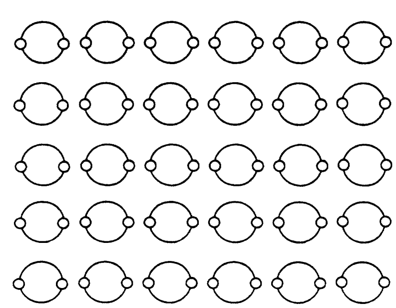

FIG. 4. Landau orbits in real space. For $\theta=\pi/2$ two half fermions are put in each orbit in the superfluid state.

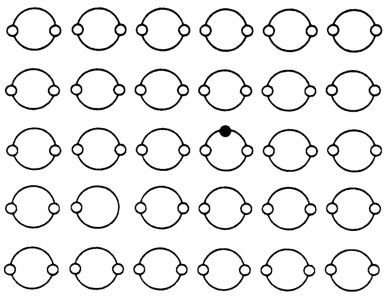

FIG. 5. One particle moves from one Landau orbit to another to obtain a state with a nonuniform distribution.

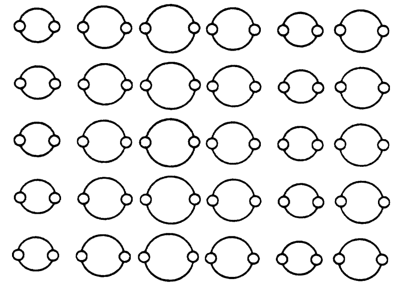

FIG. 6. The low-lying excitations correspond to a "breathing mode" of Landau orbits.

kinetic energy is equal to half of the Fermi energy: $E_F/2=\hbar^2\pi n/m$. If the fractional-statistics-particle density is not uniform the fractional-statistics particles will flow from high-density region to low-density region. In other words, there is an exchange pressure in the fractional-statistics-particle fluid to keep the fractional-statistics-particle density unform. In our mean-field approach to the fractional-statistics-particle fluid, the first Landau level has an energy $\frac{1}{2}\hbar\omega_c=\hbar^2\theta n/m$, which corresponds to the average kinetic energy of a fractional-statistics particle. Thus the "magnetic" field in the mean-field theory correctly (at least qualitatively) represents the frustration in the fractional-statistics-particle system. We expect that the correct treatment of the mean-field theory should lead to a qualitatively correct physical picture for the fractional-statistics-particle fluid at low temperatures.

In a sense, a fractional-statistics-particle gas resembles a fermion gas in having an exchange pressure due to quantum statistics. But it also resembles a Bose gas in not having a Fermi surface and so it does not have low-lying particle-hole excitations across the Fermi surface. Thus, the low-lying excitations of the fractional-statistics-particle fluid are likely to be only phonons with linear rather than quadratic dispersion.

We see that to obtain a nondegenerate ground state it is crucial to allow the magnetic field to react to the bosons rather than having the bosons move in a fixed and static magnetic field. Thus, the approximation may be called an "extended mean-field approximation."

In the ground state, the magnetic field b is constant. The low-lying long-wavelength excitation corresponds to a "breathing mode" in which b and the particle density vary slowly over space and time (Fig. 6).

Let us thus consider a hydrodynamic treatment. This is made possible by the simple observation that the picture given earlier determines the potential energy completely. The energy of the state considered here is

$$N(\tfrac{1}{2}\hbar\omega_c)=N\hbar^2\theta n/m = A\hbar^2\theta n^2/m \ .$$

Thus, the energy of the liquid E contains a term $\int d^2x\, \hbar^2\theta n^2/m$. Including the kinetic energy the energy functional E must have the form

$$E=\int d^2x\,(\tfrac{1}{2}mnv^2+\gamma n^2) \tag{4.4}$$

with $\gamma=\hbar^2\theta/m$. The energy E is to be minimized with a fixed total number of particles.

The pressure is easily calculated,

$$P=-\frac{\partial}{\partial A}\gamma A n^2=\gamma n^2 \ . \tag{4.5}$$

The hydrodynamics equation is thus

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\partial)=-\frac{1}{mn}\partial P=-\frac{2\gamma}{m}\partial n \ . \tag{4.6}$$

Linearizing (4.6) around the ground state with uniform density n_0 and the equation of continuity

$$\frac{\partial n}{\partial t}+\partial(n\mathbf{v})=0 \ , \tag{4.7}$$

we find the wave equation

$$\frac{\partial^2 n}{\partial t^2}+\frac{2\gamma n_0}{m}\nabla^2 n=0 \ , \tag{4.8}$$

so that the speed of sound is given by

$$c_s^2=2\gamma n_0/m=2\hbar^2\theta n_0/m^2 \ . \tag{4.9}$$

The form of this result could have been determined by dimensional analysis. Thus, the low-lying elementary excitation of the system is expected to be a gapless phonon.

It was crucial here that the potential energy is proportional to n^2 rather than to n (in which case c_s would clearly vanish). This is a direct consequence of the "magnetic" interaction between the bosons, or in other words, the exchange pressure of the fractional statistics as discussed earlier.

We may calculate the coherence length if we can regard (4.4) as the Ginzburg-Landau theory. The potential term in the Ginzburg-Landau theory can be identified as

$$V(n)=\mu n+\gamma n^2 \ , \tag{4.10}$$

where μ is the chemical potential to cause $V(n)$ to become minimized at $n=n_0$. Using a standard formula, we find the coherence length ξ_0 to be

$$\xi_0^2=\frac{\hbar^2}{2m|\mu|}=\frac{\hbar^2}{4m\gamma n_0}=\frac{1}{4\theta n_0} \ . \tag{4.11}$$

The coherence length obtained in Ref. 20 is larger than our value by a factor of ~ 2.

It is not entirely clear to us how the discussion, given here for $\theta=\pi/q$, would proceed for $\theta=p\pi/q$ or for θ irrational. We suspect that the statistical mechanics of the fractional-statistics-particle liquid have an extremely complicated dependence on θ, similar to the dependence of the energy or the filling factor ν in the fraction quantum Hall effect. This was already suggested by the result obtained in Ref. 1 that the second virial coefficient as a function of θ has a cusp at $\theta=0$. In the following, we stick to $\theta=\pi/q$.

V. SUPERFLUIDITY AND SUPERCONDUCTIVITY

The relation between b and n in (2.1) may be written as

$$j_0 \equiv n = \frac{1}{4\theta}\epsilon_{ij} f_{ij} \ . \tag{5.1}$$

We are led to introduce an electric component f_{0k} by writing

$$j_\mu = \frac{1}{4\theta}\epsilon_{\mu\nu\lambda} f_{\nu\lambda} \ , \tag{5.2}$$

so that in addition to (5.1) we make the identification

$$j_i \equiv n v_i = \frac{1}{2\theta}\epsilon_{ij} f_{0j} \ . \tag{5.3}$$

(We should say a word abut the notation. In the preceding sections, in keeping with the standard practice from the theory of the quantum Hall effect to label particles by the index i, we label spatial coordinates by $\alpha,\beta,\ldots,$. Here we revert to the more standard notation of labeling space-time coordinates as $\mu,\nu,\ldots,$ and spatial coordinates as $i,j,\ldots,$.)

In that case we recognize the potential energy in (4.4) as

$$\int d^2x\, \gamma n^2 = \int d^2x \frac{\gamma}{8\theta^2} f_{ij}^2 \tag{5.4}$$

and the kinetic energy as

$$\int d^2x \tfrac{1}{2} m n v^2 = \int d^2x \frac{\frac{1}{8}(m/\theta^2) f_{0k}^2}{n_0+\epsilon_{ij}f_{ij}/4\theta} \simeq \int d^2x \frac{m}{8n_0\theta^2} f_{0k}^2 \left[1 - \frac{1}{4n_0\theta}\epsilon_{ij}f_{ij} + \cdots \right] . \tag{5.5}$$

Combining (5.4) and (5.5) and keeping terms to leading order in the field strength we see that the dynamics is described by none other than the Maxwell Lagrangian

$$\mathcal{L} = \frac{1}{16\pi g^2} f_{\mu\nu}^2 \tag{5.6}$$

if we use length and time units so that the speed of sound c_s is equal to unity. This allows us to use a relativistic notation. The "fine-structure constant" g^2 is equal to

$$g^2 = \frac{\theta m}{2\pi\hbar^2} \ . \tag{5.7}$$

The theory describes a gapless excitation, namely, the Maxwell "photon" but in fact the phonon here. The higher-order terms such as $\sim f_{0k}^2(\epsilon_{ij}f_{ij})$ break "Lorentz" invariance but not gauge invariance. They are important only at short distances. It is crucial that we do not obtain the Chern-Simons term $\epsilon^{\mu\nu\lambda}a_\mu f_{\nu\lambda}$, since it would generate a gauge invariant mass for the phonon. (For a recent discussion of the Chern-Simons term, see Refs. 7 and 25). In our long-wavelength low-frequency treatment of the semion liquid the T and P violation explicit in the microscopic theory (1.4) has been "averaged over." Physically, this is quite reasonable in our picture of semions pairing into bosons: we expect T and P violation to appear only at high frequencies and short distances at which the microscopic structure is revealed. In the effective Lagrangian, T and P violation should appear in only higher-dimensional terms, and not in the Chern-Simons term (which has one lower dimension than the Maxwell term). We will discuss this point in more detail later.

It is now easy to show that superconductivity occurs by coupling in the electromagnetic field. We recall that the semions are electrically charged and thus the current $j_\mu=(n,nv_i)$ is in fact the electromagnetic current. The three spatial dimensional electromagnetic current density is given by $(e/d)j_\mu$, where d is the interplane distance. (Recall that high-temperature superconducting materials consist of weakly coupled layers.) The current-current correlation function is easily calculated in momentum space:

$$\begin{aligned}\frac{e^2}{d^2}\langle j_\mu(k) j_\nu(-k)\rangle &= \frac{e^2}{16\theta^2 d^2}\epsilon_{\mu\lambda\sigma}\epsilon_{\nu\rho\tau}\langle f_{\lambda\sigma}(k) f_{\rho\tau}(-k)\rangle = \frac{e^2}{4\theta^2 d^2}\epsilon_{\mu\lambda\sigma}\epsilon_{\nu\rho\tau}k_\lambda k_\rho\langle a_\sigma(k) a_\tau(-k)\rangle \\ &= \frac{\pi e^2}{\theta^2 d^2} g^2 (g_{\mu\nu}k^2 - k_\mu k_\nu)/k^2 \\ &= \frac{\pi e^2}{\theta^2 d^2} g^2 (g_{\mu\nu} + G_{\mathrm{df}}) \ ,\end{aligned} \tag{5.8}$$

where $G_{\rm df}$ is the gauge degree of freedom. We obtain the Meissner effect and thus superconductivity. In coordinate space (5.8) can be written as

$$\frac{e^2}{d^2}\langle j_\mu(x)j_\nu(0)\rangle=\frac{\pi e^2}{\theta^2 d^2}g^2\delta^{(3)}(x)$$
$$=\frac{\pi e^2}{\theta^2 d}g^2\delta^{(4)}(x)\ . \quad (5.9)$$

The δ function in $1+3$ dimensions is given effectively by $\delta^{(4)}(x)=\delta^{(3)}(x)/d$. Putting back the speed of the sound c_s we get

$$\frac{e^2}{d^2}\langle j_i(x)j_j(0)\rangle=\frac{\pi e^2}{\theta^2 d}g^2c_s^2\delta^{(4)}(x)$$
$$=\frac{e^2 n_0}{md}\delta^{(4)}(x)\ , \quad (5.10)$$

which is the expected result. We see that the London penetration depth is

$$\lambda_L^2=\frac{mc^2 d}{4\pi e^2 n_0}\ , \quad (5.11)$$

where c is the speed of light. The ratio between the London penetration depth and the coherence length is given by κ:

$$\kappa^2=\frac{\lambda_L^2}{\xi_0^2}=\frac{mc^2 d}{e^2}\frac{\theta}{\pi}\ . \quad (5.12)$$

The reader may have recognized that the gauge potentials a_μ and A_μ are dual to each other in the sense discussed in Ref. 25. Our effective theory (5.6) can be regarded as the dual form[26] of the conventional Ginzburg-Landau theory in $(2+1)$-dimensional space-time.

Although the effective theory (5.6) gives rise to the Meissner effect, the value of the flux quantum is not determined by the theory. This is because (5.6) only describes the local small fluctuations in the superfluid. In the following we would like to show that in any quantum fluid state of the charge e fractional-statistics-particle system, hc/qe flux always has finite energy if the fractional-statistics particles have $\theta=p\pi/q$. First, according to the argument of Byers and Yang[27] hc/e flux has finite energy because the fractional-statistics-particle wave function acquires a phase 2π after a fractional-statistics particle goes around the flux. We also know that the fractional-statistics-particle wave function acquires a phase $2p\pi/q$ after a fractional-statistics particle goes around another fractional-statistics particle. Because p and q are incommensurable, there exists an integer n such that $np/q=(-1/q)\mathrm{mod}1$. Thus the bound state of the n fractional-statistics particles and hc/qe flux has a finite energy because a fractional-statistics particle going around this bound state acquires a phase of 2π. We would like to stress that in the above we only show that hc/qe flux has a finite energy, this does not imply that hc/qe is the smallest flux quantum. When both p and q are odd integers, a $2q$-particle bound state is a boson, since the statistics angle for such a bound state is given by $(2q)^2\theta$. The condensation of these bosons lead to a superconducting state, and the smallest flux quantum in this superconducting state is $hc/2qe$. We do not know whether there exists a superconducting state for fractional-statistics particles with odd p and q such that the flux quantum is hc/qe. For fermions ($p=q=1$), Yang[28] showed that the flux quantum is always less than or equal to $hc/2e$ in the superconducting state of a charge e fermion system. When one of p and q is an even integer, q fractional-statistics-particle bound states are bosons. In the boson condensed state the flux quantum is hc/qe, which coincides with the value obtained from general consideration.

From the discussion in this section we see that the compressibility and the superfluidity of quantum fluid are closely related. We demonstrate that a compressible quantum fluid with phonons as the only low-lying excitations is actually a superfluid. Here by superfluid we mean a quantum fluid with Meissner effect, i.e., the current-current correlation is given by (5.8) for small momenta. Landau's argument[21] assures that the quantum fluid under consideration supports dissipationless flow, but it does not imply the existence of off diagonal long-range order[29,28] or Meissner effect, which is the essence of superfluidity. Our dual relation (5.2) is needed.

Due to the importance of this result it is worthwhile to give another derivation of the relation between compressibility and superfluidity. We start with the Hamiltonian of the phonon system

$$H=\int \tfrac{1}{2}mn_0v^2d^2x$$
$$+\int \tfrac{1}{2}\varphi(x-x')n(x)n(x')d^2x\,d^2x'\ , \quad (5.13)$$

where $\varphi(x)$ is some short-ranged interaction satisfying

$$\int d^2x\tfrac{1}{2}\varphi(x)=\gamma\ . \quad (5.14)$$

In momentum space, we have the linearized version of the Hamiltonian

$$H=\int\frac{d^2k}{(2\pi)^2}\left[\frac{m}{2n_0k^2}\dot n_k\dot n_{-k}+\tfrac{1}{2}\varphi_k n_k n_{-k}\right]$$
$$=\int\frac{d^2k}{(2\pi)^2}\left[\frac{n_0k^2}{2m}\pi_k\pi_{-k}+\tfrac{1}{2}\varphi_k n_k n_{-k}\right]\ , \quad (5.15)$$

where π_{-k} is the canonical conjugate of n_k

$$[\pi_{-k},n_{k'}]=-i\hbar(2\pi)^2\delta^3(k-k')\ . \quad (5.16)$$

Introducing the phonon annihilation operator

$$a_k=\frac{1}{\sqrt{2\hbar}}\left[\left[\frac{\omega_k}{\varphi_k}\right]^{1/2}\pi_k-i\left[\frac{\varphi_k}{\omega_k}\right]^{1/2}n_k\right] \quad (5.17)$$

we may write H as

$$H=\int\frac{d^2k}{(2\pi)^2}\hbar\omega_k a_k^\dagger a_k+\text{ const}\ , \quad (5.18)$$

where

$$\omega_k^2=\frac{\varphi_k n_0}{m}k^2\ , \quad (5.19)$$

$$[a_k, a_{k'}^\dagger] = (2\pi)^2 \delta(k-k') \ . \tag{5.20}$$

The current is given by

$$\mathbf{j}_k = -\frac{i\mathbf{k}}{k^2}\dot{n}_k = -i\mathbf{k}\frac{n_0}{m}\pi_{-k} \ . \tag{5.21}$$

The time-ordered product of the currents is found to be

$$\begin{aligned}\tilde{K}^{ij}(t;k) &\equiv \langle T[j_k^i(t) j_{-k}^j(0)]\rangle \\ &= \tfrac{1}{2}k^i k^j \frac{n_0^2 \varphi_k}{m^2\omega_k} e^{-i\omega_k|t|} \ .\end{aligned} \tag{5.22}$$

In frequency space

$$\tilde{K}^{ij}(\omega,k) = i\frac{k^i k^j n_0^2 \varphi_k}{m^2}\frac{1}{\omega^2-\omega_k^2} \ . \tag{5.23a}$$

Similarly, we may obtain the time-ordered products between the current and the density

$$\tilde{K}^{0i} = -i\frac{k^i n_0}{m}\frac{\omega}{\omega^2-\omega_k^2} \ , \tag{5.23b}$$

$$\tilde{K}^{00} = i\frac{k^2 n_0}{m}\frac{1}{\omega^2-\omega_k^2} \ . \tag{5.23c}$$

We find that

$$\begin{aligned} k^\mu \tilde{K}_{\mu 0} &= 0 \ , \\ k^\mu \tilde{K}_{\mu i} &= \frac{i n_0}{m} k^i \ . \end{aligned} \tag{5.24}$$

However, the current-current correlation function $K^{\mu\nu}$ is required by current conservation to satisfy the Ward identity

$$k^\mu K_{\mu\nu} = 0 \ . \tag{5.25}$$

Thus, $K^{\mu\nu}$ differs from $\tilde{K}^{\mu\nu}$ by an additional term:

$$K^{\mu\nu} = \tilde{K}^{\mu\nu} + \Delta K^{\mu\nu} \ . \tag{5.26}$$

(The existence of $\Delta K^{\mu\nu}$ corresponds to the appearance of the Schwinger term in the charge-current commutation relation, as is well-known from the field-theory literature.[30])

Since the poles in $K^{\mu\nu}$ and $\tilde{K}^{\mu\nu}$ must have the same residue and location, $\Delta K^{\mu\nu}$ can only be a polynomial function of ω and $\mathbf{k}$. Let $\Delta K_{00} = a$, $\Delta K_{0i} = b k_i$, and $\Delta K_{ij} = c k_i k_j + d\delta_{ij}$, where a,b,c,d are polynomials in ω and $\mathbf{k}$. We see immediately that Eqs. (5.24) and (5.25) imply that

$$d = \frac{i n_0}{m} + c\mathbf{k}^2 - a\omega^2 \ . \tag{5.27}$$

Thus, at zero frequency and momentum $d \to i n_0/m$ and the correlation function

$$K_{ij} \to \tilde{K}_{ij} + i\frac{n_0}{m}\delta_{ij} \ . \tag{5.28}$$

The δ_{ij} term describes the Meissner effect. (In the field-theory literature, $\Delta K_{\mu\nu}$ is determined completely by the further requirement that it behaves suitably at high frequency and momentum. Here, however, since we are dealing with an effective theory, we do not impose this additional constraint.)

We remark that the one-phonon state $a_k^\dagger|0\rangle$ is just $i\sqrt{2\varphi_k/\hbar\omega_k}\, n_k^\dagger|0\rangle$. It corresponds to the wave function

$$\psi(x) = \sum_{i=1}^{N} e^{i\mathbf{k}\cdot\mathbf{x}_i}\psi_0(x) \ ,$$

which is just the variational wave function considered by Bijl and Feynman.[31]

From the preceding discussion, it is clear that the relation between compressibility and superfluidity should be valid in any dimension, even though the mathematical manipulations used at the beginning of this section are only valid in $(2+1)$-dimensional space-time.

VI. VORTICES

We can solve Maxwell's equations $\partial_\mu f^{\mu\nu} = 0$ by writing

$$f^{\mu\nu} = \epsilon^{\mu\nu\lambda}\partial_\lambda \eta \ , \tag{6.1}$$

where η is a real scalar field. We then have

$$\epsilon^{\mu\nu\lambda}\partial_\mu f_{\nu\lambda} = \partial^2\eta \ , \tag{6.2}$$

and the Bianchi identity $\epsilon^{\mu\nu\lambda}\partial_\mu f_{\nu\lambda} = 0$ is satisfied if

$$\partial^2 \eta = 0 \ . \tag{6.3}$$

Thus, the gapless degree of freedom in $f_{\mu\nu}$, the phonon, is represented by the real field η.

The current j_μ is equal to

$$j_\mu = \frac{1}{2\theta}\partial_\mu \eta \propto \chi^\dagger \partial_\mu \chi \ , \tag{6.4}$$

where χ is a complex field with a phase proportional to η and with $|\chi| = 1$. A constant density is represented by an oscillatory phase in $\chi \propto e^{i\omega t}$.

A defect or vortex is present at a point x_0 if the circulation $\oint dx_i \partial_i \eta$ around that point is nonzero and quantized. At the core of the vortex, $|\chi|$ vanishes.

Since the total number of fractional-statistics particles is given by $N = (1/4\theta)\int d^2x\, \epsilon_{ij} f_{ij}$, adding a fractional-statistics particle to the system is equivalent to adding a 2θ flux:

$$\begin{aligned} 2\theta &= \delta\int d^2x\, f_{12} \\ &= \delta\oint d\mathbf{x}\cdot\mathbf{a} \ . \end{aligned} \tag{6.5}$$

In other words, the Lagrangian in (5.6) describes compact "electrodynamics."

As is well known, this theory contains instantons,[32] namely the hedgehog in Euclidean 3 space. The hedgehog density $h(x)$ is given by

$$h(x) = e^{\mu\nu\lambda}\partial_\mu f_{\nu\lambda} = \partial^2\eta \ . \tag{6.6}$$

Denote the phenomenological field that creates and annihilates vortices by the complex field $\Phi(x)$. We now show that Φ represents an "electric" charge with respect to $f_{\mu\nu}$. Assuming that Φ is "electric," we can write the effective Lagrangian, by gauge invariance, as

$$L=\int d^2x\left[\frac{1}{16\pi g^2}f_{\mu\nu}^2+\tfrac{1}{2}|(\partial_t+i\tilde{q}a_0)\Phi|^2-\tfrac{1}{2}v_V^2|(\partial_i+i\tilde{q}a_i)\Phi|^2-\tfrac{1}{2}\Delta_V^2|\Phi|^2\right]. \tag{6.7}$$

The Φ particles correspond to the vortices in the superfluid. To see this we notice that a density of the Φ particle $i\Phi^*\overleftrightarrow{\partial}_t\Phi$ generates an "electric" field of the U(1) gauge field:

$$f_{0i}(x)=2\tilde{q}g^2\frac{x^i}{x^2}Q_\Phi\ , \tag{6.8}$$

where $Q_\Phi=\int i\Phi^*\overleftrightarrow{\partial}_t\Phi d^2x$ is the number of Φ particles. The superfluid velocity for such a configuration is given by

$$v^i=\frac{\epsilon_{ij}f_{0j}}{2n\theta}=\frac{\tilde{q}g^2}{n\theta}\frac{\epsilon_{ij}x^j}{x^2}Q_\Phi\ . \tag{6.9}$$

This corresponds to a vortex with circulation $(2\tilde{q}\theta\hbar/m)Q_\Phi$, after we restore a factor of c_s^2. Since the circulation is quantized in unit $\hbar/m$, $\tilde{q}$ must equal to $1/2\theta$. Because of the finite amount of circulation, to create a vortex (or a Φ particle) cost an energy that diverges logarithmically with the size of the system. Due to the finite-energy gap Δ_V for creating vortex-antivortex pair, the vortex excitations have little effect on the low-energy superfluid properties, but as we will see further, they have important effects at short distances.

The gap parameter Δ_V may become negative as various parameters in the theory are changed. The system may thus enter into a Higgs phase in which the statistical gauge potential acquires a mass and becomes short ranged. This would then a represent a statistics-changing phase transition.[23]

We note that in the presence of Φ, $\partial^\mu f_{\mu\nu}$ no longer vanishes and so the field η cannot be defined [cf. (6.1)]. Thus, a vortex is present where η is not defined, as it should be. The vortex field Φ is dual to the χ field in the context discussed in Refs. 26 and 25.

VII. FINITE TEMPERATURE

Thus far, the discussion has been for zero temperature. At finite temperature, we expect a normal fluid component to form. Microscopically, configurations such as those shown in Fig. 7 would be excited by thermal fluctuations. The energy of excitation is of order $\hbar\omega_c\sim n/m$. Thus, we expect the critical temperature to be

$$T_c\sim n/m\ . \tag{7.1}$$

Let us look at this in more detail in order to develop a two-fluid picture in the spirit of London and Tisza. We first identify the superfluid as the state described earlier, i.e., the state in which each Landau orbit contains q fractional-statistics particles. The low-frequency excitation of the superfluid (phonon) corresponds to the breathing mode of the Landau orbit. We now construct the high-energy excitation, i.e., the normal fluid component,

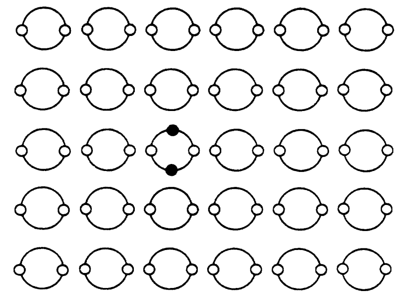

FIG. 7. Normal fluid component is created by adding two particles to a Landau orbit ($\theta=\pi/2$).

by adding q particles to the Landau orbit ψ_0 at the origin (Fig. 7). The added particles behave like a 2π flux tube to the particles far away from the origin. Therefore the q added particles have no effect on the wave function far away from the origin. As a direct consequence, the added particles have a finite energy. Certainly we can add the q particles to any Landau orbit we like. The added particles behave like a quasiparticle moving on the background of the superfluid and carrying finite entropy. Thus we may identify the added particles as the normal fluid component.

Another way to construct the normal fluid component is to use Laughlin's fluxoid argument. We fix the gauge field f_{12} to be constant, and pierce the space at the origin by an infinitely thin fluxoid. Then we turn on a 2π flux through the fluxoid slowly. The initial uniform superfluid state evolves adiabatically into a new state. Due to the "Hall" effect, q particles are accumulated near the fluxoid after the 2π flux has been turned on. These additional q particles are identified as an element of the normal fluid. The quasiparticle that we constructed earlier can be regarded as the roton in the superfluid.

The energy of the roton can be estimated by substituting the density function with the added particles into (4.4). We find the energy can be written as $E_{\rm rot}=\eta\pi n_s\hbar^2/m$, where n_s is the superfluid density and η is the numerical constant of order 1, which is independent of q. (Actually $\eta=3$ from this crude calculation.)

The free energy of the two-fluid model can be roughly written as

$$f=\frac{\pi\hbar^2}{qm}(n_s^2+\eta n_s n_n)-\frac{1}{q}Tn_n\ln\frac{n}{n_n} \tag{7.2}$$

if the normal fluid density $n_n\ll n$. The entropy in (7.2) is calculated from the roton (the q-particle bound state) gas. As the temperature increases, more and more rotons are excited and the superfluid density decreases. At a critical temperature T_c, the superfluid density vanishes.

From (7.2) we find that near $T=0$ the normal fluid density is

$$n_n=ne^{-(\eta-2)\pi n\hbar^2/mT-1}\ . \tag{7.3}$$

The gap for the normal fluid excitation can be read off from (7.3):

$$\Delta_n = \pi(\eta-2)\frac{n\hbar^2}{m} \ . \tag{7.4}$$

Near $T_c(n_s \ll n)$ the physical picture is quite different. The situation becomes very complicated and the roton picture breaks down because of overlapping between rotons. Instead, we develop a simple picture based on a mean-field approach for the normal fluid. The free energy is essentially determined by the normal fluid. Due to the frustrations introduced by the statistics, the normal fluid naively behaves like a gas of bosonic particles in the "magnetic" field with $\hbar\omega_c = 2\pi n\hbar^2/qm$. The free energy of such a boson gas can be easily calculated. There is no superfluid phase for this system because the first Landau level is highly degenerate (N/q fold degeneracy). In this simple picture, to obtain superfluidity we require that there is a nondegenerate single-body energy level lying below the first Landau level. The bosons in the ground state are identified as superfluid, and the bosons in the excited state are normal fluid. The energy difference between the ground state and the first Landau level corresponds to the condensation energy of the superfluid. The condensation energy of the superfluid is difficult to calculate. However it is reasonable to assume that the condensation energy is a fraction of the energy of the first Landau level $(\zeta\pi\hbar^2/qm)n$ (where $0 \lesssim \zeta \lesssim 1$). Thus the thermodynamic potential Ω may take a form (for $n_s \ll n$)

$$\Omega = T\ln(1-e^{[\mu+(\zeta/2)\hbar\omega_c]/T}) + T\frac{N}{q}\sum_{l=0}^{\infty}\ln(1-e^{(\mu-l\hbar\omega_c)/T}) \ . \tag{7.5}$$

The chemical potential is determined by requiring $N = -\partial\Omega/\partial\mu$. At the critical temperature $\mu + (\zeta/2)\hbar\omega_c = 0$. We obtain

$$q = \sum_{l=0}^{\infty}\frac{1}{e^{[l+(1/2)\zeta]\hbar\omega_c/T_c}-1} \ . \tag{7.6}$$

If ζ is independent of q, $T_c \to 1/\ln q$ as $q \to \infty (\theta \to 0)$, which is a reasonable result since T_c for the free boson gas vanishes. If ζ is small ($\zeta \lesssim \ln[1+1/q)]$ we get

$$T_c = \frac{\zeta\pi n\hbar^2}{qm\ln(1/q+1)} \ . \tag{7.7}$$

We find that (for small ζ)

$$\frac{\Delta_n}{T_c} = \frac{\eta-2}{\zeta} q \ln\left[\frac{1}{q}+1\right] \ . \tag{7.8}$$

If we take $\eta=3$, $\zeta=0.25$, and $q=2$ (semion), we find that $\Delta_n/T_c = 3.2$. For La-Cu-O with 10% doping, $T_c \simeq 200 m_e/m$ K where m_e denotes the mass of the electron. We see that the fractional-statistics-particle superconductor behaves very much like the conventional s-wave BCS superconductor as far as the thermodynamics properties are concerned.

VIII. TIME REVERSAL AND PARITY VIOLATION

As remarked earlier, because fractional-statistics-particle gas breaks time-reversal symmetry and parity, the superfluid should show some T and P breaking effects at short distances. Since the vortex is sensitive to short-distance physics, we can include the T and P breaking effect of the fractional-statistics-particle by including a coupling between the superfluid current and the vortex current in our theory. The effective Lagrangian (6.9) becomes

$$L = \int d^2x \left[\frac{1}{16\pi g^2}f_{\mu\nu}^2 + \left|\left[\partial_t + ia_0 + i\frac{\lambda}{\omega_c}\epsilon^{0\mu\nu}f_{\mu\nu}\right]\Phi\right|^2 + \left|\left[\partial_i + ia_i + i\frac{\lambda}{\omega_c}\epsilon^{i\mu\nu}f_{\mu\nu}\right]\Phi\right|^2 - \Delta_V^2|\Phi|^2\right] \ . \tag{8.1}$$

We have set $v_V = c_s = 1$ here for convenience. The possibility of including the λ coupling was pointed out in Ref. 34. Because the coupling between the superfluid current $j^\mu = \epsilon^{\mu\nu\lambda}f_{\mu\lambda}$ and the vortex current $\Phi^* D^\mu \Phi$, the superfluid density is different near the core of the vortex and the antivortex. Thus the vortex and the antivortex have different energies. If the fractional-statistics particles are charged as they are in high-temperature superconductors, this means that the lower critical field H_{c1} is different for the magnetic field with opposite orientation.

Integrating out the vortex field Φ in (8.1), we obtain the following low-energy effective Lagrangian of the superfluid

$$L = \int d^2x \left[\frac{1}{16\pi g^2}f_{\mu\nu}^2 + \frac{1}{48\pi\Delta_V}\tilde{f}_{\mu\nu}^2\right], \quad \tilde{f}_{\mu\nu} = f_{\mu\nu} + \frac{\lambda}{\omega_c}(\partial_\mu\epsilon_{\nu\rho\sigma}f_{\rho\sigma} - \partial_\nu\epsilon_{\mu\rho\sigma}f_{\rho\sigma}) \ . \tag{8.2}$$

The Lagrangian contains a T and P breaking term of the form

$$\epsilon_{\mu\nu\lambda}a_\mu\partial^2\partial_\nu a_\lambda \ . \tag{8.3}$$

As is consistent with our earlier discussion, this is not the Chern-Simons term but part of a higher derivative Chern-Simons term. Thus, in addition to the term giving rise to the Meissner effect in (5.8), the superfluid current-current correlation also contains a T and P breaking term

$$\langle j_\mu(k) j_\nu(-k)\rangle = \frac{\pi}{\theta^2(g^{-2}+\frac{1}{3}\Delta_V^{-1})}\left[g_{\mu\nu} + \frac{2\lambda}{\omega_c(3g^{-2}+\Delta_V^{-1})\Delta_V}\epsilon_{\mu\nu\rho}k_\rho\right] + \cdots \ . \tag{8.4}$$

This T and P breaking term leads to T and P violation in the propagation of light in the medium. We pointed out in Ref. 11 that when polarized light is reflected from such a T and P breaking superconductor the polarization is rotated. The rotation angle is given here by

$$\varphi=\frac{2\lambda}{(1+3\Delta_V/g^2)}\frac{\omega_p^2\omega^2}{(\omega_p^2-\omega^2)^{3/2}}\ , \tag{8.5}$$

where ω_p is the plasma frequency. We see that (8.5) and the T and P breaking Ginzburg-Landau theory discussed in Ref. 11 give the same physical results. Thus in some sense (8.2) can be regarded as the dual theory of the T and P breaking Ginzburg-Landau theory. The calculation given here is meant to be illustrative. We have arbitrarily set the vortex velocity to be equal to the phonon velocity. Also, in principle, there could be two different λ couplings in (8.1).

We note from (8.5) that the rotation φ vanishes as the vortex gap Δ_V goes to infinity. This restates the fact that in this treatment T and P violation comes from the short-distance physics introduced effectively by the vortex field.

In Ref. 11, the role of the vortex field is effectively played by the spinon pairing order parameter, denoted there by ϕ_{SS}. The Landau-Ginzburg theory given there cannot be compared in a one-to-one fashion with the more microscopic theory given here. However, the underlying physics is similar in spirit. As the gap Δ_{SS} for the spinon field ϕ_{SS} goes to infinity, or equivalently as the condensate value for ϕ_{SS} (denoted by v in Ref. 11) goes to zero, we see from Eq. (8) in Ref. 11 that the T and P violation, and hence the rotation of polarization φ disappear. This consistent with the treatment given here.

Halperin[35] and Wilczek[36] have pointed out to us that since an electric field constant in space couples only to the center of mass of the fractional-statistics-particle gas, the rotation of polarization we discussed in Ref. 11 and in this paper should vanish to leading order in frequency for the ideal fractional-statistics-particle gas described in Eq. (1.1), in which the charge density ρ_e and the mass density ρ_m are strictly tied together.

We expect the holon fluid in the T and P breaking spin liquid state not to be ideal. In particular, as described in our previous work,[9] the holons interact with the (electromagnetically) neutral spinons via the statistical gauge potential a_μ. The statistical gauge potential would acquire in general a Maxwell term

$$\frac{1}{16\pi g_s^2}f_{\mu\nu}^2\ , \tag{8.6}$$

relevant only at short distances. We expect g_s^2 to be proportional to the spinon gap Δ_{SS}. As a result of this Maxwell term, the holons are dressed by an electrically neutral cloud of statistical gauge quanta and spinons.

At energy scales near the spinon gap Δ_{SS} or the statistical gauge field gap $1/g_s^2$, the charge density ρ_e and the mass density ρ_m are no longer tied together and a constant electric field no longer couples exclusively to the center of the mass of the holon field. In our discussion here, the deviation from an ideal gas is described phenomenologically by the parameter λ. Since the spinon gap and the gauge field gap are about the same order as the superconducting gap (they are all order a few hundred Kelvin), λ is expected of order 0(1).

We may remark in passing here that T and P violation and superconductivity are logically distinct. For instance, in this paper, the normal phase is described by a gas of fractional-statistics particles and T and P are certainly violated. The calculation of φ in our Landau-Ginzburg approach[11] breaks down for ω larger than the gap, but this does not imply that the rotation disappears at high ω. We expect that the polarization of light reflected off material to be rotated even in the normal, i.e., nonsuperconducting, phase.

IX. SHORT-DISTANCE PHYSICS

From the discussion in Sec. II, we see that the superfluidity and the compressibility of quantum fluid are closely related. We believe that existence of a gapless phonon implies superfluidity. Mathematically, the compressibility of the fractional-statistical-particle fluid is reflected by the fact that the U(1) gauge field in (5.6) is massless. Our discussion about the superfluidity of the fractional-statistical-particle fluid relies crucially on the masslessness of these gauge fields.

The effective Lagrangian in (5.6) is only a leading order approximation of the full effective theory of the fractional-statistics-particle fluid. A more complete effective Lagrangian should contain higher- and higher-dimensional interaction terms, such as $f_{\mu\nu}^4$, interaction of the vortex field Φ, etc. It would not be obviously meaningful to calculate the quantitative effects of these higher-dimensional terms since in going to the fractional-statistical-particle gas we have already averaged over the short-distance lattice scale physics of the actual solid-state system. However, we do have to argue that these high-energy effects cannot change our conclusions qualitatively.

In particular, an important question would be whether the U(1) phonon field a_μ remains massless after we turn on these short-distance interactions. If the U(1) field would obtain a mass after we turn on arbitrarily small interactions, this would mean the superfluidity and the compressibility discussed in this paper for the fractional-statistical-particle fluid may just be an artifact of the mean-field approximation, and these properties may have nothing to do with the real fractional-statistical-particle fluid. However, due to gauge symmetry, the U(1) field can never obtain a mass in perturbation theory if the interaction respects T and P. There are only three possible ways for a_μ to obtain a mass term and we will discuss them in turn.

(i) Through the Higgs mechanism: this may happen, e.g., if Δ_V^2 in (6.9) become negative. In this case creation of the vortex-antivortex pairs lowers the energy of the system. As a consequence the vacuum becomes a superfluid of the vortex. The density fluctuation of fractional-statistics particles acquire an energy gap, or in other words, the fractional-statistical-particles fluid becomes incompressible.[26] If the fractional-statistical particles carry charge, such a vortex condensed state is an insulator. However, if we start with positive Δ_V^2, interactions can change sign of Δ_V^2 only when the interactions are strong enough. Weak interactions cannot give a mass to a_μ.

(ii) Through the Polyakov instantons: a Polyakov instanton is a monopole, or more precisely, hedgehog of the U(1) field if we view the (2+1)-dimensional space-time as a three-dimensional space. After an instanton event, the total number of the U(1) flux is changed

$$\frac{1}{4\theta}\left[\int_{t=+\infty} d^2x\, \epsilon^{ij} f_{ij} - \int_{t=-\infty} d^2x\, \epsilon^{ij} f_{ij}\right] = q \ . \tag{9.1}$$

Our normalization corresponds to unit monopole charge. Polyakov[32] showed that the instantons may induce a mass term for the U(1) field [and the U(1) gauge field becomes linearly confining). However, in our two-fluid model, (9.1) implies that the total number of the particles in the superfluid is changed by q. Because the conservation of the number of particles, some normal fluid must be created by the instanton. Since the normal fluid excitations has a finite-energy gap, the instantons are confined in space-time. For example, for a pair of instanton and anti-instanton, (Fig. 8) the action is given by

$$e^{-S_{\text{pair}}} \lesssim e^{-\Delta_n t_0} \ , \tag{9.2}$$

where t_0 is the time separation of the two instantons and Δ_n is the gap for the normal fluid. Because the instanton is linearly confined, it cannot generate any mass for the U(1) field.

(iii) Through the Chern-Simons term: the U(1) field may obtain a topological mass from the Chern-Simons term $\epsilon_{\mu\nu\lambda} a_\mu f_{\nu\lambda}$. In the leading order approximation our effective theory for the fractional-statistical-particle gas does not contain the Chern-Simons term. However, this does not imply the superfluid state of the fractional-statistical-particle fluid respects the T and$P$ symmetries. The T and P breaking effects of the fractional-statistical-particle fluid may appear in the higher-order interaction terms in the effective theory. Now the crucial question is whether the radiative corrections from those T and P breaking interaction terms may generate the Chern-Simons term. To our knowledge, there is no general

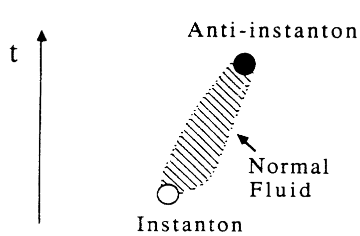

FIG. 8. A pair of the instantons. Normal fluid is created by instanton and annihilated by anti-instanton.

proof that the Chern-Simons term does not receive any radiative corrections in a T and P breaking theory. However, the experience from many specific T and P breaking models strongly suggests that the Chern-Simons term does not receive any perturbative corrections.[37–40] The Chern-Simons term may be changed only when the interactions are strong enough to drive the system across a critical point and into a new phase.[40]

We would like to mention that in the presence of the Chern-Simons term the fractional-statistical-particle fluid is an incompressible quantum Hall fluid because the gauge potential a acquires a topological mass. The quantum Hall conductance is determined by the coefficient in front of the Chern-Simons term. A (true) magnetic field is generated in the system.[11] In the presence of a magnetic field the quantum Hall state of the fractional-statistical-particle fluid (including fermion fluid and boson fluid) is described by the following effective theory

$$\mathcal{L} = \frac{1}{16\pi g^2} f_{\mu\nu}^2 + \alpha \epsilon_{\mu\nu\lambda} a_\mu f_{\nu\lambda} + \frac{e}{4\theta} \epsilon_{\mu\nu\lambda} A_\mu f_{\nu\lambda} \ , \tag{9.3}$$

which can be thought of as the dual form of the effective theory studied in Ref. 18. By varying a_μ we find that the magnetic field $B \equiv \epsilon_{ij} \partial A_j$ is proportional to the fractional-statistical-particle density.

In the preceding discussion we argue that the superfluid phase of the fractional-statistical-particle fluid is a stable phase in the sense that superfluidity persists for any small perturbations of the Hamiltonian. The superfluid phase exists in a finite region in the parameter space. When we turn on a strong enough interaction, the superfluid state may change into new states. We have discussed two possible phases connected to the superfluid phase. One is the incompressible fluid or insulating state. Another is the quantum Hall state. The third possible phase connecting to the superfluid phase is the crystal phase. The fractional statistical-particle fluid may crystalize in low-density limit if there is a repulsive interaction, e.g., Coulomb interaction, between fractional-statistical particles.

X. FERMION PICTURE

In our discussion we studied the low-temperature properties of fractional-statistical-particle fluid with $\theta = \pm\pi/q$, by regarding the fractional-statistical-particle as a bound state of a boson and a $\pm\pi/q$ flux tube. We may use a similar method to study fractional-statistical-particle fluid with $\theta = \pi \pm \pi/q$ by treating the fractional-statistical particle bound state of a fermion and a $\pm\pi/q$ flux tube. In the mean-field approach,[10,15,20] the superfluid component is identified as the particles in the lowest q filled Landau levels. The normal fluid component is identified as the fermion excitations in the $(q+1)$th Landau level or higher levels. The low-lying excitations of the superfluid still correspond to the breathing modes of the Landau orbits, and is expected to have a gapless linear dispersion.

Notice that once again it is crucial to have the extended mean-field approximation. If the mean field is regarded as fixed and static, then one would conclude erroneously that the introduction of an excitation requires a finite amount of energy corresponding to the gap between the qth and the $(q+1)$th level. In fact, the introduction of particles also increases the magnetic field and allows more particles to be accommodated in each level in precisely such a way that we do not have to cross the gap between the qth and the $(q+1)$th level.

Interestingly enough, the description of the origin of the gapless excitation is quite different in the fermion picture than in the boson picture. In the boson picture, as explained in Sec. IV, the extended mean-field approximation is necessary to lift the degeneracy of the ground state.

XI. CONCLUSION

In conclusion, we believe that we understand the behavior of the fractional-statistics liquid at low temperature and high density. The salient feature is its superfluidity. Using a dual construction, we showed that the charged fractional-statistics liquid exhibits a Meissner effect and we calculated the London penetration depth. The finite temperature behavior is described by a two-fluid picture. Experimentally, $2\Delta/T_c$ is found to range from 3 to 10, which is consistent with our picture. (The 2Δ used conventionally in the superconductivity literature corresponds to Δ_n here.)

Physically, we believe that the low-temperature fractional-statistics liquid resembles both the free fermion gas and the boson gas in some respects, but yet is unlike either of these two gases. The frustration experienced by the wave function leads to an exchange pressure due to quantum statistics, just as is the case for the fermion gas. In particular, the energy density ε is proportional to the number density n squared: $\varepsilon \simeq (\hbar^2\theta/m)n^2$, for small θ. The n^2 dependence is characteristic for a Fermi liquid, for which $\epsilon = (\pi\hbar^2/m)n^2$. But yet the fractional-statistics liquid is unlike the fermion gas in that there is no Fermi surface and no low-energy particle-hole excitation across the Fermi surface. The fractional-statistics liquid exhibits behavior reminiscent of Bose-Einstein condensation. In contrast to the free Bose gas however, the only-low-lying excitation does not disperse quadratically but rather linearly. This behavior is reminiscent of the hard-core Bose gas, but at the same time the linear dispersion may be attributed to the Fermilike exchange pressure.

Upon first hearing of the presence of a gapless excitation, one might immediately ask what continuous symmetry is broken. In our low-energy long-wavelength field theoretic description (5.6) of the fractional-statistics liquid, the broken continuous symmetry is hidden. The gapless phonon excitation is guaranteed by gauge invariance. In the point particle description however, the phonon results from broken global U(1) invariance related to the conservation of the particle numbers. The "off-diagonal long-range order" here may be expressed schematically in a second quantized formalism in which $\varphi(x)$ is a field for creating a semion. Then we expect the "effective pairing operator"

$$O(\omega)=\int d^2x_1\, d^2x_2 f(\omega;x_1,x_2)\varphi(x_1)\varphi(x_2)$$

to have a nonzero ground-state expectation value. (Here the bound-state wave function $f(\omega;x_1,x_2)$ acquires a phase $e^{i\pi/2}$ upon interchanging x_1 and x_2.) This is just a formal way of saying semions pair into bosons.

Our treatment depends crucially on an extended mean-field approximation, which goes beyond the mean-field approximation in that the particles and the mean "magnetic" field the particles move in are allowed to influence each other dynamically.

We also studied the possible T and P breaking effect in the superfluid state of fractional-statistical particles. The T and P breaking effects are found to vanish in the limit $k=0$ and $\omega=0$. This is consistent with the picture that the superfluidity is due to semion pair condensation and that the semion pair behaves as a boson with no long-distance T and P breaking effect. However, we do find some T and P breaking effects for nonzero k and ω. In this case we are probing the short-distance properties inside the semion pair, and the T and P breaking effects are expected to appear because fractional-statistics breaks T and P.

In this paper, we have studied the physical picture of the fractional-statistics liquid in some detail. However, our discussion is only semi-quantitative, and remains to be done in deriving a more quantitative theory from the microscopic Hamiltonian.

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APPENDIX

Following Laughlin's lead in developing his theory of the fractional Hall effect, we may be tempted to write down the wave function

$$\psi(z_1,\ldots,z_N)=\prod_{i<j}(z_i-z_j)^{\theta/\pi} . \tag{A1}$$

This wave function is in fact formally an eigenstate of the Hamiltonian

$$H'=\sum_i \frac{p_i^2}{2m}=-\frac{1}{2m}\sum_i \frac{\partial^2}{\partial z_i^*\partial z_i} .$$

However, it has an enormous angular momentum $N(N-1)\theta/2\pi$ (as we can see readily by letting $z_i \to e^{i\alpha}z_i$). To reduce the angular momentum we can introduce $\tilde{N}$ "effective particles" located at $\omega_1,\ldots,\omega_{\tilde{N}}$. Consider the wave function

$$\psi(w,z)=\phi(w)\prod_{ia}(z_i-w_a)^{-1}\psi(z) , \tag{A2}$$

where $\psi(z)$ is the $\psi(z_1,\ldots,z_n)$ of (3.1) and where

$$\phi(w)=\prod_{a<b}(w_a-w_b)^{\pi/\theta}\ . \tag{A3}$$

The angular momentum of the system is then

$$L=\frac{N(N-1)\theta}{2\pi}-N\tilde{N}+\tilde{N}(\tilde{N}-1)\pi/2\theta\ . \tag{A4}$$

For a suitable choice of $\tilde{N}$ we can reduce the angular momentum L from $O(N^2)$ to $O(N)$ (or even zero). We find to leading order $\tilde{N}=\theta N/\pi$. In particular, for half fermions, we have $\tilde{N}=N/2$ and we see from the form of $\phi(\omega)$ that these "w particles" behave like bosons upon interchange. This suggests that somehow we may be able to trade the degrees of freedom in z_i for the degrees of freedom in w_a.

This discussion is however not quite correct since the system would not have a uniform density unless it is contained in a finite region. In other words, we lose Laughlin's plasma analogy which guarantees the uniform density of the Hall gas in his theory. In other words, the wave function in (A1), for instance, is not normalizable without some "magnetic exponential" of the form $\exp(-\xi\sum_l|z_l|^2)$.

The singularity contained in the factor $(z_i-w_a)^{-1}$ in (A2) should be regularized, say to have the form

$$(z-w)^{-1}\rightarrow\frac{(z-w)^*}{|z-w|^2+l^2}\ . \tag{A5}$$

The parameter l is a measure of the size of the pairs. We expect that to minimize the energy l would have to increase, to something of the order of the interparticle separation $n^{-1/2}$.

One possible interpretation of the wave function is to treat the ω's to be parameters to be integrated over. Thus, the N-particle wave function of the Hamiltonian in (1.4) may have the form

$$\Psi(z)=\int d\omega_1,\ldots,d\omega_{\tilde{N}}\,\psi(\omega,z)\ ,$$

with $\phi(\omega)$ and $\psi(\omega,z)$ as given here but with $\psi(z)$ possibly different from (A.1). For half fermions ($\theta=\pi/2$) and $\tilde{N}=N/2$, we can carry out the ω integration by observing that the result of the integration should have a dimension like $z^{-N^2/2}$ and be totally symmetric upon interchange of any pairs of z's. We find that Ψ can only be

$$\Psi(z)=\left[\sum_P\sum_{\substack{i<j\\ i,j\in P}}\frac{1}{(z_i-z_j)^2}\right]\psi(z)\ .$$

Here P denotes a partition of the N z_i's into two equal sets each with $N/2$ z's. The summation is over all possible partitions. We feel that $\Psi(z)$ may offer a starting point for a variational calculation.

[1] D. Arovas, J. R. Schrieffer, F. Wilczek, and A. Zee, Nucl. Phys. **B251**, 117 (1985).
[2] F. Wilczek, Phys. Rev. Lett. **49**, 957 (1982).
[3] J. M. Leinaas and J. Myrheim, Nuovo Cimento **37B**, 1 (1977).
[4] F. Wilczek and A. Zee, Phys. Rev. Lett. **51**, 2250 (1983).
[5] Y. S. Wu and A. Zee, Phys. Lett. **147B**, 325 (1984).
[6] V. Kalmeyer and R. Laughlin, Phys. Rev. Lett. **59**, 2095 (1987).
[7] X. G. Wen and A. Zee, Phys. Rev. Lett. **63**, 461 (1989).
[8] X. G. Wen and A. Zee, Nucl. Phys. B (to be published).
[9] X. G. Wen, F. Wilczek, and A. Zee, Phys. Rev. B **39**, 11 413 (1989).
[10] R. Laughlin, Science **242**, 525 (1988).
[11] X. G. Wen and A. Zee, Phys. Rev. Lett. **62**, 2873 (1989).
[12] Y. S. Wu, Phys. Rev. Lett. **52**, 111 (1984); **52**, 2102 (1984).
[13] J. March-Russell and F. Wilczek, Phys. Rev. Lett. **61**, 2066 (1988).
[14] A. Comtet, Y. Georgelin, and S. Ouvry (unpublished).
[15] R. Laughlin, Phys. Rev. Lett. **60**, 2677 (1988).
[16] S. Girvin and A. MacDonald, Phys. Rev. Lett. **56**, 1252 (1982); R. E. Prange and S. M. Girvin, *The Quantum Hall Effect* (Springer-Verlag, Berlin, 1986), Chap. 10.
[17] E. H. Rezayi and F. D. H. Haldane, Phys. Rev. Lett. **61**, 1985 (1988).
[18] S. C. Zhang, T. H. Hansson, and S. Kivelson, Phys. Rev. Lett. **62**, 82 (1989).
[19] N. Read, Phys. Rev. Lett. **62**, 86 (1989).
[20] A Fetter, C. Hanna, and R. Laughlin (unpublished).
[21] L. D. Landau, J. Phys. (Moscow) **XI**, 91 (1947).
[22] R. P. Feynman, *Statistical Mechanics* (Benjamin, New York, 1972).
[23] I. M. Khalatnikov, *An Introduction to the Theory of Superfluidity* (Benjamin, New York, 1965).
[24] N. N. Bogoliubov, J. Phys. (Moscow) **11**, 23 (1947).
[25] X. G. Wen and A. Zee, Phys. Rev. Lett. **62**, 1937 (1989).
[26] M. Fisher and D. H. Lee (unpublished).
[27] N. Byers and C. N. Yang, Phys. Rev. Lett. **7**, 46 (1961).
[28] C. N. Yang, Rev. Mod. Phys. **34**, 694 (1962).
[29] O. Penrose, Philos. Mag. **42**, 1373 (1951); O. Penrose and L. Onsager, Phys. Rev. **104**, 576 (1956).
[30] J. Schwinger, Phys. Rev. Lett. **3**, 296 (1959); S. L. Adler and R. F. Dashen, *Current Algebras* (Benjamin, New York, 1968), pp. 136 and 296.
[31] G. D. Mahan, *Many Body Physics* (Plenum, New York, 1981).
[32] A. M. Polyakov, Nucl. Phys. **B120**, 429 (1977).
[33] X. G. Wen and A. Zee, J. Phys. (Paris) **50**, 1623 (1989).
[34] S. K. Paul and A. Khare, Phys. Lett. B **174**, 420 (1986); **193**, 253 (1987).
[35] B. Halperin (private communication).
[36] F. Wilczek (private communication).
[37] S. Coleman and B. Hill, Phys. Lett. **159B**, 184 (1985).
[38] M. Bernstein and T. J. Lee, Phys. Rev. D **32**, 1020 (1985).
[39] Y. Kao and M. Suzuki, Phys. Rev. D **31**, 2137 (1985).
[40] G. W. Semenoff, P. Sodano, and Y. S. Wu, Phys. Rev. Lett. **62**, 715 (1989).