Replica symmetry breaking in the spin-glass model on lattices with finite connectivity: Application to graph partitioning

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A systematic way to construct replica symmetry-breaking solutions of the spin glass on random lattices with finite (fixed or average) connectivity is presented. The method generalizes Parisi's scheme to the case of infinitely many-order parameters $q_{\alpha\beta}, q_{\alpha\beta\gamma\delta}, \ldots$. A systematic expansion in inverse powers of the connectivity ($=M+1$) is performed. At finite temperatures the expansion is in powers of $1/M$, and at zero temperature in powers of $1/\sqrt{M}$. The q's with larger number of indices contribute at higher orders in the expansion parameter. At zero temperature the results apply to the graph bipartitioning problem and are compared with numerical simulation. The agreement is of the order of $\sim 1\%$, for the range $9 \le M \le 20$, much closer than the replica symmetric solution.

I. INTRODUCTION

The theory of spin glasses on lattices with finite connectivity attracts much current interest. Since Parisi[1] proposed his solution to the Sherrington-Kirkpatrick[2] (SK) infinite-ranged model there have been many attempts to extend the theory to short-ranged systems.[3] In the SK model each site is connected to any other site whereas Bravais lattices of real systems have finite connectivity. An important question which is still not entirely settled is whether real spin glasses have many coexisting thermodynamic states as is the case for the infinite-ranged model.

Besides, spin glasses on random lattices with finite connectivity are related to some well known optimization problems like graph partitioning[4-8] and coloring.[7] Such random lattices are characterized by many of the simplifying features of mean field theory because small loops are rare.[5,8] If two points A and B are both directly connected to a point C by bonds, then the probability that A and B are directly connected to each other is $O(1/N)$, where N is the number of lattice sites.

Previous treatments of such models,[5,6] except in the vicinity of T_c (Refs. 9 and 10) and for special limiting cases,[11,12] have used the assumption of a single thermodynamic state, or in a more technical language assumed replica symmetry (RS) of the order parameters. But evidence has been accumulating[9,10,13] that in many of the systems under consideration RS has to be broken, and the problem we want to address is how to construct a broken replica scheme at any temperature, including $T=0$, where the connection between the ground-state energy of the frustrated system on the random lattices and the cost function of the optimization problems have been established.[5]

The physical meaning of replica symmetry breaking in these systems is the coexistence of many thermodynamics (Gibbs) states in the spin-glass phase at any temperature T, which are organized in a tree-like (ultrametric) structure.[3] Let us remind the reader that the Ising spin-glass Hamiltonian is given by

$$\mathcal{H} = -\sum_{(ij)} J_{ij}\sigma_i\sigma_j - h_{\text{ext}}\sum_i \sigma_i \ , \tag{1.1}$$

where $\sigma_i = \pm 1$, J_{ij} are random interactions between a pair of spins (ij) on the lattice, and h_{ext} is the constant external field which will be put to zero in most of the discussion. The replica trick[3] amounts to replicating the spin variables σ_a, $\alpha=1,\ldots,n$ and using the identity

$$\overline{\ln Z} = \lim_{n\to 0}\frac{\overline{Z^n}-1}{n} = \lim_{n\to 0}\frac{1}{n}\ln\overline{Z^n} \tag{1.2}$$

to carry out the quenched average over the disorder. Here Z is the partition function, Z^n is the partition function involving the replicated spins, and the bar stands for average over the disorder.

The order parameter of the Ising spin glass (SG) in the infinite-range case is denoted by $q_{\alpha_1\alpha_2}$ and Parisi[1] has shown how to construct an RS-breaking scheme by appropriately parametrizing the $n\times n$ matrix $q_{\alpha_1\alpha_2}$ as $n\to 0$. In the finite connectivity case the system is characterized by infinitely many-order parameters $q_{\alpha_1\alpha_2}, q_{\alpha_1\alpha_2\alpha_3\alpha_4}, q_{\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6}, \ldots$, and the problem is how to construct an RS-breaking scheme including all of them. These order parameters constitute a measure for the averaged joint overlap of several thermodynamic states labeled by $\alpha_1,\alpha_2,\alpha_3,\ldots$, etc, similar to the interpretation of $q_{\alpha\beta}$ in the infinite-ranged model.[3] In order to construct a systematic scheme of RS breaking involving all these order parameters we developed the method of $1/M$ expansion where $M+1$ is the connectivity of the lattice (number of nearest neighbors). Thus we expand about the infinite-ranged model, and q's with higher number of indices enter at higher and higher order in the $1/M$ expansion. In fact at $T=0$ the expansion parameter turns out to be $1/\sqrt{M}$, at least for finite-stage RS breaking.[14] Thus the $1/M$ expansion at finite temperature diverges as $T\to 0$. Thus we have two separate expansions—one at finite and one at zero temperature. Nevertheless the expansion at finite temperature is rather well behaved up to a temperature of order 0.1–0.2 of T_c

and the free energy can be extrapolated to $T=0$ to agree well with the results obtained directly at $T=0$. A short version of this work has been already submitted for publication.[15] Here we give a more detailed and complete exposition of the results.

Our results at zero temperature for the graph bipartitioning problem on a lattice with fixed connectivity approach the results of numerical simulations[8] to within $\sim 1\%$ which is much closer than previous estimates which assumed RS.[5,6] This is for the range $9 \le M \le 20$ where higher order corrections in the $1/M$ expansion are expected to be small. The remaining error may be due to three sources: higher order $1/M$ corrections, corrections from the fact that we considered first stage RS breaking versus infinite RS breaking, and errors in the simulations themselves that tend to overestimate the cost.

Two kinds of lattices can be treated by our method:
(i) lattices with an average finite connectivity,[9] in which the bond distribution is given by

$$P(J)=\left[1-\frac{\alpha}{N}\right]\delta(J)+\frac{\alpha}{N}\rho(J) , \tag{1.3}$$

where α is the average finite connectivity at each site and $\rho(J)$ is a normalized distribution not containing a δ function at $J=0$, and (ii) random lattices with fixed finite connectivity $\alpha=M+1$, for which the bonds' strength is given by a probability distribution $\rho(J)$. Such lattices[8] can be constructed by building the connectivity matrix a_{ij} in which the matrix elements $a_{ij}=a_{ji}$ are chosen at random to be 0 or 1 with the sole constraint

$$\sum_j a_{ij}=M+1 . \tag{1.4}$$

Such random lattices are locally similar to a Bethe lattice, since small loops are rare, but the difference is that the random lattice has no boundary whereas the Bethe lattice does. In the Bethe lattice boundary conditions serve at the same time to introduce frustration and possibly to select one or more thermodynamic states. For a discussion of the role of boundary conditions on the Bethe lattice (BL) see Refs. 16–18. Numerical evidence suggests that the BL with "closed" boundary conditions behaves similarly to the random lattice[17,18] but, since we are mainly interested in the latter, this issue will not concern us further here.

The method of the $1/M$ expansion works equally well in both lattices (i) and (ii) discussed above. Our discussion will concentrate more on case (ii) of fixed connectivity since (a) there are numerical simulation results of graph bipartitioning with which we can compare and (b) this case is more closely related to hypercubic lattices which have finite fixed value and the equations involved are identical to the Bethe approximation to such a lattice.[19]

The paper is organized as follows. In Sec. II we discuss the large connectivity expansion at finite temperature. In Sec. III we discuss the expansion at zero temperature. Section IV contains concluding remarks. A number of Appendices discuss some technical details.

II. THE LARGE CONNECTIVITY EXPANSION AT FINITE TEMPERATURE

Our starting point will be the equation for the global order parameter[20] $g_n(\{\sigma_\alpha\})$ first derived by Mottishaw[10] for the Bethe lattice. Since he does not give details of the derivation we give a concise summary in Appendix A. Our normalization of g_n for finite n differs from his for convenience of the calculation. The equation reads (in the absence of an external field)

$$g_n(\{\sigma_\alpha\})=\int dJ\,\rho(J)\mathrm{Tr}_{\{\tau_\alpha\}}\exp\left[\beta J\sum_{\alpha=1}^{n}\sigma_\alpha\tau_\alpha\right]g_n^M(\{\tau_\alpha\})\Big/\mathrm{Tr}_{\{\tau_\alpha\}}g_n^M\{\tau_\alpha\} ; \tag{2.1}$$

here $\sigma_\alpha, \alpha=1,\ldots,n$ are the replicated spin variables and $M+1$ is the number of neighbors. At finite temperature, $g_n\{\sigma_\alpha\}$ can be parametrized in the form

$$g_n\{\sigma_\alpha\}=\sum_{r=0}^{\infty}b_r\sum_{(\alpha_1,\ldots,\alpha_r)}q_{\alpha_1,\ldots,\alpha_r}\sigma_{\alpha_1,\ldots,\alpha_r} , \tag{2.2}$$

where

$$b_r=\langle\cosh^n\beta J\tanh^r\beta J\rangle , \tag{2.3}$$

the average being with respect to $\rho(J)$. For the case of the even distribution

$$\rho(J)=\tfrac{1}{2}[\delta(J+J_0)+\delta(J-J_0)] , \tag{2.4}$$

only b_r with even r survive. Our method works for any distribution $\rho(J)$ but (2.4) will be used because of its simplicity and because of its relevance for the graph partitioning problem. In the averaged quantities the index 0 in J_0 will be omitted. Using the identity

$$\exp\left[\beta J\sum_\alpha\sigma_\alpha\tau_\alpha\right]=\cosh^n\beta J\sum_n\tanh^r\beta J\sum_{(\alpha_1,\ldots,\alpha_r)}\sigma_{\alpha_1}\tau_{\alpha_1}\cdots\sigma_{\alpha_r}\tau_{\alpha_r} \tag{2.5}$$

in (2.1) it becomes clear that $q_{\alpha_1,\ldots,\alpha_r}$ satisfy

$$q_{\alpha_1,\ldots,\alpha_r}=\mathrm{Tr}_{\sigma_\alpha}\sigma_{\alpha_1}\cdots\sigma_{\alpha_r}g^M(\{\sigma_\alpha\})/\mathrm{Tr}_{\sigma_\alpha}g^M(\{\sigma_\alpha\})\ . \tag{2.6}$$

Together with Eq. (2.2) this constitutes an equation for $q_{\alpha_1,\ldots,\alpha_r}$.

Before we proceed to evaluate g^M in the large-M limit we present an expression for the free energy of the system. This is derived in Appendix B. The result is

$$n\beta f=M\ln\mathrm{Tr}_{\sigma_\alpha}g_n^{M+1}(\{\sigma_\alpha\})-\frac{M+1}{2}\ln\int dJ\,\rho(J)\mathrm{Tr}_{\sigma_\alpha}\mathrm{Tr}_{\tau_\alpha}g_n^M(\sigma_\alpha)g_n^M(\tau_\alpha)\exp\left[\beta J\sum_\alpha\sigma_\alpha\tau_\alpha\right]\ . \tag{2.7}$$

If we make a variation

$$g(\{\sigma_\alpha\})\to g(\{\sigma_\alpha\})+\delta g(\{\sigma_\alpha\})\ , \tag{2.8}$$

stationarity of the free energy gives

$$\frac{\mathrm{Tr}_{\sigma_\alpha}g_n^M(\{\sigma_\alpha\})\delta g_n(\{\sigma_\alpha\})}{\mathrm{Tr}_{\sigma_\alpha}g_n^{M+1}(\{\sigma_\alpha\})}=\frac{\int dJ\rho(J)\mathrm{Tr}_{\sigma_\alpha}\mathrm{Tr}_{\tau_\alpha}\exp\left[\beta J\sum_\alpha\sigma_\alpha\tau_\alpha\right]g_n^M(\{\tau_\alpha\})g_n^{M-1}(\{\sigma_\alpha\})\delta g_n(\{\sigma_\alpha\})}{\int dJ\rho(J)\mathrm{Tr}_{\sigma_\alpha}\mathrm{Tr}_{\tau_\alpha}g_n^M(\{\sigma_\alpha\})g_n^M(\{\tau_\alpha\})\exp\left[\beta J\sum_\alpha\tau_\alpha\sigma_\alpha\right]}\ . \tag{2.9}$$

Since this equation holds for any variation δg_n one must have

$$g_n(\{\sigma_\alpha\})=\mathcal{N}\int dJ\rho(J)\mathrm{Tr}_{\tau_\alpha}\exp\left[\beta J\sum_\alpha\sigma_\alpha\tau_\alpha\right]g_n^M(\{\tau_\alpha\}) \tag{2.10}$$

with

$$\mathcal{N}=\mathrm{Tr}_{\sigma_\alpha}g_n^{M+1}(\{\sigma_\alpha\})\Big/\int\delta J\rho(J)\mathrm{Tr}_{\sigma_\alpha}\mathrm{Tr}_{\tau_\alpha}g_n^M(\{\sigma_\alpha\})g_n^M(\{\tau_\alpha\})\exp\left[\beta J\sum_\alpha\tau_\alpha\sigma_\alpha\right]\ . \tag{2.11}$$

Suppose $\hat{g}_n(\{\sigma_\alpha\})$ is a particular solution of this equation. Then for any constant c, $c\hat{g}_n(\{\sigma_\alpha\})$ also satisfies the same equation. But notice that in this case $\hat{\mathcal{N}}$, which is the value of $\mathcal{N}$ for $\hat{g}_n$, changes also $\hat{\mathcal{N}}\to c^{1-M}\hat{\mathcal{N}}$. Thus it is possible to choose c such that $\mathcal{N}=[\mathrm{Tr}c^M\hat{g}_n^M(\{\sigma_\alpha\})]^{-1}$. Thus Eq. (2.10) always has a solution which satisfies Eq. (2.1). Also any solution of (2.1) is also a solution of (2.10). Since the free energy is independent of the normalization of g_n we are free to use Eq. (2.1) instead of (2.10) and (2.11). To implement the $1/M$ expansion we scale the coupling

$$J=\tilde{J}/\sqrt{M}\ , \tag{2.12}$$

and build g^M either by using Eq. (2.1) or from the parametrized form (2.2). Using the shorthand $\lambda\equiv\beta\tilde{J}$ we obtain

$$g_n^M=[\cosh^n(\lambda/\sqrt{M})]^M\left[1+\frac{\lambda^2}{M}\left[1-\frac{2}{3}\frac{\lambda^2}{M}\right]\sum_{(\alpha,\beta)}q_{\alpha\beta}\sigma_\alpha\sigma_\beta+\frac{\lambda^4}{M^2}\sum_{(\alpha\beta\gamma\delta)}q_{\alpha\beta\gamma\delta}\sigma_\alpha\sigma_\beta\sigma_\gamma\sigma_\delta+\cdots\right]^M\ , \tag{2.13}$$

and hence

$$\begin{aligned}g_n^M\propto\exp\left[\lambda^2\sum_{(\alpha\beta)}q_{\alpha\beta}^{(0)}\sigma_\alpha\sigma_\beta\right]\Bigg[&1-\frac{2}{3M}\lambda^4\sum_{(\alpha\beta)}q_{\alpha\beta}^{(0)}\sigma_\alpha\sigma_\beta+\frac{1}{M}\lambda^2\sum_{(\alpha\beta)}q_{\alpha\beta}^{(1)}\sigma_\alpha\sigma_\beta+\frac{\lambda^4}{M}\sum_{(\alpha\beta\gamma\delta)}q_{\alpha\beta\gamma\delta}^{(0)}\sigma_\alpha\sigma_\beta\sigma_\gamma\sigma_\delta\\&-\frac{1}{2M}\lambda^4\sum_{(\alpha\beta)}q_{\alpha\beta}^{(0)2}-\frac{1}{M}\lambda^4\sum_{(\alpha\beta\gamma)}(q_{\alpha\beta}^{(0)}q_{\beta\gamma}^{(0)}\sigma_\alpha\sigma_\gamma+q_{\alpha\beta}^{(0)}q_{\alpha\gamma}^{(0)}\sigma_\beta\sigma_\gamma+q_{\alpha\gamma}^{(0)}q_{\beta\gamma}^{(0)}\sigma_\alpha\sigma_\beta)\\&-\frac{1}{M}\lambda^4\sum_{(\alpha\beta\gamma\delta)}(q_{\alpha\beta}^{(0)}q_{\gamma\delta}^{(0)}+q_{\alpha\gamma}^{(0)}q_{\beta\delta}^{(0)}+q_{\alpha\delta}^{(0)}q_{\beta\gamma}^{(0)})\sigma_\alpha\sigma_\beta\sigma_\gamma\sigma_\delta+\cdots\Bigg]\ ,\end{aligned} \tag{2.14}$$

where we denoted

$$q_{\alpha\beta}=q_{\alpha\beta}^{(0)}+\frac{1}{M}q_{\alpha\beta}^{(1)}+\cdots\ , \tag{2.15}$$

and similarly for $q_{\alpha\beta\gamma\delta}$. In order to calculate the free energy we first use Eq. (2.1) to express (2.7) in a more convenient form:

$$n\beta f=\tfrac{1}{2}M\ln(\mathrm{Tr}g_n^{M+1}/\mathrm{Tr}g_n^M)-\tfrac{1}{2}(\ln\mathrm{Tr}g_n^{M+1}+\ln\mathrm{Tr}g_n^M)\ . \tag{2.16}$$

This free energy is no longer stationary with respect to g_n and is valid only for g_n normalized according to (2.1). Substituting (2.14) in this expression and simplifying further using Eq. (2.6) we obtain $\beta f\equiv\beta f_0+\beta f_1/M+O(1/M^2)$ with

$$\beta f_0 = -(\lambda^2/4) + (\lambda^2/2n)\sum_{(\alpha\beta)} q_{\alpha\beta}^{(0)2} - (1/n)\ln \mathrm{Tr}\exp\left[\sum_{\alpha\beta}\lambda^2 q_{\alpha\beta}^{(0)}\sigma_\alpha\sigma_\beta\right],$$
$$\beta f_1 = -(\lambda^2/4) + (\lambda^4/24) - (\lambda^2/2n)(1-5\lambda^2/3)\sum_{(\alpha\beta)} q_{\alpha\beta}^{(0)^2} - (\lambda^4/2n)\sum_{(\alpha\beta\gamma\delta)} q_{\alpha\beta\gamma\sigma}^{(0)2}$$
$$+(3\lambda^4/n)\sum_{(\alpha\beta\gamma)} q_{\alpha\beta}^{(0)} q_{\beta\gamma}^{(0)} q_{\gamma\alpha}^{(0)} + \lambda^4/n \sum_{(\alpha\beta\gamma\delta)} (q_{\alpha\beta}^{(0)} q_{\gamma\delta}^{(0)} + 2\,\mathrm{perm})q_{\alpha\beta\gamma\delta}^{(0)} . \quad (2.17)$$

The leading term βf_0 is just the expression for the free energy density in the infinite-ranged SK model.[2] The term βf_1 introduces the correction due to finite connectivity. Notice that to this order f depends only on $q^{(0)}$ and not on $q^{(1)}$. First we evaluated f in the replica symmetric case. In that case

$$\beta f_0 = -\frac{\lambda^2}{4}(1-q_2)^2 - \int \frac{dz}{\sqrt{2\pi}} \ln[2\cosh(\lambda\sqrt{q_2}z)]e^{-z^2/2} ,$$
$$\beta f_1 = \frac{\lambda^2}{4}\left[1-\frac{5}{3}\lambda^2\right] q_2^2 + \frac{1}{8}\lambda^4 q_4^2 + \lambda^4 q_2^3 - \frac{3}{4}\lambda^4 q_4 q_2^2 - \frac{1}{4}\lambda^2 + \frac{1}{24}\lambda^4 , \quad (2.18)$$

and q_2, q_4 are given by

$$q_2 = \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} \exp(-z^2/2)\tanh^2\lambda\sqrt{q_2}z , \quad (2.19)$$

$$q_4 = \int_{-\infty}^{\infty} \frac{dz}{\sqrt{2\pi}} \exp(-z^2/2)\tanh^4\lambda\sqrt{q_2}z . \quad (2.20)$$

Equation (2.19) has been solved numerically for q_2 at different temperatures and the solution has been used to compute q_4 from (2.20). We then evaluated the free energy from Eq. (2.18). The result for $M=\infty$ and $M=10$ is displayed in Fig. 1. We see that at very low temperature f_1 tends to diverge. This has been verified analytically by calculating the correction for q_2 and q_4 away from $T=0$. We have found

$$q_2 = 1 - \frac{1}{\beta}\sqrt{2/\pi} - \frac{1}{\beta^2}\frac{1}{\pi} + O(1/\beta^3) , \quad (2.21)$$

$$q_4 = 1 - \frac{1}{\beta}\frac{4}{3}\sqrt{2/\pi} - \frac{1}{\beta^2}\frac{4}{3\pi} + O(1/\beta^3) . \quad (2.22)$$

It is then found that f_1 diverges linearly with β as $\beta \to \infty$. But for a wide range of temperature the $1/M$ expansion is well behaved, and the extrapolation of the results obtained prior to the runaway divergence to $T=0$ tends to agree with the result of the calculation performed exactly at $T=0$ in Sec. III. We will see that the reason for the divergence is that the large M expansion at $T=0$ is powers of $1/\sqrt{M}$ instead of $1/M$. This phenomenon occurs also at first stage of RS breaking that we will consider next. It is interesting to find out whether it will persist at infinite order of RS breaking.

Since it has been shown[10] that the RS solution is unstable we have to break the symmetry. Near T_c one can introduce continuous order parameter functions $q_2(x)$, $q_4(x,y,z)$, etc., and evaluate those in powers of $T-T_c$. In the entire temperature range we can obtain a solution up to a given stage of RS breaking. In the infinite-range case Parisi[1] has shown that the one stage replica symmetry breaking already improves significantly the value of the ground-state energy as compared to numerical simulations, and also renders the value of the entropy to be very close to zero (still on the negative side).

We have thus considered first stage RS breaking for the finite connectivity lattice. In that case one parametrizes the replica index α as $\alpha=(K,\gamma)$, where K is the box label, $K=1,\ldots,n/m$, and $\gamma=1,\ldots,m$ is a label within the box. One classifies the values of $q_{\alpha_1,\ldots,\alpha_r}$ according to the number of spin indices in the same box K.[11] For example, for $q_{\alpha_1\alpha_2}$ there are two values q_2 and q_{11} referring to one box with two spins or two boxes with one spin in each. For $q_{\alpha_1\alpha_2\alpha_3\alpha_4}$ there are five values q_4, q_{22}, q_{31}, q_{211}, q_{1111}, etc.

It is then easy to prove the following identities in the limit $n \to 0$:

$$\frac{1}{n}\sum_{(\alpha\beta)} q_{\alpha\beta}^2 = \frac{m-1}{2}q_2^2 - \frac{m}{2}q_{11}^2 , \quad (2.23)$$

$$\frac{1}{n}\sum_{(\alpha\beta\gamma)} q_{\alpha\beta}q_{\beta\gamma}q_{\gamma\alpha} = \frac{(m-1)(m-2)}{6}q_2^3 - \frac{m(m-1)}{2}q_2 q_{11}^2 + \frac{m^2}{3}q_{11}^3 ,$$
$$\frac{1}{n}\sum_{\alpha\beta\delta\gamma} q_{\alpha\beta\gamma\delta}^2 = \frac{(m-1)(m-2)(m-3)}{24}q_4^2 - \frac{m(m-1)^2}{8}q_{22}^2 - \frac{m(m-1)(m-2)}{6}q_{31}^2 + \frac{m^2(m-1)}{2}q_{211}^2 - \frac{m^3}{4}q_{1111}^2 , \quad (2.24)$$

$$\frac{1}{n}\sum_{(\alpha\beta\gamma\delta)} q_{\alpha\beta\gamma\delta}(q_{\alpha\beta}q_{\gamma\delta}+q_{\alpha\gamma}q_{\beta\delta}+q_{\alpha\delta}q_{\beta\gamma})=\tfrac{1}{8}(m-1)(m-2)(m-3)q_4q_2^2$$
$$-\frac{m(m-1)^2}{8}q_{22}(q_2^2+2q_{11}^2)-\frac{m(m-1)(m-2)}{2}q_{31}q_2q_{11}$$
$$+\frac{m^2(m-1)}{2}q_{211}(q_2q_{11}+2q_{11}^2)-\tfrac{3}{4}m^3q_{1111}q_{11}^2 . \tag{2.25}$$

When these relations are substituted into the free energy we finally obtain:

$$\beta f_0=-\frac{\lambda^2}{4}[1+mq_{11}^2+(1-m)q_2^2-2q_2]-\ln 2$$
$$-\frac{1}{\sqrt{2\pi}}\frac{1}{m}\int dz\, e^{-z^2/2}\ln\left[\int\frac{dy}{\sqrt{2\pi}}\exp(-y^2/2)\cosh^m(\lambda\sqrt{q_{11}}z+\lambda\sqrt{q_2-q_{11}}y)\right] , \tag{2.26}$$

$$\beta f_1=-(\lambda^2/4)+(\lambda^4/24)-(\lambda^2/4)(1-5\lambda^2/3)[(m-1)q_2^2-mq_{11}^2]$$
$$-(\lambda^4/48)[(m-1)(m-2)(m-3)q_4^2-3m(m-1)^2q_{22}^2-4m(m-1)(m-2)q_{31}^2$$
$$+12m^2(m-1)q_{211}^2-6m^3q_{1111}^2]+(\lambda^4/2)[(m-1)(m-2)q_2^3-3m(m-1)q_2q_{11}^2+2m^2q_{11}^3]$$
$$+(\lambda^4/8)[(m-1)(m-2)(m-3)q_4q_2^2-m(m-1)^2q_{22}(q_2^2+2q_{11}^2)-4m(m-1)(m-2)q_{31}q_2q_{11}$$
$$+4m^2(m-1)q_{211}(q_2q_{11}+2q_{11}^2)-6m^3q_{1111}q_{11}^2] . \tag{2.27}$$

Since we have used the equations of "motion," (2.27) is no longer stationary with respect to the q's but gives the correct value of the free energy. The total energy is still stationary with respect to the parameter m. Equation (2.26) coincides with the result obtained by Parisi[1] for the infinite-ranged model.

In the case of first-stage RS breaking the function g_n depends only on the variables

$$\sigma_K=\sum_{\gamma=1}^{m}\sigma_{K\gamma} . \tag{2.28}$$

It is convenient to introduce the effective field distribution defined by

$$P_n^{(M)}(\{h_K\})=\frac{1}{g_n^M(\{0\})}\int\prod_K\frac{ds_K}{2\pi}\exp\left[+i\sum_K s_Kh_K\right]g_n^M\left[\left[\frac{is_K}{\beta}\right]\right] . \tag{2.29}$$

The trace in Eq. (2.6) is easily evaluated and one obtains [the index (0) on the q's has been dropped for simplicity]

$$q_2=\int\frac{dH}{\sqrt{2\pi q_{11}}}\exp\left[-\frac{H^2}{2q_{11}}\right]\frac{\int_{-\infty}^{\infty}dh\exp\left[-\frac{(h-H)^2}{2(q_2-q_{11})}\right]\cosh^m\beta h\tanh^2\beta h}{\int_{-\infty}^{\infty}dh\exp\left[-\frac{(h-H)^2}{2(q_2-q_{11})}\right]\cosh^m\beta h} , \tag{2.30}$$

$$q_{11}=\int\frac{dH}{\sqrt{2\pi q_{11}}}\exp\left[-\frac{H^2}{2q_{11}}\right]\frac{\left[\int_{-\infty}^{\infty}dh\exp\left[\frac{-(h-H)^2}{2(q_2-q_{11})}\right]\cosh^m\beta h\tanh\beta h\right]^2}{\left[\int dh\exp\left[-\frac{(h-H)^2}{2(q_2-q_{11})}\right]\cosh^m\beta h\right]^2} . \tag{2.31}$$

The q's with four indices are given to leading order by similar equations in terms of q's with two indices. For example,

$$q_4=\int\frac{dH}{\sqrt{2\pi q_{11}}}\exp\left[-\frac{H^2}{2q_{11}}\right]\frac{\int dh\exp\left[-\frac{(h-H)^2}{2(q_2-q_{11})}\right]\cosh^m\beta h\tanh^4\beta h}{\int dh\exp\left[-\frac{(h-H)^2}{2(q_2-q_{11})}\right]\cosh^m\beta h} , \tag{2.32}$$

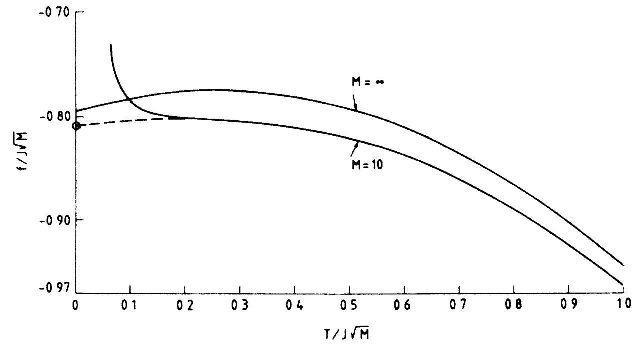

FIG. 1. Plot of the rescaled free energy vs rescaled temperature for $M=\infty$ (SK model) and $M=10$ with no RS breaking. The dashed line is the extrapolation to zero temperature. The value we obtain directly at $T=0$ is encircled.

and similarly for $q_{22}^{(0)}$, $q_{31}^{(0)}$, $q_{211}^{(0)}$, and $q_{1111}^{(0)}$. Thus after solving (2.30) and (2.31) for $q_2^{(0)}$ and $q_{11}^{(0)}$ and obtaining m by extremizing (2.26) one calculates $q_4^{(0)}$ from (2.32) and similarly the other q's with four indices. Then one can evaluate the $1/M$ correction for the free energy from Eq. (2.27). Again when $\beta\rightarrow\infty$ the $1/M$ correction diverges as we will see in the next section. Numerical evaluation of the free energy in the one-step RS breaking at finite temperature will be presented elsewhere.[21] Note the similarity of Eqs. (2.30) and (2.31) to the results of Mezard *et al.*[22] for the infinite-ranged model, derived from a different approach.

In the case of a random lattice with an average finite connectivity the factors g_n^M in Eqs. (2.1) and (2.6) are replaced by

$$\exp[(M+1)(g_n\{\sigma_\alpha\}-1)]\ . \tag{2.33}$$

The parametrization (2.2) still holds, and Eqs. (2.13) and (2.14) are replaced by the corresponding equations using (2.33). The expression for the free energy is given by Mottishaw and De Dominicis.[23]

III. THE $1/\sqrt{M}$ EXPANSION AT ZERO TEMPERATURE

A. Replica symmetric case

We have seen that the generalized order parameter $g_n(\{\sigma_\alpha\})$ satisfies Eq. (2.1). We first show how to solve this equation in the RS case. In that case g_n depends only on the variable $\hat\sigma=\sum\sigma_\alpha$ and in the limit $n\rightarrow 0$ can be shown similarly to Ref. 20 (see also Appendix C) to satisfy the equation

$$g_0\{\hat\sigma/\beta\}=\int dJ\,\rho(J)\int(ds/2\pi)g_0^M\{is/\beta\}\int du\,\exp(isu)\exp\left[-\hat\sigma\frac{1}{\beta}\tanh^{-1}(\tanh\beta\tilde J/\sqrt{M}\tanh\beta u)\right]\ . \tag{3.1}$$

Defining

$$\gamma_0(x)=g_0(\hat\sigma/\beta) \tag{3.2}$$

and using the identity

$$\lim_{\beta\rightarrow\infty}\frac{1}{\beta}\tanh^{-1}(\tanh\beta J\tanh\beta u)=\mathrm{sgn}J\,\mathrm{sgn}\,u\,\min(|u|,|J|)\ , \tag{3.3}$$

we obtain for $\rho(J)$ being a $\pm J$ distribution with $J=\tilde J/\sqrt{M}$,

$$\gamma_0(x)=\int\frac{ds}{2\pi}\gamma_0^M(is)\left[2\pi\delta(s)\cosh\frac{x\tilde J}{\sqrt{M}}-\frac{2\sinh(\tilde J/\sqrt{M})is}{is}\cosh\frac{x\tilde J}{\sqrt{M}}+\frac{\sinh(\tilde J/\sqrt{M})(is+x)}{is+x}+\frac{\sinh(\tilde J/\sqrt{M})(is-x)}{is-x}\right]\ . \tag{3.4}$$

We now expand the various terms inside the large parentheses in powers of $1/\sqrt{M}$ and use the fact that $\gamma_0(0)=1$ to obtain

$$\gamma(x)=1+\frac{\tilde J^2x^2}{2M}+\frac{\tilde J^4x^4}{24M^2}-\frac{2}{3}\frac{\tilde J^3x^2}{M\sqrt{M}}\int\frac{ds}{2\pi}\gamma^M(is)+O\left[\frac{1}{M^2\sqrt{M}}\right]\ . \tag{3.5}$$

Taking the Mth power of both sides and solving for γ^M self-consistently we obtain

$$\gamma^M(x)=e^{(1/2)\tilde J^2x^2}\left\{1-\frac{1}{\sqrt{M}}\frac{2}{3\sqrt{2\pi}}\tilde J^2x^2+\frac{1}{M}\left[\left(\frac{1}{9\pi}-\frac{1}{12}\right)\tilde J^4x^4-\frac{2}{9\pi}\tilde J^2x^2\right]\right\} \tag{3.6}$$

up to $O(1/M)$.

The effective field distribution $P_0^{(M)}(h)$ is given by

$$P_0^{(M)}(h)=\int \frac{dx}{2\pi}\exp(ihx)\gamma^M(ix)=\frac{1}{(2\pi\tilde{J}^2)^{1/2}}\exp\left[-\frac{h^2}{2\tilde{J}^2}\right]$$
$$\times\left\{1+\frac{1}{\sqrt{M}}\frac{2}{3\sqrt{2\pi}}\left[1-\frac{h^2}{\tilde{J}^2}\right]+\frac{1}{M}\left[\left[\frac{1}{9\pi}-\frac{1}{12}\right](3-6h^2/\tilde{J}^2+h^4/\tilde{J}^4)+\frac{2}{9\pi}(1-h^2/\tilde{J}^2)\right]\right\}. \quad (3.7)$$

In terms of $P_0^{(M)}(h)$ the free energy density is given by (for $\beta\to\infty$)

$$F=M\int_{-\infty}^{\infty}dh\,P_0^{(M+1)}(h)|h|-\frac{M+1}{2}\int dh_1dh_2P_0^{(M)}(h_1)P_0^{(M)}(h_2)$$
$$\times\frac{1}{2}\left[\max\left[|h_1+h_2|+\frac{\tilde{J}}{\sqrt{M}},|h_1-h_2|-\frac{\tilde{J}}{\sqrt{M}}\right]+(\tilde{J}\to-\tilde{J})\right]. \quad (3.8)$$

After some algebra we finally obtain

$$f/J\sqrt{M}=-\sqrt{2/\pi}\left[1-\frac{1}{\sqrt{M}}\frac{1}{\sqrt{18\pi}}+\frac{1}{M}\left[\frac{7}{12}-\frac{1}{9\pi}\right]+\cdots\right]. \quad (3.9)$$

B. One-step RS breaking

In the first stage of RS breaking g_n depends only on the variables σ_K defined in Eq. (2.28). We have shown (see Appendix C) that it satisfies the equation

$$g_n\{\sigma_K/\beta\}=\mathcal{N}^{-1}\int dJ\,\rho(J)\int\prod_K(ds_K/2\pi)g_n^M(is_K/\beta)\int\prod_K du_K\exp\left[i\sum_K s_Ku_K\right]$$
$$\times\exp\left[-\sum_K\sigma_K\frac{1}{\beta}\tanh^{-1}(\tanh\beta J\tanh\beta u_K)\right]$$
$$\times\exp\left[\frac{m}{2}\sum_K\ln(\cosh^2\beta J\cosh^2\beta u_K-\sinh^2\beta J\sinh^2\beta u_K)\right], \quad (3.10)$$

$$\mathcal{N}=\int\prod_K\frac{ds_K}{2\pi}g_n^M\{is_K/\beta\}\int\prod_K du_K\exp\left[i\sum s_Ku_K\right]\cosh^m\beta u_K. \quad (3.11)$$

If we now define

$$\gamma_n\{x_K\}=g_n\{\sigma_K/\beta\},$$

we find, in the limit $\beta\to\infty$,

$$\gamma_n(\{x_K\})=\mathcal{N}^{-1}\int dJ\,\rho(J)\int\prod_K\frac{ds_K}{2\pi}\gamma_n^M(\{is_K\})\int\prod_K du_K\exp\left[i\sum s_Ku_K\right]\exp\left[-\sum_K x_K\,\mathrm{sgn}J\,\mathrm{sgn}u_K\min(|u_K|,|J|)\right]$$
$$\times\exp\left[\mu\sum_K\max(|u_K|,|J|)\right], \quad (3.12)$$

where

$$\mu=\lim_{\beta\to\infty}m\beta \quad (3.13)$$

and

$$\mathcal{N}=\int\prod_K\frac{ds_K}{2\pi}\gamma_n^M(\{is_K\})\int\prod_K du_K\exp\left[i\sum s_Ku_K\right]\exp\left[\mu\sum_K|u_K|\right]. \quad (3.14)$$

Equation (3.13) is based on the assumption that, like in the infinite-ranged model, $m\propto1/\beta$ for large β. We now scale $J=\tilde{J}/\sqrt{M}$ and expand in $1/\sqrt{M}$. We defined

$$b_K=\int du_K\exp(is_Ku_K)\exp(\mu|u_K|), \quad (3.15)$$

$$a_K=-\frac{1}{b_K}\int du_K\exp(is_Ku_K+\mu|u_K|)\mathrm{sgn}(u_K), \quad (3.16)$$

and in terms of those quantities γ_n^M becomes

$$\gamma_n^M(x)=\exp\left[\tilde{J}^2\sum_{(KK')}\langle a_K a_{K'}\rangle x_K x_{K'}+\tfrac{1}{2}\tilde{J}^2\sum_K x_K^2+\tilde{J}^2\mu\sum_K\langle b_K^{-1}\rangle\right]$$
$$\times\left\{1-\frac{2}{3}\frac{|\tilde{J}|^3}{\sqrt{M}}\sum_K\langle b_K^{-1}\rangle x_K^2+\frac{2}{3}\frac{|\tilde{J}|^3\mu^2}{\sqrt{M}}\sum_K\langle b_K^{-1}\rangle\right.$$
$$+\frac{1}{M}\left[\frac{2}{9}\tilde{J}^6\sum_K\left[\langle b_K^{-1}\rangle^2-\frac{3}{8\tilde{J}^2}\right]x_K^4-\tfrac{1}{3}\tilde{J}^4\sum_{(K_1K_2)}\langle a_{K_1}a_{K_2}\rangle(x_{K_1}x_{K_2}^3+x_{K_1}^3x_{K_2})\right.$$
$$+\tfrac{2}{9}\tilde{J}^6\sum_{(K_1K_2)}\left[2\langle b_{K_1}^{-1}\rangle\langle b_{K_2}^{-1}\rangle-\frac{9}{4\tilde{J}^2}\langle a_{K_1}a_{K_2}\rangle^2\right]x_{K_1}^2x_{K_2}^2$$
$$-\tilde{J}^4\sum_{(K_1K_2K_3)}\langle a_{K_1}a_{K_2}\rangle^2(x_{K_1}x_{K_2}x_{K^3}^2+x_{K_1}x_{K_2}^2x_{K_3}+x_{K_1}^2x_{K_2}x_{K_3})$$
$$-3\tilde{J}^4\sum_{(K_1K_2K_3K_4)}(\langle a_{K_1}a_{K_2}\rangle^2-\tfrac{1}{3}\langle a_{K_1}a_{K_2}a_{K_3}a_{K_4}\rangle)x_{K_1}x_{K_2}x_{K_3}x_{K_4}$$
$$-\frac{\mu\tilde{J}^4}{6}\sum_K\langle b_K^{-1}\rangle x_K^2+\tilde{J}^4\mu\sum_{(K_1K_2K_3)}\langle a_{K_1}a_{K_2}b_{K_3}^{-1}\rangle(x_{K_1}x_{K_2}+x_{K_1}x_{K_3}+x_{K_2}x_{K_3})$$
$$+\tfrac{2}{3}\tilde{J}^4\sum_{(K_1K_2)}\langle a_{K_1}b_{K_2}^{-1}is_{K_2}\rangle x_{K_1}x_{K_2}-\frac{\mu\tilde{J}^4}{12}\sum_K(\langle s_K^2b_K^{-1}\rangle-3\mu^2\langle b_K^{-1}\rangle)$$
$$\left.\left.+\tilde{J}^4\mu^2\sum_{(K_1K_2)}\langle b_{K_1}^{-1}b_{K_2}^{-1}\rangle\right]+\cdots\right\}, \tag{3.17}$$

where we defined

$$\langle A\rangle=\frac{\int\prod_K\frac{ds_K}{2\pi}\gamma_n^M(\{is_K\})\prod_K b_K(s_K)A(\{s_K\})}{\int\prod_K\frac{ds_K}{2\pi}\gamma_n^M(\{is_K\})\prod_K b_K(s_K)}\ . \tag{3.18}$$

The different averages can also be expanded in powers of $1/\sqrt{M}$:

$$\langle A\rangle=\langle A\rangle_0+\frac{1}{\sqrt{M}}\langle A\rangle_1+\cdots\ . \tag{3.19}$$

We have used Eq. (2.16) to calculate the free energy (see Appendix D for details).
We have made used the identities (see Appendix D)

$$-\langle s_K^2\rangle=\mu^2+2\mu\langle b_K^{-1}\rangle\ , \tag{3.20a}$$

$$-\langle s_K s_{K'}\rangle=\mu^2\langle a_K a_{K'}\rangle\ , \tag{3.20b}$$

$$\langle s_K^4\rangle=\mu^4+2\mu^3\langle b_K^{-1}\rangle-2\mu\langle b_K^{-1}s_K^2\rangle\ , \tag{3.20c}$$

$$\langle s_{K_1}s_{K_2}^3\rangle=\mu^4\langle a_K a_{K'}\rangle+2\mu^2\langle a_{K_1}b_{K_2}^{-1}is_{K_2}\rangle\ , \tag{3.20d}$$

$$\langle s_{K_1}^2s_{K_2}^2\rangle=\mu^4+4\mu^3\langle b_K^{-1}\rangle+4\mu^2\langle b_{K_1}^{-1}b_{K_2}^{-1}\rangle\ , \tag{3.20e}$$

$$\langle s_{K_1}s_{K_2}s_{K_3}^2\rangle=\mu^4\langle a_{K_1}a_{K_2}\rangle+2\mu^3\langle a_{K_1}a_{K_2}b_{K_3}^{-1}\rangle\ , \tag{3.20f}$$

$$\langle s_{K_1}s_{K_2}s_{K_3}s_{K_4}\rangle=\mu^4\langle a_{K_1}a_{K_2}a_{K_3}a_{K_4}\rangle\ . \tag{3.20g}$$

The final answer is

$$
\begin{aligned}
f/\sqrt{M}J = {} & \frac{\mu}{4}(1-\langle a_{K_1}a_{K_2}\rangle_0^2)-\frac{1}{\mu}\int\frac{dz}{\sqrt{2\pi}}\exp(-z^2/2)\ln\left[\int\frac{dy}{\sqrt{2\pi}}\exp(-y^2/2)\right.\\
& \left.\times\exp[\mu|y(1-\langle a_{K_1}a_{K_2}\rangle_0)^{1/2}+z(\langle a_{K_1}a_{K_2}\rangle_0)^{1/2}|]\right]\\
& +\frac{1}{\sqrt{M}}\frac{2}{3}\langle b_K^{-1}\rangle_0^2+\frac{1}{M}\Big(-\langle b_K^{-1}\rangle_0-\tfrac{1}{6}\langle b_K^{-1}s_K^2\rangle_0+\tfrac{4}{9}\langle b_K^{-1}\rangle_0^2\langle b_K^{-1}s_K^2\rangle_0-\frac{\mu}{4}\\
& -\tfrac{4}{3}\mu\langle a_{K_1}a_{K_2}\rangle_1\langle b_K^{-1}\rangle_0\langle a_{K_1}b_{K_2}^{-1}is_{K_2}\rangle_0-\mu\langle b_{K_1}^{-1}b_{K_2}^{-1}\rangle_0\langle a_{K_1}a_{K_2}\rangle_0^2\\
& -\tfrac{2}{3}\mu\langle a_{K_1}a_{K_2}\rangle_0\langle a_{K_1}b_{K_2}^{-1}is_{K_2}\rangle_0-\frac{\mu}{4}\langle a_{K_1}a_{K_2}\rangle_1^2+\mu\langle a_{K_1}a_{K_2}\rangle_1^2\langle b_{K_1}^{-1}b_{K_2}^{-1}\rangle_0\\
& +\frac{\mu}{6}\langle b_K^{-1}\rangle_0^2+\tfrac{8}{9}\mu\langle b_K^{-1}\rangle_0^2\langle b_{K_1}^{-1}b_{K_2}^{-1}\rangle_0+\frac{\mu}{4}\langle a_{K_1}a_{K_2}\rangle_0^2+\tfrac{4}{9}\mu^2\langle b_K^{-1}\rangle_0^3\\
& +\tfrac{4}{3}\mu^2\langle a_{K_1}a_{K_2}\rangle_1\langle b_K^{-1}\rangle_0\langle a_{K_1}a_{K_2}b_{K_3}^{-1}\rangle_0-\tfrac{1}{12}\mu^2\langle b_K^{-1}\rangle_0-\mu^2\langle b_K^{-1}\rangle_0\langle a_{K_1}a_{K_2}\rangle_0^2\\
& +\mu^2\langle a_{K_1}a_{K_2}\rangle_1^2\langle b_K^{-1}\rangle_0-2\mu^2\langle a_{K_1}a_{K_2}\rangle_1^2\langle a_{K_1}a_{K_2}b_{K_3}^{-1}\rangle_0+2\mu^2\langle a_{K_1}a_{K_2}\rangle_0^2\langle a_{K_1}a_{K_2}b_{K_3}^{-1}\rangle_0\\
& -\tfrac{3}{4}\mu^3\langle a_{K_1}a_{K_2}a_{K_3}a_{K_4}\rangle_0\langle a_{K_1}a_{K_2}\rangle_0^2+\frac{\mu^3}{24}+\mu^3\langle a_{K_1}a_{K_2}\rangle_0^3-\tfrac{5}{12}\mu^3\langle a_{K_1}a_{K_2}\rangle_0^2\\
& +\tfrac{1}{8}\mu^3\langle a_{K_1}a_{K_2}a_{K_3}a_{K_4}\rangle_0^2+\frac{\mu^3}{4}\langle a_{K_1}a_{K_2}\rangle_1^2+\tfrac{3}{4}\mu^3\langle a_{K_1}a_{K_2}\rangle_1^2\langle a_{K_1}a_{K_2}a_{K_3}a_{K_4}\rangle_0\\
& -\mu^3\langle a_{K_1}a_{K_2}\rangle_1^2\langle a_{K_1}a_{K_2}\rangle_0\Big)\ .
\end{aligned}
\tag{3.21}
$$

The parameter μ should also be expanded:

$$\mu=\mu_0+\mu_1/\sqrt{M}+\cdots\ . \tag{3.22}$$

We evaluated μ_0 and $\langle a_{K_1}a_{K_2}\rangle_0$ by extremizing the leading-order term in (3.21) and obtained

$$\mu_0=1.35\ , \tag{3.23}$$

$$q_{11}^{(0)}\equiv\langle a_{K_1}a_{K_2}\rangle_0=0.476\ . \tag{3.24}$$

The fact that $\mu\neq\mu_0$ introduces a correction when f is evaluated with μ_0:

$$\Delta=-\frac{1}{M}\frac{1}{2}\frac{[f_1'(\mu_0)]^2}{f_0''(\mu_0)}\ , \tag{3.25}$$

where f_0 and f_1 are leading term and first order correction on the right-hand side of (3.21). We have evaluated the different averages in Eq. (3.21). For example,

$$\langle b_K^{-1}\rangle_0=\frac{1}{\sqrt{2\pi}}\int\frac{dw}{\sqrt{2\pi}}\exp(-w^2/2)\left[\int\frac{dy}{\sqrt{2\pi}}\exp(-y^2/2)\exp[\mu_0(1-q_{11}^{(0)})^{1/2}|y+w(q_{11}^{(0)})^{1/2}|]\right]^{-1}\ . \tag{3.26}$$

Only for $\langle a_{K_1}a_{K_2}\rangle_1$ one has to solve an equation to express it in terms of other leading-order averages.

The numerical values of the different averages are

$$
\begin{aligned}
& \langle b_K^{-1}\rangle_0=0.121,\quad \langle b_K^{-1}s_K^2\rangle_0=0.154\ ,\\
& \langle a_{K_1}a_{K_2}a_{K_3}a_{K_4}\rangle_0=0.342,\quad \langle b_{K_1}^{-1}b_{K_2}^{-1}\rangle_0=0.0189\ ,\\
& \langle a_{K_1}b_{K_2}^{-1}is_{K_2}\rangle_0=-0.0507\ ,\\
& \langle a_{K_1}a_{K_2}b_{K_3}^{-1}\rangle_0=0.0350,\quad \langle a_{K_1}a_{K_2}\rangle_1=0.0741\ ,\\
& f_1'(\mu_0)=-0.0214,\quad f_0''(\mu_0)=-0.0328\ .
\end{aligned}
\tag{3.27}
$$

In terms of these quantities the final result for the ground-state energy density is

$$E_0/\sqrt{M}J=-0.765+\frac{1}{\sqrt{M}}0.010-\frac{1}{M}0.390+O\left[\frac{1}{M\sqrt{M}}\right]\ . \tag{3.28}$$

The leading term coincides with the result for the infinite-ranged model with one-step RS breaking.[1]

IV. CONCLUDING REMARKS

Banavar *et al.*[8] obtained numerical results for the graph bipartitioning problem on random graphs with finite fixed connectivity. Mezard *et al.*[5] argued that in that case the cost function is related to the ground-state energy of the Ising spin glass on such a lattice on the

TABLE I. Values of $E_0/J\sqrt{M}$ for different values of M as discussed in the text.

M	1	2	3	4
9	−0.792	−0.810	−0.811	−0.805
10	−0.789	−0.809	−0.808	−0.801
11	−0.786	−0.806	−0.805	−0.797
19	−0.777		−0.797	−0.783
20	−0.776		−0.796	−0.782
∞	−0.763	−0.798	−0.798	−0.765

basis of the expectation that the effective field distribution is even under $h \to -h$ when $M+1 > 2\ln 2$. The relation between the two problems is

$$\frac{c}{N}=\frac{M+1}{4}+\frac{E_0}{2J}\ , \tag{4.1}$$

where c is the cost function, N is the number of sites on the lattice, and E_0 is ground-state energy density of the SG on such a lattice. In Table I we displayed different results for $E_0/J\sqrt{M}$. In Column 1 we display the numerical results of Ref. 8. These results were shown in Ref. 8 to fit the empirical formula

$$E_0/J\sqrt{M}=-\frac{1}{\sqrt{M}}\frac{M+1}{2}\frac{c}{(M-1+c^2)^{1/2}}$$
$$=0.763-\frac{1}{M}0.256+\cdots \tag{4.2}$$

with $c=1.5266$.

In Column 2 we show results obtained by Mezard and Parisi[5] using RS and not including a continuous part in the effective field distribution. In Column 3 we display our result in the RS case as given by Eq. (3.9) and in Column 4 the result with one-step RS breaking as obtained from Eq. (3.28). The result with one-step RS breaking approaches closer the numerical result of Ref. 8 than the RS results. The coefficient of the $1/\sqrt{M}$ term is much smaller in the one-step RS breaking (3.24) as compared to RS case Eq. (3.9). It will be interesting to find out if this trend continues when more steps of RS breaking are introduced.

Of course the theory which we considered still differs from a regular cubic lattice which contains many small loops, and one cannot infer from our results that in that case too (say for $d=5$ which corresponds to $M=9$), the solution with RS breaking comes closer to the exact value of the ground-state energy of the spin glass. We should also mention that a calculation for a one-step RS breaking for the case of $M=2$ and 3 has been done by Wong and Sherrington.[24] However, they assumed zero overlap between different replicas that in our language corresponds to $g_K(\{\sigma_K\})=\prod_K f(\sigma_K)$ which is not the case in our calculation. Lastly, the method of $1/M$ expansion presented in this paper can be extended to the case of the Potts spin glass which is related to the problem of graph q partitioning where q is the number of Potts states.[7,13,25]

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APPENDIX A

Consider an Ising spin glass on a Cayley tree. The replicated partition function is given by

$$Z^n=\sum_{\sigma_\alpha}\prod_\alpha P(\sigma_\alpha)\ , \tag{A1}$$

where

$$\prod_\alpha P(\sigma_\alpha)=\exp\left[\beta\sum_\alpha\sum_{\langle ij\rangle}J_{ij}\sigma_i^\alpha\sigma_j^\alpha+\beta h\sum_\alpha\sum_i\sigma_i^\alpha\right]\ . \tag{A2}$$

Denoting by σ_0^α the spin at the center, and by $s^{(l)}$ the spins (other than σ_0) on the lth subtree, we have

$$\prod_\alpha P(\sigma^\alpha)=\exp\left[\sum_\alpha\beta h\sigma_0^\alpha\right]\prod_{l=1}^{M+1}\prod_\alpha Q_{(m)}(\sigma_0^\alpha|s^{(l)\alpha})\ , \tag{A3}$$

with

$$Q_{(m)}(\sigma_0^\alpha|s^\alpha)=\exp(\beta J_{10}\sigma_0^\alpha s_1^\alpha+\beta h s_1^\alpha)$$
$$\times\prod_{l=1}^{M}Q_{(m-1)}(s_1^\alpha|t^{(l)\alpha})\ , \tag{A4}$$

m being the number of shells in the tree. s_1 is the spin adjacent to σ_0. $t^{(l)}$ are the spins, other than s_1 on the lth subtree.

Denoting by

$$g_{(m)}(\{\sigma_0^\alpha\})=\left\langle\sum_{s^{(l)\alpha}}\prod_\alpha Q_{(m)}(\sigma_0^\alpha|s^{(l)\alpha})\right\rangle\ , \tag{A5}$$

where $\langle\ \rangle$ denotes average over the disorder, it follows that

$$\langle Z^n\rangle=\mathrm{Tr}_{\sigma_0^\alpha}\exp\left[\beta h\sum_\alpha\sigma_0^\alpha\right]g_{(m)}^{M+1}(\{\sigma_0^\alpha\})\ , \tag{A6}$$

and that g_m satisfies

$$g_{(m)}(\{\sigma_0^\alpha\})=\int dJ\,\rho(J)\mathrm{Tr}_{s_1^\alpha}$$
$$\times\exp\left[\beta h\sum_\alpha s_1^\alpha+\beta J\sum\sigma_0^\alpha s_1^\alpha\right]$$
$$\times g_{(m-1)}^M(\{s_1^\alpha\})\ . \tag{A7}$$

On the Bethe lattice in the thermodynamic limit, one is interested in a shell independent solution of (A7) which for the case $h=0$ yields Eqs. (2.1). Since the random lattice with fixed connectivity behaves locally like a tree, the same equation holds there (see, e.g., Ref. 5 and 8).

In Appendix B and in Eqs. (2.8)–(2.10) another derivation of this equation is obtained using the Bethe approximation. This approximation becomes exact for a Bethe lattice or a random lattice with fixed connectivity.

APPENDIX B

In this Appendix we derive the expression for the free energy density in terms of the global order parameter $g(\{\sigma_\alpha\})$. Since the random lattice with fixed connectivity looks locally like a tree, the expression for the free energy is the same as for the Bethe lattice. For such a lattice the pair (Bethe) approximation becomes exact, in the absence of loops. We generalize the approach of Katsura[19] to the case of many coexisting thermodynamic states. The introduction of several replicas of effective fields is similar in spirit to the treatment of the infinite-ranged model in Ref. 22, although we use replicas explicitly. Katsura expressed the free energy density in the form

$$F=\sum_i F_i^{(1)}+\sum_{\langle ij\rangle}(F_{ij}^{(2)}-F_i^{(1)}-F_j^{(1)}) , \quad \text{(B1)}$$

where

$$F_i^{(1)}=-\frac{1}{\beta}\overline{\ln \mathrm{Tr}\rho_i^{(1)}} , \quad \text{(B2)}$$

$$F_{ij}^{(2)}=-\frac{1}{\beta}\overline{\ln \mathrm{Tr}\rho_{ij}^{(2)}} , \quad \text{(B3)}$$

with

$$\rho_i^{(1)}=\exp\left[\beta\sum_{k=1}^{M+1}\eta_{ik}\sigma_i\right] , \quad \text{(B4)}$$

$$\rho_{ij}^{(2)}=\exp\left[\beta{\sum_{k\neq j}^{M}}'\eta_{ik}\sigma_i+\beta{\sum_{l\neq i}^{M}}'\eta_{jl}\sigma_j+\beta J_{ij}\sigma_i\sigma_j\right] , \quad \text{(B5)}$$

from which one readily obtains

$$\beta f=M\overline{\ln \mathrm{Tr}\rho_i^{(1)}}-\frac{M+1}{2}\overline{\ln \mathrm{Tr}\rho_i^{(2)}} . \quad \text{(B6)}$$

In (A4) and (A5), η_{ij} is the effective field at site i due to its neighbor at site j. Denoting by $P^{(M)}(h)$ the distribution of the effective fields h where ($P^{(M)}$ is assumed to be independent of the site)

$$h_i=\sum_j^M \eta_{ij} , \quad \text{(B7)}$$

one obtains, after averaging over the bond distribution $\rho(J)$,

$$\beta f=M\int_{-\infty}^{\infty}dh\,P^{(M+1)}(h)\ln(\cosh\beta h)-\frac{M+1}{2}\times\int dJ\,\rho(J)dh_1dh_2P^{(M)}(h_1)P^{(M)}(h_2)\ln\times[2\cosh(\beta h_1+\beta_2)e^{\beta J}+2\cosh(\beta h_1-\beta h_2)e^{-\beta J}] ; \quad \text{(B8)}$$

compare also with Bownman and Levin.[26] Now consider a replicated spin system with spins $\{\sigma_i^\alpha\}$, $\alpha=1,\dots,n$. One can define analogously the effective fields η_{ij}^α and the averages become

$$\overline{\ln \mathrm{Tr}\rho_i^{(1)}}\to\lim_{n\to 0}\frac{1}{n}\ln\left[\overline{\mathrm{Tr}_{\{\sigma_i^\alpha\}}\exp\left[\beta\sum_k\sum_\alpha\eta_{ik}^\alpha\sigma_i^\alpha\right]}\right] , \quad \text{(B9)}$$

and similarly for $\rho_{ij}^{(2)}$.

Denoting by

$$h_i^\alpha=\sum_j^M\eta_{ij}^\alpha \quad \text{(B10)}$$

and by $P_n^{(M)}(h^1,\dots,h^n)$, the corresponding effective field distributions, we obtain

$$n\beta f=M\ln\int\prod_\alpha dh^\alpha P_n^{(M+1)}(\{h^\alpha\})2^n\prod_{\alpha=1}^n\cosh\beta h^\alpha-\frac{M+1}{2}\ln\int dJ\,\rho(J)\int\prod_\alpha dh_1^\alpha dh_2^\alpha P_n^{(M)}\{h_1^\alpha\}P_n^{(M)}\{h_2^\alpha\}2^n\prod_\alpha[\exp(\beta J)\cosh(\beta h_1^\alpha+\beta h_2^\alpha)+\exp(-\beta J)\cosh(\beta h_1^\alpha-\beta h_2^\alpha)] . \quad \text{(B11)}$$

If we now define

$$g_n^M(\{i\sigma^\alpha\})=\int\prod_\alpha dh^\alpha epx\left[-i\beta\sum_\alpha h^\alpha\sigma^\alpha\right]P_n^{(M)}(\{h^\alpha\}) , \quad \text{(B12)}$$

then in terms of g_n the free energy density becomes

$$n\beta f=M\ln \mathrm{Tr}_{\sigma_\alpha}g_n^{M+1}(\{\sigma_\alpha\})-\frac{M+1}{2}\ln\int dJ\,\rho(J)\mathrm{Tr}_{\sigma_\alpha}\mathrm{Tr}_{\tau_\alpha}\exp\left[\beta J\sum_\alpha\sigma_\alpha\tau_\alpha\right]\times g^M(\{\sigma_\alpha\})g^M(\{\tau_\alpha\}) , \quad \text{(B13)}$$

which is the desired result. Notice that equation (B12) is consistent with the relation

$$P^{(M)}(\{h^\alpha\})=\int\prod_{i=1}^M\prod_\alpha d\eta_i^\alpha\prod_{i=1}^M P^{(1)}(\{\eta_i^\alpha\})\times\prod_\alpha\delta\left[h^\alpha-\sum_{i=1}^M\eta_i^\alpha\right] , \quad \text{(B14)}$$

which is required of the field distribution. Wong and Sherrington[24] also derive an expression for the free energy in the fixed connectivity case. Their expression can be shown to yield the same value of the free energy as ours when use of the equations of motion (stationarily conditions) is made. Our expression has the advantage to be simpler and expressed directly in terms of $g_n(\{\sigma_\alpha\})$.

APPENDIX C

Let us prove the following identity ($\sigma_\alpha=\pm1$):

$$\sum_{r=0}^{\infty}\sum_{(\alpha_1,\ldots,\alpha_r)} X_{\alpha_1}\ldots X_{\alpha_r}\sigma_{\alpha_1}\cdots\alpha_{\alpha_r}=\exp\left[\sum_\alpha \sigma_\alpha \tanh^{-1}X_\alpha+\tfrac{1}{2}\sum_\alpha \ln(1-X_\alpha^2)\right]; \tag{C1}$$

consider first the formula

$$\exp\left[\sum_\alpha y_\alpha\sigma_\alpha\right]=\prod_{\alpha=1}^{n}(\cosh y_\alpha)\sum_r\sum_{(\alpha_1,\ldots,\alpha_r)}\tanh y_{\alpha_1}\cdots\tanh y_{\alpha_r}\sigma_{\alpha_1}\cdots\sigma_{\alpha_r}. \tag{C2}$$

This is obtained by writing

$$\exp\left[\sum_\alpha y_\alpha\sigma_\alpha\right]=\sum_r\sum_{(\alpha_1,\ldots,\alpha_r)} b_{\alpha_1\cdots\alpha_r}\sigma_{\alpha_1}\cdots\sigma_{\alpha_r},$$

multiplying both sides by $\sigma_{\beta_1},\ldots\sigma_{\beta_s}$ and taking the trace. Now define $y_\alpha=\tanh^{-1}X_\alpha$. Equation (C1) follows immediately.

Consider now the case of one-step RS breaking. g_n depends on the variables

$$\sigma_K=\sum_{\gamma=1}^{m}\sigma_{K_\gamma}, \tag{C3}$$

which can be considered to be continuous variables. We can then define the Fourier transform

$$P_n^{(M)}(\{h_K\})=\int\prod_K\frac{ds_K}{2\pi}\exp\left[i\sum s_K h_K\right]g_n^M(\{is_K/\beta\}). \tag{C4}$$

Equation (2.1) then becomes

$$g_n(\{\sigma_K\})=\mathcal{N}^{-1}\int dJ\,\rho(J)\int\prod_K du_K P^{(M)}(\{u_K\})\mathrm{Tr}_{\tau_\alpha}\exp\left[\beta\sum_K\tau_K u_K\right]\exp\left[\beta J\sum_\alpha \tau_\alpha\sigma_\alpha\right]. \tag{C5}$$

Using (C2) we have

$$\exp\left[\beta J\sum_\alpha\sigma_\alpha\tau_\alpha\right]=(\cosh\beta J)^n\sum_r\tanh^r\beta J\sum_{(\alpha_1,\ldots,\alpha_r)}\sigma_{\alpha_1}\cdots\sigma_{\alpha_r}\tau_{\alpha_1}\cdots\tau_{\alpha_r}, \tag{C6}$$

but

$$\exp\left[\beta\sum_K\tau_K u_K\right]=\exp\left[\beta\sum_\alpha\tau_\alpha u_\alpha\right] \tag{C7}$$

with $u_\alpha=u_K$ for $\alpha=(K\gamma)$. Hence using (C2) again

$$\mathrm{Tr}_{\tau_\alpha}\tau_{\alpha_1}\cdots\tau_{\alpha_r}\exp\left[\beta\sum\tau_K u_K\right]=\prod_{\alpha=1}^{n}(2\cosh\beta u_\alpha)\prod_{(\alpha_1,\ldots,\alpha_r)}\tanh\beta u_\alpha. \tag{C8}$$

Therefore

$$\begin{aligned}\mathrm{Tr}_{\tau_\alpha}\exp\left[\beta\sum_K\tau_K u_K\right]\exp\left[\beta J\sum_K\tau_\alpha u_\alpha\right]&=(\cosh\beta J)^n\prod_\alpha(2\cosh u_\alpha)\sum_r\tanh^r\beta J\sum_{(\alpha_1,\ldots,\alpha_r)}\tanh\beta u_{\alpha_1}\cdots\tanh\beta u_{\alpha_r}\sigma_{\alpha_1}\cdots\sigma_{\alpha_r}\\&=(\cosh\beta J)^n\prod_\alpha(2\cosh\beta u_\alpha)\exp\left[\sum_\alpha\sigma_\alpha\tanh^{-1}(\tanh\beta J\tanh\beta u_\alpha)\right.\\&\qquad\left.+\tfrac{1}{2}\sum_\alpha\ln(1-\tanh^2\beta J\tanh^2\beta u_\alpha)\right]\\&=\exp\sum_K\sigma_K\tanh^{-1}(\tanh\beta J\tanh\beta u_K)\exp\left[\frac{m}{2}\sum_K\ln(\cosh^2\beta J\cosh^2\beta u_K\right.\\&\qquad\left.-\sinh^2\beta J\sinh^2\beta u_K)\right],\end{aligned} \tag{C9}$$

where we have used Eq. (C1). From (C5) and (C9) we finally obtain Eq. (3.10). Equation (3.1) follows as a special case by choosing $m=n$ and $K=1$ and taking the limit $n\to 0$.

APPENDIX D

In this appendix we explain some of the details leading to (3.20) and to the formula (3.21) which represent the ground-state energy density. The identities (3.20) are derived with the aid of the formula

$$\langle (is_{K_0})^l \rangle = \left[\int \prod_K dh_K \exp(\mu|h_K|) \frac{d^l}{dh_{K_0}^l} \int \prod_K \frac{ds_K}{2\pi} \exp\left[i\sum h_K s_K \right] \gamma^M(\{is_K\}) \right] \left[\int \prod_K \frac{ds_K}{2\pi} \prod_K b_K(s_K) \gamma^M(\{is_K\}) \right]^{-1},$$

$$= (-1)^l \left[\int \prod_K dh_K \frac{d^l}{dh_{K_0}^l} \exp(\mu|h_{K_0}|) \prod_{K\neq K_0} \exp(\mu|h_K|) \int \prod_K \frac{ds_K}{2\pi} \exp\left[i\sum h_K s_K \right] \gamma^K(\{is_K\}) \right]$$

$$\times \left[\int \prod_K \frac{ds_K}{2\pi} \prod_K b_K(s_K) \gamma^M(\{is_K\}) \right]^{-1}. \quad \text{(D1)}$$

The various derivatives of $\exp(\mu|h|)$ can be easily calculated, e.g.,

$$\frac{d}{dh}\exp(\mu|h|) = \mu\, \mathrm{sgn} h\, \exp(\mu|h|), \quad \text{(D2)}$$

$$\frac{d^2}{dh^2}\exp(\mu|h|) = [2\mu\delta(h) + \mu^2 \mathrm{sgn}^2 h]\exp(\mu|h|). \quad \text{(D3)}$$

To derive Eq. (3.21) we parametrize γ in the following way:

$$\gamma = 1 + \frac{1}{M}\eta_2 + \frac{1}{M\sqrt{M}}\eta_3 + \frac{1}{M^2}\eta_4 + \cdots, \quad \text{(D4)}$$

$$\gamma^M = \gamma_0^{(M)} \left[1 + \frac{1}{\sqrt{M}}\chi_1 + \frac{1}{M}\chi_2 + \cdots \right], \quad \text{(D5)}$$

$$\gamma^{M+1} = \gamma_0^{(M)} \left[1 + \frac{1}{\sqrt{M}}\chi_1 + \frac{1}{M}\chi_2 + \frac{1}{M}\eta_2 + \cdots \right]. \quad \text{(D6)}$$

To calculate the free energy we have used Eq. (2.16)

$$\ln \mathrm{Trg}^M = \ln \int \prod_K \frac{ds_K}{2\pi} \prod_K b_K \gamma_0^{(M)} + \frac{1}{\sqrt{M}}\langle \chi_1 \rangle_0 + \frac{1}{M}\langle \chi_2 \rangle_0 - \frac{1}{2}\frac{1}{M}\langle \chi_1 \rangle_0^2 + \cdots, \quad \text{(D7)}$$

the last term being of order n^2 and can be dropped. Thus

$$\frac{1}{2n}(\ln \mathrm{Trg}^M + \ln \mathrm{Trg}^{M+1}) = \frac{1}{n}\ln \int \prod_K \frac{ds_K}{2\pi} \prod_K b_K \gamma_0^{(M)} + \frac{1}{n}\frac{1}{\sqrt{M}}\langle \chi_1 \rangle_0 + \frac{1}{nM}\langle \chi_2 \rangle_0 + \frac{1}{n}\frac{1}{2M}\langle \eta_2 \rangle_0. \quad \text{(D8)}$$

Also

$$\frac{M}{2n}\ln\frac{\mathrm{Trg}^M g}{\mathrm{Trg}^M} = \frac{M}{2n}\ln\left[1 + \frac{1}{M}\langle \eta_2 \rangle + \frac{1}{M\sqrt{M}}\langle \eta_3 \rangle + \frac{1}{M^2}\langle \eta_4 \rangle + \cdots \right]$$

$$= \frac{1}{2n}\langle \eta_2 \rangle + \frac{1}{2n}\frac{1}{\sqrt{M}}\langle \eta_3 \rangle + \frac{1}{2n}\frac{1}{M}\langle \eta_4 \rangle + \cdots; \quad \text{(D9)}$$

combining these relations and using the identities (3.20) leads to Eq. (3.21).

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