Dynamics of sine-Gordon solitons under random perturbations: Weak additive large-scale white noise

Peter Biller and Francesco Petruccione
Fakultät für Physik der Universität Freiburg, Hermann-Herder-Strasse 3, D-7800 Freiburg im Breisgau, Federal Republic of Germany
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We test the soundness of stochastic perturbation theory in the case of large-scale random perturbations of sine-Gordon solitons. To this end both the stochastic sine-Gordon equation and the stochastic perturbation theory equations are solved numerically. For the former we develop an explicit numerical scheme to integrate randomly perturbed nonlinear wave equations. The method is shown to work very efficiently, allowing a substantial improvement in the quality of the simulations. In the case under study, the predictions of the perturbation analysis are in good agreement with the numerical computations of the sine-Gordon equation.

I. INTRODUCTION

The dynamics of solitons under random perturbations is a topic of great interest in various areas of physics. These interests range from concrete applications in solid-state physics to more speculative approaches in elementary particle physics.[1] The development of reliable calculation tools is therefore an essential preliminary task. As a consequence of the nonlinear nature of the equations that show soliton solutions one has to rely upon approximation methods. In the excellent review by Bass *et al.*[2] the state of the art in perturbation theory has been summed up. Unfortunately the only criterion to test the validity of an approximation is to compare it with exact results. These can in principle be obtained by numerical means. To numerically solve a randomly perturbed nonlinear wave equation one has to generate an ensemble of realizations of the stochastic process under consideration and evaluate the properties of interest as ensemble averages.

There has already been some pioneering numerical work by Vazquez and collaborators[3,4] in studying the dynamics of sine-Gordon solitons under random perturbations. Their numerical integration uses an implicit finite difference scheme that turns out to be rather time consuming. As a consequence, their numerical approach does not allow a definite answer on the validity of the approximations, due to the insufficient statistics of their numerical simulations. Moreover, the stochastic perturbation equations are solved only in a limiting case. The comparison of the numerical results for the sine-Gordon equation and the predictions of perturbation theory is thus restricted to this case.

In this paper we present an alternative numerical approach based on well-established methods in the integration of nonlinear deterministic wave equations and stochastic differential equations. The proposed explicit algorithm is shown to work much more efficiently than the implicit one used by Vazquez *et al.*[3,4] We use the algorithm to investigate the dynamics of a sine-Gordon soliton with and without dissipation under an additive large-scale random noise. By also solving numerically the stochastic differential equations arising in stochastic perturbation theory a definite judgement on the soundness of this approximate approach is possible. A further advantage of the suggested numerical algorithm is that it can easily be implemented for other types of perturbation or other nonlinear wave equations.

Many interesting applications in condensed matter physics can be formulated with the help of stochastic differential equations. The perturbed sine-Gordon equation describes, for instance, the propagation of nonlinear waves in a semiconductor with a superlattice in which there are deviations from periodicity.[5] Another example is the propagation of magnetic flux in a Josephson junction in the presence of contact inhomogeneities.[6]

The present work concentrates on the formulation and the test of the new numerical approach. As a case study we investigate the most simple case, namely the sine-Gordon equation with additive large-scale white noise. The application to more concrete physical problems are planned to be handled in forthcoming papers.

In Sec. II we sum up some analytical and perturbation theory results for a sine-Gordon soliton under weak larger-scale stochastic perturbation. In Sec. III we describe in detail the new numerical algorithm. Then we present the numerical results and compare them with the perturbation theory in Sec. IV. A short summary concludes the paper.

II. THEORY

The system we are going to investigate is supposed to be governed by the perturbed sine-Gordon equation for the real field $\Phi(x,t)$

$$\Phi_{tt}-\Phi_{xx}+\sin\Phi+\alpha\Phi_t+F=0\ , \tag{1}$$

where $F(x,t)$ is a random perturbation and $\alpha\Phi_t$ represents a dissipation term (the lower indices t and x are symbols for the partial derivatives in time and space).

It is assumed here that the space correlation of the noise is of a large scale compared to the soliton width. In this limit the noise can be treated as independent of the space variable x, i.e., $F(x,t)=F(t)$. The correlation in time is chosen to be that of Gaussian white noise with vanishing average

$$\langle F(t)\rangle=0 \tag{2}$$

and two-time covariance function

$$\langle F(0)F(t)\rangle=2D\delta(t) . \tag{3}$$

In the absence of both dissipation and random perturbation Eq. (1) has the following soliton-antisoliton solutions:

$$\Phi_{\pm}=4\arctan\{\exp[\pm\gamma(x-X)]\} , \tag{4}$$

where $\gamma=(1-U^2)^{-1/2}$. U is the velocity of the unperturbed soliton and $X=X_0+Ut$ locates its center.

Let us briefly recapitulate some results of the stochastic perturbation analysis approach to Eq. (1).[2] In order to characterize the solution of Eq. (1) one introduces the energy E and the momentum P which are defined as

$$E=\int_{-\infty}^{\infty} dx\,\varepsilon , \tag{5}$$

where

$$\varepsilon=\tfrac{1}{2}\Phi_t^2+\tfrac{1}{2}\Phi_x^2+(1-\cos\Phi) , \tag{6}$$

and

$$P=-\int_{-\infty}^{\infty} dx\,\Phi_x\Phi_t . \tag{7}$$

With the help of the energy E and of the momentum P one defines the quantities X and U as

$$X=\int_{-\infty}^{\infty} dx\,x\varepsilon/E , \tag{8}$$

and

$$U=P/E . \tag{9}$$

For a soliton solution X is the location of the center of mass and U is the velocity of the soliton. The idea of the perturbation analysis[3,7] is to assume that the major effect of the perturbations consisting of dissipation and random noise is to modulate the center and the velocity of the soliton, while its shape is preserved. The natural ansatz for Φ is thus

$$\Phi=4\arctan(\exp\{\gamma(t)[x-X(t)]\}) , \tag{10}$$

with $\gamma(t)=[1-U(t)^2]^{-1/2}$. An additional ansatz is made for Φ_t,

$$\Phi_t=-2U(t)\gamma(t)\operatorname{sech}\{\gamma(t)[x-X(t)]\} . \tag{11}$$

Comparing Eq. (11) with the time derivative of Eq. (10) one observes that a term proportional to $\dot{U}(t)$ is omitted in Eq. (11). This term is zero for the nondissipative case $\alpha=0$. With this ansatz it is possible to deduce a set of closed ordinary stochastic differential equations for the introduced soliton parameters $X(t)$ and $U(t)$. In the case of small additive noise with a scale much larger than the soliton width the perturbation analysis leads to the following equations:[7]

$$\dot{U}=-\alpha U(1-U^2)+\frac{\pi}{4}(1-U^2)^{3/2}F(t) , \tag{12}$$

$$\dot{X}=U . \tag{13}$$

In Eq. (12) the term with multiplicative noise has to be interpreted in the Stratonovich sense.[8] The nonlinear ordinary stochastic differential equations (12) and (13) are easier to be handled than the stochastic partial differential equation (1). The two equations (12) and (13) can be solved analytically in the limit of small velocity $U\ll 1$.[3] For an arbitrary velocity U the equivalent Fokker-Planck equation was treated in a WKB approximation in Ref. 8. In the present paper we solve Eqs. (12) and (13) numerically. To this end they are first written down in the equivalent Ito interpretation[9]

$$\dot{U}=-\alpha U(1-U^2)-\tfrac{3}{16}\pi^2 DU(1-U^2)^2 + \frac{\pi}{4}(1-U^2)^{3/2}F(t) , \tag{14}$$

$$\dot{X}=U . \tag{15}$$

The numerical simulation of these equations as well as that of the stochastic sine-Gordon equation is described in the next section.

III. SIMULATION

A. The stochastic sine-Gordon equation

Introducing the field $\psi(x,t)=\Phi_t(x,t)$, Eq. (1) is equivalent to the system of equations

$$\Phi_t=\psi , \tag{16}$$

$$\psi_t=\Phi_{xx}-\sin\Phi-\alpha\psi+F . \tag{17}$$

Let us first discuss the simulation of the stochastic sine-Gordon equation (1) for the case $\alpha=0$, i.e., vanishing dissipation. For this case equations (16) and (17) are discretized as follows:

$$\Phi_i^{n+1}=\Phi_i^n+\psi_i^{n+1/2}\Delta t , \tag{18}$$

$$\psi_i^{n+1/2}=\psi_i^{n-1/2}+(\Phi_{i+1}^n-2\Phi_i^n+\Phi_{i-1}^n)\Delta t/(\Delta x)^2 - \sin[\tfrac{1}{2}(\Phi_{i+1}^n+\Phi_{i-1}^n)]\Delta t+\eta^n(\Delta t)^{1/2} . \tag{19}$$

Here and in the following the upper index denotes the time discretization, while the lower one represents the space discretization. Δt is the time step and Δx the space step. The quantities η^n in Eq. (19) are Gaussian random numbers with the correlations

$$\langle\eta^n\rangle=0 , \tag{20}$$

$$\langle\eta^n\eta^m\rangle=2D\delta_{nm} . \tag{21}$$

Note that in the above equations there appears no lower index on η^n because the random number is the same for all space points for the large-scale noise considered in this work. Eliminating the quantities $\psi_i^{n\pm1/2}$ in (18) and (19) one obtains the following explicit scheme:

$$\Phi_i^{n+1}=2\Phi_i^n-\Phi_i^{n-1}+(\Phi_{i+1}^n-2\Phi_i^n+\Phi_{i-1}^n)(\Delta t/\Delta x)^2 -\sin[\tfrac{1}{2}(\Phi_{i+1}^n+\Phi_{i-1}^n)](\Delta t)^2+\eta^n(\Delta t)^{3/2} . \quad (22)$$

We consider first the deterministic case $F\equiv 0$ (i.e., $\eta^n\equiv 0$). Then Eq. (22) is a slight modification of the well-known leapfrog scheme.[10] One can prove that for the choice $\Delta x=\Delta t$ this scheme is stable. To this end it is essential that the nonlinear sine term is not evaluated at Φ_i^n but at $\frac{1}{2}(\Phi_{i+1}^n+\Phi_{i-1}^n)$. For the stochastic case we then obtain with $\Delta x=\Delta t$

$$\Phi_i^{n+1}=\Phi_{i+1}^n+\Phi_{i-1}^n-\Phi_i^{n-1} -\sin[\tfrac{1}{2}(\Phi_{i+1}^n+\Phi_{i-1}^n)](\Delta t)^2+\eta^n(\Delta t)^{3/2} . \quad (23)$$

We remark that the grid consists of two independent subgrids, one with $n+i$ even, the other with $n+i$ odd. This structure of the algorithm is conserved if we approximate the additional dissipation term $-\alpha\psi$ in Eq. (17) as $-\alpha(\psi_i^{n+1/2}+\psi_i^{n-1/2})/2$ which leads to an additional term $-\alpha(\Phi_i^{n+1}-\Phi_i^{n-1})\Delta t/2$ in Eq. (23). Note that then the term Φ_i^{n+1} appears on both sides of the discretized evolution equation. Solving for this quantity one ends up with the scheme

$$\Phi_i^{n+1}=\{\Phi_{i+1}^n+\Phi_{i-1}^n-\Phi_i^{n-1}-\sin[\tfrac{1}{2}(\Phi_{i+1}^n+\Phi_{i-1}^n)](\Delta t)^2+\alpha\Phi_i^{n-1}\Delta t/2+\eta^n(\Delta t)^{3/2}\}/(1+\alpha\Delta t/2) . \quad (24)$$

This explicit scheme is much faster than the implicit Vazquez-Strauss algorithm.[3,11] Furthermore, only one random number is used at each time step. Where we take η^n in Eq. (24) the authors of Ref. 3 use $\frac{1}{2}(\eta^{n+1}+\eta^{n-1})$. This may lead to undesired correlations in the perturbations because the stochastic increments in the equations for Φ^n and $\Phi^{n\pm 2}$ are not independent.

The discretized versions of the energy, momentum, velocity, and center of the soliton are chosen to be

$$E^n=\sum_i\left[\frac{1}{8\Delta t}(\Phi_i^{n+1}-\Phi_i^{n-1})^2+\frac{1}{8\Delta t}(\Phi_{i+1}^n-\Phi_{i-1}^n)^2+\{1-\cos[\tfrac{1}{2}(\Phi_{i+1}^n+\Phi_{i-1}^n)]\}\Delta t\right] , \quad (25)$$

$$P^n=-\sum_i\frac{1}{4\Delta t}(\Phi_i^{n+1}-\Phi_i^{n-1})(\Phi_{i+1}^n-\Phi_{i-1}^n) , \quad (26)$$

$$U^n=P^n/E^n , \quad (27)$$

$$X^n=\frac{1}{E^n}\sum_i x_i\left[\frac{1}{8\Delta t}(\Phi_i^{n+1}-\Phi_i^{n-1})^2+\frac{1}{8\Delta t}(\Phi_{i+1}^n-\Phi_{i-1}^n)^2+\{1-\cos[\tfrac{1}{2}(\Phi_{i+1}^n+\Phi_{i-1}^n)]\}\Delta t\right] . \quad (28)$$

In the Vazquez-Strauss scheme of Ref. 3 the discretized energy is exactly constant for $\alpha\equiv 0$ and $F\equiv 0$. For the same parameters it is well known that in the modified leapfrog scheme the energy is nearly constant within very small errors. This is also confirmed in our simulations.

Equation (24) is the most important equation of this section. It allows the determination of the quantities Φ_i^{n+1} if all the quantities Φ_i^n and Φ_i^{n-1} are known. The simulation is started with a soliton of the form as in Eq. (4) with appropriate initial conditions U_0 and X_0 for U and X. This fixes Φ_i^0 and Φ_i^1 ($i=1,2\ldots$). The iteration of Eq. (24) leads to a random trajectory which is one realization of the stochastic process. The simulation is stopped at the final time T. A large number of trajectories has to be generated and the interesting quantities are evaluated as mean values over all realizations, for example

$$\langle U^n\rangle=\frac{1}{N}\sum_{j=1}^{N}U_j^n , \quad (29)$$

where j labels the realizations and N is the total number of realizations. As a measure for the statistical errors we choose the root-mean-square deviations from the mean values, for example

$$\langle\Delta U^n\rangle=\left[\frac{1}{N}[\langle(U^n)^2\rangle-\langle U^n\rangle^2]\right]^{1/2} . \quad (30)$$

It is well known[12] that the numerically evaluated moments of a stochastic differential equation depend on the chosen time step Δt. This dependence is predicted to be linear for small time steps for the first order Euler algorithm we use in Eq. (19). For this reason all the runs are repeated with different time steps. The final results with their errors are obtained by a linear extrapolation to $\Delta t=0$.[12]

Finally, we give some quantitative information concerning our simulations. Following Ref. 3, the dissipation constant is chosen to be $\alpha=0$ or $\alpha=0.1$ and the diffusion constant D is $D=1.25\times 10^{-6}$ in all runs. For this small value it turns out that the distortion of the soliton shape and the generation of a phonon field are negligible. All trajectories were simulated until the final time $T=12.5$. The space interval was $[-30,30]$ and the boundary conditions were simulated as $\Phi(\pm 30,t)=0$. In contrast to the simulations in Ref. 3 the noise was not localized in a part of the interval but "active" in the whole interval. The time steps were chosen to be $\Delta t=0.03$, 0.05, and 0.075. The initial conditions $U_0=0.1$ ($X_0=-0.6$), $U_0=0.5$ ($X_0=-3$), and $U_0=0.9$

($X_0=-5.4$) were investigated. Due to the simple explicit scheme a large number of trajectories can be simulated. All our runs were done in double precision for a statistics of 2000 trajectories. For the time step $\Delta t=0.05$ this requires approximately one hour of central processing unit (CPU) time on an IBM 3090. We stress that in Ref. 3, where the implicit scheme was used, the results were obtained by averaging over an ensemble of only 30 trajectories. As a consequence the statistical errors are very large so that a comparison of the data with the approximate perturbation results is possible only qualitatively.

B. The perturbation analysis equations

In the perturbation analysis approach one has to simulate the ordinary coupled stochastic differential equations (14) and (15). As was already pointed out these equations were solved analytically in the limit $U\to 0$.[3] Here they are numerically solved for arbitrary U. The discretized version of the Ito stochastic differential equation is

$$X^{n+1}=X^n+U^n\Delta t\ , \tag{31}$$

$$U^{n+1}=U^n-\alpha U^n[1-(U^n)^2]\Delta t - \tfrac{3}{16}\pi^2 D U^n[1-(U^n)^2]^2\Delta t + \frac{\pi}{4}[1-(U^n)^2]^{3/2}\eta^n(\Delta t)^{1/2}\ . \tag{32}$$

The quantities η^n are Gaussian random numbers with the correlations given by Eqs. (20) and (21). The simulation starts with $U^0=U_0$ and $X^0=X_0$ and is integrated in exactly the same way as described in Sec. III A. For the time step $\Delta t=0.05$ one can simulate 150 000 trajectories in about 15 min CPU time on an IBM 3090. The relative errors of the data are negligible and are thus ignored in the figures.

IV. RESULTS

Let us begin with the center of mass position $X(t)$. The mean value of this quantity can be seen in Fig. 1 for two different initial conditions U_0 (with a corresponding X_0) both with and without dissipation. The solid lines are the approximate perturbation results. For all curves the results of the sine-Gordon equation lie exactly on the approximation curves with negligible error bars. The linear curves for $\alpha=0$ indicate that the velocity of the soliton is nearly constant. This can more clearly be seen in Fig. 2 where the mean velocity $\langle U(t)\rangle$ is plotted. For the case of dissipation the function $\langle U(t)\rangle$ is decaying and the perturbation results are practically identical to the results obtained from the stochastic sine-Gordon equation. For $\alpha=0$ the stochastic perturbation theory predicts a very small decrease of the velocity [see Eq. (14) or Ref. 8]. This trend is also suggested by our numerical data but the statistical errors are too great to obtain an accurate comparison.

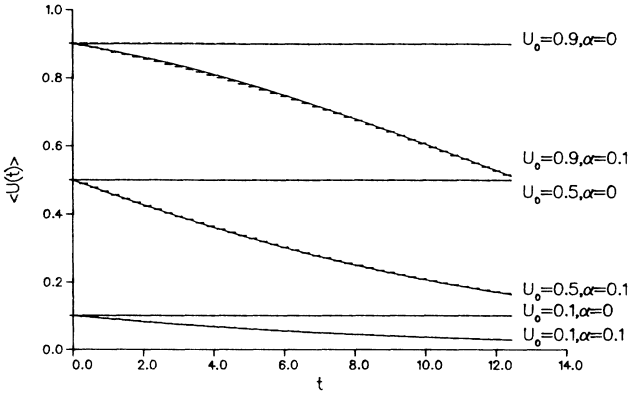

FIG. 2. The mean velocity $U(t)$ vs time for different initial conditions with and without dissipation.

After the discussion of the mean values of the functions $X(t)$ and $U(t)$ let us next turn to the variances of these quantities. In Fig. 3 one can see the quantity $\langle[U(t)-\langle U(t)\rangle]^2\rangle$ for the initial velocity $U_0=0.1$ with and without dissipation. The approximate results (solid lines) fit the exact results (error bars) well. This is also the case for the initial condition $U_0=0.5$ in Fig. 4. For $U_0=0.9$, on the other hand, the agreement is only good

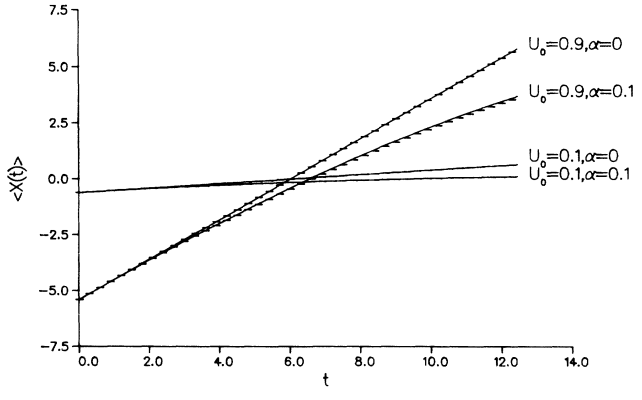

FIG. 1. The mean position of the center of mass $X(t)$ vs time t for different initial conditions with and without dissipation.

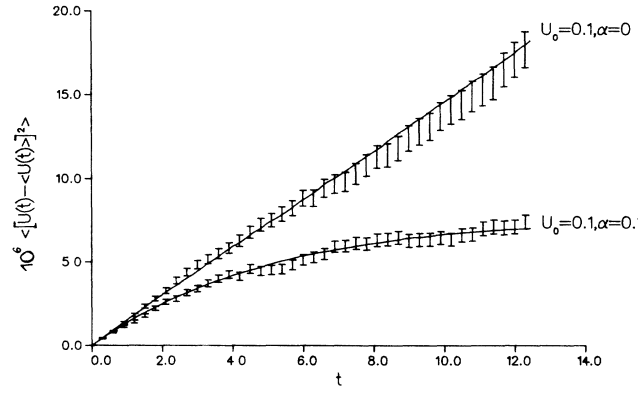

FIG. 3. The function $\langle[U(t)-\langle U(t)\rangle]^2\rangle$ vs time for the initial velocity $U_0=0.1$ with and without dissipation.

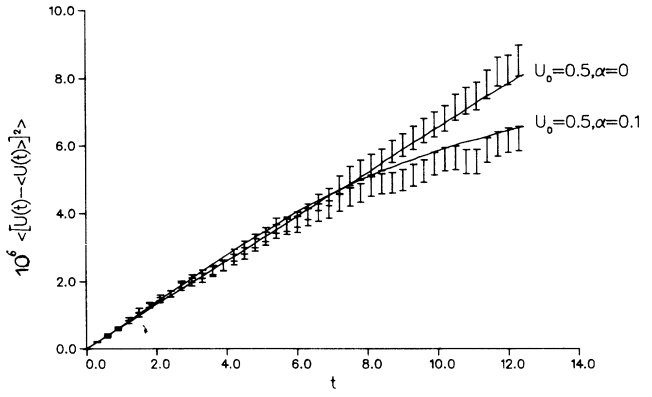

FIG. 4. The function $\langle [U(t)-\langle U(t)\rangle]^2\rangle$ vs time for the initial velocity $U_0=0.5$ with and without dissipation.

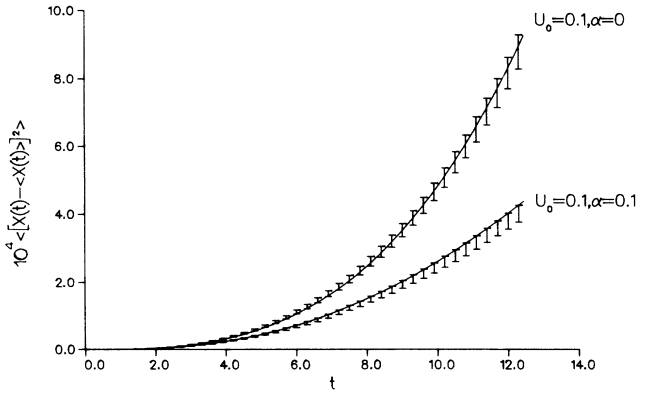

FIG. 6. The function $\langle [X(t)-\langle X(t)\rangle]^2\rangle$ vs time for the initial velocity $U_0=0.1$ with and without dissipation.

in the case of vanishing dissipation $\alpha=0$ as can be seen in Fig. 5. For the curve with $\alpha=0.1$ there is a region for intermediate times where the perturbative approximation gives values slightly different from the exact numerical results of the stochastic sine-Gordon equation (1). This is possibly due to the fact that the term proportional to $\dot{U}$ is ignored in the ansatz for Φ_t in Eq. (11). This term occurs only in the dissipation case $\alpha\neq 0$.

The same behavior is found for the quantity $\langle [X(t)-\langle X(t)\rangle]^2\rangle$. The agreement of exact and approximate results is fairly good for the small initial velocity $U_0=0.1$ in Fig. 6 and the intermediate initial velocity $U_0=0.5$ in Fig. 7. However, there is a clear mismatch of the curves for the approximate and the exact theory for the initial velocity $U_0=0.9$ for the case of dissipation, as can be seen in Fig. 8.

Summing up, one can say that the stochastic perturbation theory leads to very good results for weak additive large-scale noise especially in the case of no dissipation. The assumption that the soliton structure is preserved and only the parameters X and U are modulated by the perturbations is thus very good. For large initial velocities and dissipation the perturbation theory results differ slightly from the exact numerical results of the stochastic sine-Gordon equation (1). Note that such a conclusion could not be drawn in Ref. 3 because there the error bars are of the order 10 times larger than in our calculations.

V. SUMMARY AND OUTLOOK

Nonlinear wave equations with random perturbations can be treated analytically only in the general frame of a perturbation theory. As this topic is of great interest in many areas of physics it is important to know the soundness of the approximations. To this end reliable numerical methods to integrate the exact equations are unavoidable.

In this paper, we presented an explicit numerical scheme to integrate the perturbed sine-Gordon equation. The method we suggested can easily be implemented on a computer. It allows a more efficient computation of the trajectories of the underlying stochastic process. In this

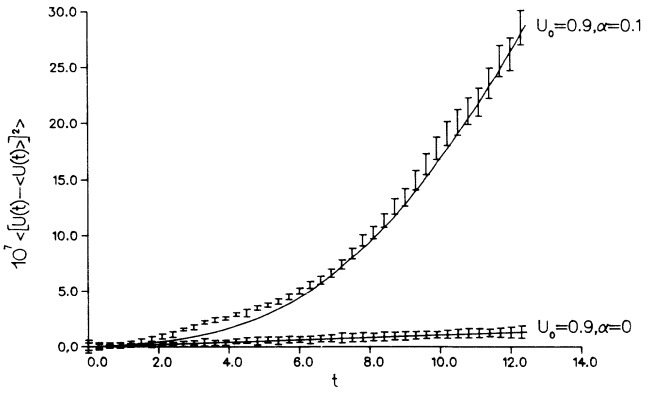

FIG. 5. The function $\langle [U(t)-\langle U(t)\rangle]^2\rangle$ vs time for the initial velocity $U_0=0.9$ with and without dissipation.

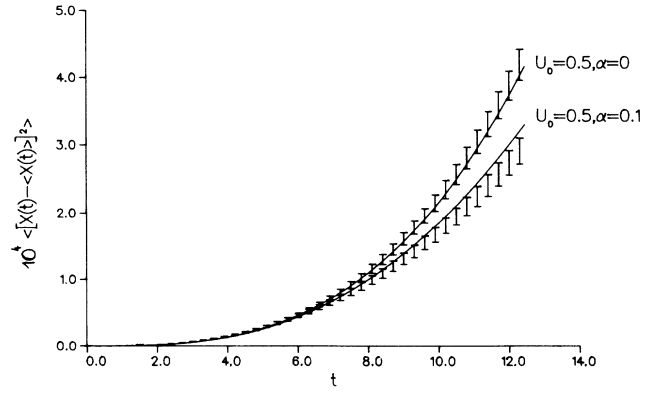

FIG. 7. The function $\langle [X(t)-\langle X(t)\rangle]^2\rangle$ vs time for the initial velocity $U_0=0.5$ with and without dissipation.

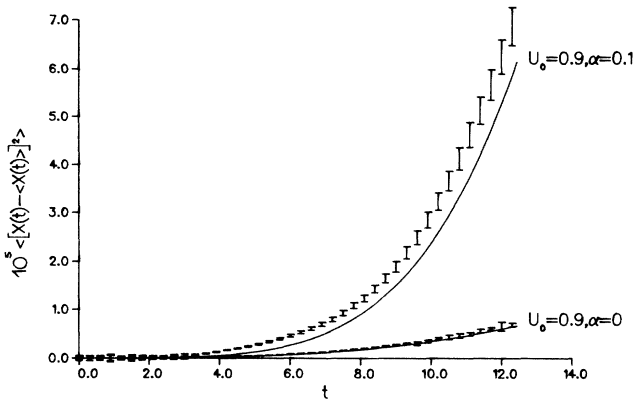

FIG. 8. The function $\langle [X(t)-\langle X(t)\rangle]^2\rangle$ vs time for the initial velocity $U_0=0.9$ with and without dissipation.

way much better statistics can be obtained, compared to previous approaches. Our method is well suited for both large-scale and small-scale additive and multiplicative random perturbations and can be applied to other nonlinear wave equations.

With the new explicit algorithm we investigated the dynamics of sine-Gordon solitons under large-scale weak additive random perturbations. We compared the numerical results with the predictions of perturbation theory. In the cases under consideration the agreement was found to be very good. Due to the good statistics of our simulation we are also able to show the limits of the perturbation approach.

As we already pointed out, the algorithm is very flexible and allows also the investigation of less trivial perturbations. Work is in progress to study the effects of multiplicative and small-scale noise.

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