Critical properties of a dilute gas of vortex rings in three dimensions and the λ transition in liquid helium

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We compute the critical properties of a dilute gas of circular and elliptical vortex rings in a three-dimensional superfluid using a scale-dependent mean-field approximation of the Kosterlitz-Thouless type. A set of scaling differential equations is derived and solved numerically. The system has a second-order phase transition with superfluid density critical exponent $\nu=0.57$ and specific-heat critical exponent $\alpha=0.64$ for circular rings and $0.50<\nu<0.53$, $0.94<\alpha<0.99$ for elliptical rings of small eccentricity. The relevance of these results for the understanding of the λ transition in liquid helium is discussed.

INTRODUCTION

Vortex-unbinding phase transitions of the type first discussed by Kosterlitz and Thouless[1] have been highly successful in the description of the critical properties of two-dimensional superfluids.[2] In three dimensions, it is easy enough to set up the analogous problem: Compute the thermodynamical properties of a system of loops, with interaction energy given by

$$H=\frac{\rho}{8\pi}\sum_{i,j}\Gamma_i\Gamma_j\oint\oint\frac{dl_i dl_j}{|\mathbf{X}_i-\mathbf{X}_j|} . \tag{1}$$

Of course, this is easier said than done since the phase space for the system is the space of all closed curves ("strings") in three dimensions, with a far from trivial interaction. Numerical simulations have been attempted,[3] and they suggest that vortex filaments do play a crucial role at criticality. A different approach has been followed by Williams,[4] in which the phase space is drastically reduced by considering vortex filaments of circular shape only. Here, each state of a given loop is labeled not by an infinite number of parameters but by six: three for position, two for orientation, and one, radius, for size. An exact renormalization-group calculation is still not possible, but a scale-dependent mean-field approach can be successfully completed, and Williams obtained a value $\nu=0.53$ for the superfluid density critical exponent ν. In this paper we follow an approach that is similar in spirit to that of Williams but differs in quantitative detail. In this way we improve the computation of ν to $\nu=0.57$. We are also able to compute the specific-heat critical exponent $\alpha=0.64$. In addition we redo everything for vortex rings of elliptical shape with varying small eccentricity, obtaining $0.50<\nu<0.53$ and $0.94<\alpha<0.99$.

The energy associated with an isolated vortex ring of radius R and circulation Γ, both in classical and quantum fluids, is

$$U_0(R)=\frac{\rho\Gamma^2 R}{2}[\ln(R/\tau)+C] , \tag{2}$$

where ρ is the liquid density and C a constant whose exact value depends on what goes on inside the vortex core, of radius τ. For a superfluid, the value $C=0.464$ has been proposed,[5] and expression (2) is valid for a thin vortex ring, that is, for $R\gg\tau$. As we are interested in the critical behavior of the system, what goes on at a microscopic scale should not matter and the precise values of τ and C should be irrelevant as far as the universal quantities are concerned. A system of noninteracting vortex rings will clearly not have a phase transition, as the energy that it costs to get one grows monotonically with R. Interactions have to be taken into account, and now we follow the ideas of Kosterlitz and Thouless,[1] as explained in particular by Young,[6] whereby this interaction is approximately considered by way of a scale-dependent screening. This makes sense, since a vortex ring behaves in many respects as a magnetic dipole, and we expect that the self-interaction of a large ring will be screened by smaller ones, whose creation will be facilitated by the presence of the larger one, and one might expect that as in two dimensions, it will cost a finite energy to form infinitely large rings. It has also been noted[7] that for a hypothetical system of vortices whose energy was proportional to length, allowing for arbitrarily shaped vortices would mean that at a finite temperature the gain in entropy would dominate over the cost in internal energy allowing for infinitely large vortices to occur.

The basic physics we wish to emphasize is that, just as in two dimensions, a three-dimensional superfluid is characterized by a condensate wave function having variations both in amplitude and phase. The latter may be multiple valued, in which case singular regions appear: vortex points in two dimensions and closed vortex lines in three. As already mentioned, a vortex line of finite length costs (for a nonzero core cutoff) a finite amount of energy to form and, because of thermal excitation, there will be a nonzero population of such lines at any finite temperature T, with a distribution of lengths that will depend on T. As the temperature is increased the mean size of the vortex loops will also increase; a large loop will polarize the medium lowering the energy that it costs to form a smaller one, and these, in turn, will screen the self-energy of the former making it easier and easier to form longer and longer vortex loops. In this paper we show that there is a finite critical temperature T_c at which infinitely long vortex loops occur. The appearance of closed vortex lines of infinite length (or, more precisely, bounding an area of the size of the system) signals the disappearance of superfluidity, of course, because the phase of the order parameter can now change by multiples of 2π rendering the flow states unstable.

SCALE-DEPENDENT MEAN-FIELD APPROACH TO INTERACTING VORTEX RINGS

We wish to approximate the properties of a system of interacting vortex rings by way of something like a dielectric constant. An ideal incompressible fluid of density ρ, velocity $\mathbf{v}$, and vorticity $\boldsymbol{\omega}$ is described by the equations

$$\nabla\cdot\mathbf{v}=0\ ,$$

$$\nabla\wedge\mathbf{v}=\boldsymbol{\omega}\ ,$$

and its energy is

$$E=\frac{\rho}{2}\int v^2 d\mathbf{r}\ .$$

These relations are clearly analogous to the magnetostatics equations

$$\nabla\cdot\mathbf{B}=0\ ,$$

$$\nabla\wedge\mathbf{B}=\frac{4\pi}{c}\mathbf{j}\ ,$$

$$E=\frac{1}{8\pi}\int B^2 d\mathbf{r}\ ,$$

for a magnetic field $\mathbf{B}$ and current $\mathbf{j}$. This allows for a visualization of the fluid as a magnetic system with field

$$\mathbf{B}=\sqrt{4\pi\rho}\,\mathbf{v}$$

and current

$$\mathbf{j}=c\sqrt{\rho/4\pi}\,\boldsymbol{\omega}\ .$$

We now compute the energy that it costs to form a vortex ring with velocity field $\mathbf{v}_1$ in a prescribed uniform external flow $\mathbf{v}_0$ that may be produced, for instance, by a much larger vortex ring. The total energy of the fluid is

$$E=\frac{\rho}{2}\int(\mathbf{v}_0+\mathbf{v}_1)^2 d\mathbf{r}$$

from which we extract the interaction energy

$$E_{\rm int}\approx\mathbf{v}_0\cdot\int\rho\mathbf{v}_1 d\mathbf{r}=\mathbf{v}_0\cdot\mathbf{P}\ ,$$

where $\mathbf{P}$ is the vortex momentum which, for a vortex filament parametrized as $\mathbf{r}=\mathbf{r}(\sigma)$, is given by[8]

$$\mathbf{P}=\frac{\rho\Gamma}{2}\oint\mathbf{r}(\sigma)\wedge\frac{d\mathbf{r}(\sigma)}{d\sigma}d\sigma\ .$$

Thus, within the magnetic analogy, the vortex momentum is related to the magnetic moment $\mathbf{m}$ of a current loop through

$$\mathbf{m}=\frac{\mathbf{P}}{4\pi\rho}\ .$$

There is, however, an important physical difference between the magnetic and fluid systems in the *sign* of the interaction. This is due to the absence in the fluid of electromagnetic induction effects. These effects are responsible for the fact that in the magnetic system, for fixed current intensity, the force acting along some generalized coordinate ξ is

$$F_\xi=+\frac{\partial E}{\partial\xi}\ ,$$

whereas in the fluid it has the usual form

$$F_\xi=-\frac{\partial E}{\partial\xi}\ .$$

In order to study screening effects we need a "diamagnetic constant," for which we need a magnetic susceptibility, for which in turn we need the magnetic polarizability of a vortex loop measuring its response to an external flow. This "magnetic polarizability" is

$$q=\frac{\partial}{\partial v}\left\langle\frac{\mathbf{m}\cdot\mathbf{v}}{v}\right\rangle\Bigg|_{v=0}\ ,$$

where the statistical average $\langle\ \rangle$ is taken over all possible vortex orientations. One then has, for a dilute gas of vortices,

$$q=\lim_{v\to 0}\frac{\int d\Omega\,\mathbf{m}\cdot\mathbf{v}\exp(-\beta\mathbf{P}\cdot\mathbf{v})}{v^2\int d\Omega\exp(-\beta\mathbf{P}\cdot\mathbf{v})}$$

$$=-\frac{4\pi}{3}\beta\rho m^2\ .$$

For a vortex ring of radius R and circulation Γ the "magnetic moment" is $m=\Gamma R^2/4$, giving for the scale-dependent "polarizability"

$$q(R)=-\frac{\pi}{12}\beta\rho\Gamma^2 R^4\ .$$

The negative sign indicates that, contrary to a system of current loops, vortices are diamagnetic: They tend to orient themselves with their magnetic moment pointing *opposite* to the external flow. The "magnetic susceptibility" is obtained by calculating $V^{-1}\sum_i q_i n_i$, where V is the volume of the system, i labels the vortex states, q_i is the polarizability of a vortex in the ith state, and n_i is the number of vortices in that state. The sum over states is carried out replacing $\sum_i$ by $(V/\tau^6)\int R^2 dR\,d\Theta$, with Θ a solid angle. The average number of vortices in a given state is $e^{-\beta U}$ where U is the energy that it costs to put the vortex in that state. Consequently, the susceptibility χ is given by

$$\chi(R)=\frac{4\pi}{\tau^6}\int_\tau^R R'^2 dR'\,q(R')e^{-\beta U(R')}\ .$$

Only vortices smaller than R contribute to the screening of a vortex of radius R. Since the energy of a thin isolated vortex ring is given by (2), it can be regarded as a string with a line tension

$$T_0(R)=\frac{dU_0}{dR}=\frac{\rho\Gamma^2}{2}[\ln(R/\tau)+C+1]$$

due to its self-interaction. If, however, many more vortices are present, this tension will be modified with a scale-dependent "diamagnetic constant" $\epsilon(R)$:

$$T_0(R)\rightarrow T(R)=\frac{T_0(R)}{\epsilon(R)} ,$$

and the energy of a vortex becomes

$$U(R)=\mu+\int_\tau^R T(R)dR$$
$$=\mu+\frac{\rho\Gamma^2}{2}\int_\tau^R\frac{\ln(R'/\tau)+C+1}{\epsilon(R')}dR' , \quad (3)$$

where μ is the energy needed to form a vortex of radius τ. Under the assumption of a dilute gas, the self-energy of a ring of radius R is affected only by smaller vortices, acting as a magnetic medium with linear response. Using then $\epsilon=1-4\pi\chi$ we have

$$\epsilon(R)=1+\frac{4\pi^3}{3}\beta\rho\Gamma^2\int_\tau^R dR'\left[\frac{R'}{\tau}\right]^6 e^{-\beta U(R')} , \quad (4)$$

which, together with (3), contain all the relevant information for our model.

SCALING EQUATIONS AND CRITICAL BEHAVIOR

We have derived a set of (coupled, integral) equations [(3),(4)] for the effective potential $U(R)$ and the "diamagnetic constant" $\epsilon(R)$. One relevant physical quantity is the superfluid density[1] $\rho_s=\epsilon(\infty)^{-1}$ as a function of temperature. To find it, one studies numerically the asymptotic behavior of ϵ as $R\rightarrow\infty$, and tries to see whether there is a temperature where this asymptotic behavior changes qualitatively, signaling a phase transition. If such a critical temperature exists, the next task is to compute the behavior of $\epsilon(\infty)$ near the critical temperature.

A property of (3),(4) that greatly simplifies the analysis is that it can be turned into a set of two coupled ordinary, nonlinear differential equations:

$$\frac{d\epsilon}{dR}=\frac{4\pi^3}{3}\beta\rho\Gamma^2\left[\frac{R}{\tau}\right]^6 e^{-\beta U(R)}$$
$$\frac{dU}{dR}=\frac{\rho\Gamma^2[\ln(R/\tau)+C+1]}{2\epsilon(R)} , \quad (5)$$

with initial conditions $U(R=\tau)=\mu$ and $\epsilon(R=\tau)=1$. The first initial condition is the energy needed to create a vortex of minimum size, and the second is the statement that for a vortex of minimum size there are no smaller vortices, and hence no screening. The critical behavior of (5) should not depend on the precise values of the parameters or on the precise values of the initial conditions. For a numerical solution of (5) we choose $\mu=1.39\rho\Gamma^2\tau$ as given by Jones and Roberts[9] on the basis of a Landau-Ginzburg theory for an isolated vortex. A considerable simplification in the numerical search for a critical point of (5) is obtained through the change of variables

$$K=\frac{\beta\rho\Gamma^2R[\ln(R/\tau)+C+1]}{12\epsilon(R)} ,$$
$$y=\frac{16\pi^3(R/\tau)^6e^{-\beta U(R)}}{\ln(R/\tau)+C+1} , \quad (6)$$
$$l=\ln(R/\tau) ,$$

that gives the new set of equations

$$\frac{dK}{dl}=K\left[1-Ky+\frac{1}{l+C+1}\right] ,$$
$$\frac{dy}{dl}=6y\left[1-K-\frac{1}{6(l+C+1)}\right] , \quad (7)$$

with initial conditions

$$K(l=0)=\frac{1}{12}\beta\rho\Gamma^2\tau(C+1) ,$$
$$y(l=0)=\frac{16\pi^3}{C+1}e^{-\beta\mu} , \quad (8)$$

which, upon elimination of the temperature β, turn into the single condition

$$y(l=0)=\frac{16\pi^3}{C+1}\exp\left[-\frac{12\mu K(l=0)}{\rho\Gamma^2\tau(C+1)}\right] .$$

We have solved numerically the system (7) using a fourth-order Runge-Kutta method with adaptive step size. The trajectories are shown in Fig. 1, and it may be easily observed that there is a sharp change in the qualitative behavior of the system at a critical value K_{0c} of $K(l=0)$. It is given by

$$K_{0c}=0.54222 .$$

So, our first conclusion is that a system of interacting

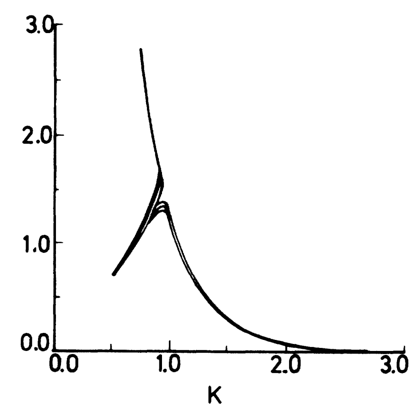

FIG. 1. Trajectories for Eqs. (7). There is a sharp transition at the critical value $K_{0c}=0.54222$.

vortex rings in three dimensions has a phase transition. The next task is to find the critical behavior. To do this, we define the critical temperature $t=1-(T/T_c)$ in terms of which one has

$$K_0=\frac{K_{0c}}{1-t} ,$$

and solve (7) for different values of K_0 in the vicinity of K_{0c}, corresponding to $t\in[10^{-3},10^{-7}]$. The resulting $\epsilon(\infty)$ as a function of reduced temperature t is adjusted to the curve

$$\ln\epsilon(\infty,t)=-\nu\ln t+\text{const}$$

by way of a linear least-squares fit. This procedure yields the critical exponent

$$\nu=0.57 .$$

For $t<0$ the quantity $\epsilon(\infty)$ diverges. As a check on the numerics, we redid the whole computation with the system (5) and the critical behavior was confirmed. A detailed discussion of the computational details, including numerical accuracy, has been given elsewhere.[10] As an additional check the scaling equations were solved with a different, well-tested, numerical algorithm[11] (PLOD) and the value of ν was confirmed. Both codes were also used to solve the scaling equations derived by Williams[4] and, although we obtain the same value as he for his $K_{0c}=0.174\,471\,449$, our computation gives a critical exponent for those equations of $\nu=0.50$, at variance with Williams's claim of $\nu=0.53$.

COMPUTATION OF THE SPECIFIC-HEAT EXPONENT

Near the transition temperature T_c the specific heat is given by

$$c_V\approx-\frac{1}{T_c}\frac{\partial^2\Omega}{\partial t^2} ,$$

where Ω is the thermodynamic potential per unit volume

$$\Omega(t)\approx-\frac{4\pi k_B T_c}{\tau^6}\int_\tau^\infty R^2dR\, e^{-\beta U(R,t)} .$$

It can be numerically evaluated as follows. Take the system of equations (5), with linearly scaled length and energy variables $x=R/\tau$ and $u=\beta U$ and define the dimensionless parameter $p=\beta\tau p_s^0\Gamma^2$. To obtain the specific heat we need the second derivatives of the grand potential Ω with respect to β or p (they being proportional). We have, up to a multiplicative constant,

$$c_v=-\frac{d^2\Omega}{dp^2}$$
$$=-\frac{d^2}{dp^2}\frac{1}{p}\int_1^\infty x^2e^{-u}dx ;$$

define a function $F(x)$ so that

$$\frac{dF}{dx}(x)=-\frac{x^2e^{-u(x)}}{p},\quad F(1)=0 .$$

Clearly, $\Omega=\lim_{x\to\infty}F(x)$. To obtain the specific heat we solve the system

$$\begin{aligned}
\frac{d\epsilon}{dx}&=\frac{4\pi^3}{3}px^6e^{-u} ,\\
\frac{du}{dx}&=\frac{p}{2\epsilon}[\ln(x)+C+1] ,\\
\frac{d\epsilon_p}{dx}&=\frac{4\pi^3}{3}x^6e^{-u}(1-pu_p) ,\\
\frac{du_p}{dx}&=\frac{[\ln(x)+C+1]}{2\epsilon^2}(\epsilon-p\epsilon_p) ,\\
\frac{d\epsilon_{pp}}{dx}&=\frac{4\pi^3}{3}x^6e^{-u}(pu_p^2-pu_{pp}-2u_p) ,\\
\frac{du_{pp}}{dx}&=\frac{[\ln(x)+C+1]}{2\epsilon^2}\left[-2\epsilon_p-p\epsilon_{pp}+\frac{2p\epsilon_p^2}{\epsilon}\right] ,\\
\frac{dF_{pp}}{dx}&=x^2e^{-u}\left[-\frac{2}{p^3}-\frac{2u_p}{p^2}-\frac{u_p^2}{p}+\frac{u_{pp}}{p}\right] ,
\end{aligned} \tag{9}$$

where differentiation with respect to the parameter p is denoted by a subindex p; in particular,

$$F_{pp}(x)=\frac{\partial^2F}{\partial p^2}(x) ,$$

and one has the relation

$$c_v=\lim_{x\to\infty}p^2F_{pp}(x) .$$

The initial conditions are $\epsilon(1)=1$, $u(1)=\beta\mu=1.39p$, $\epsilon_p(1)=0$, $u_p(1)=1.39$, $\epsilon_{pp}(1)=0$, $u_{pp}(1)=0$, and $F_{pp}(1)=0$. In this way, we obtain the "diamagnetic constant" ϵ and the specific heat c_v in just one integration. A linear least-squares fit to $c_v=t^{-\alpha}$ gives

$$\alpha=0.64 .$$

ELLIPTICAL VORTICES

In order to see what is the trend in critical behavior as the number of degrees of freedom of the system is increased, in this section we enlarge the dimension of the phase space by two, by considering ellipses of small eccentricity parametrized as

$$\mathbf{X}(\sigma)=(R\cos\sigma,R(1+e)\sin\sigma,0) . \tag{10}$$

The two additional degrees of freedom are the eccentricity $e, 0<e<E\ll1$, and the orientation of a given ellipse within its plane. The quantity E is a (small) parameter characterizing the maximum allowed eccentricity of the elliptical vortices. In principle, the critical behavior of the system depends on its value. However, as we shall see below, the critical behavior turns out to be quite insensitive to the actual value of E. Going back to (1), we take its leading behavior when the vortex is very thin. To do this, its singularity is regularized replacing $|\mathbf{X}_i-\mathbf{X}_j|^{-1}$ by $(|\mathbf{X}_i-\mathbf{X}_j|^2+\tau^2)^{-1/2}$ with τ a small cutoff whose existence is important but whose precise value should be irrelevant for the critical properties. We need to find

$$H=\frac{\rho\Gamma^2}{8\pi}\oint\oint d\sigma\, d\sigma'\frac{d\mathbf{X}}{d\sigma}\cdot\frac{d\mathbf{X}}{d\sigma'}\frac{1}{[|\mathbf{X}(\sigma)-\mathbf{X}(\sigma')|^2+\tau^2]^{1/2}}$$

to leading order when $\tau\to 0$. The dominant behavior in the double integral comes from points with $\sigma\sim\sigma'$ so we introduce the new variable $\Delta\equiv\sigma-\sigma'$ in terms of which

$$H=\frac{\rho\Gamma^2}{4\pi}\oint d\sigma|\mathbf{X}'|^2\left[\int_0^{\delta}\frac{d\Delta}{(|\mathbf{X}'|^2\Delta^2+\tau^2)^{1/2}}+O(1)\right]$$
$$=\frac{\rho\Gamma^2}{4\pi}\oint d\sigma|\mathbf{X}'|\left[\ln\frac{|\mathbf{X}'|}{\tau}+O(1)\right],$$

where $O(1)$ terms remain finite when $|\mathbf{X}'|\delta/\tau\to\infty$; and $|\mathbf{X}'|\delta$ is a typical length for the vortex filament. For a circle of radius R, $|\mathbf{X}'|=R$ and we recover (2). For an ellipse parametrized as in (10)

$$|\mathbf{X}'|=R[1+e\cos^2\sigma+O(e^2)]$$

and

$$H=\frac{\rho\Gamma^2}{4}\left[(2+e)R\ln\frac{R}{\tau}+R(e+2C+eC)\right].$$

Since we are considering ellipses of small eccentricity only, the length scale is set by R. Hence, the "diamagnetic constant" will be a function of R only: $\epsilon=\epsilon(R)$ and the effective potential will be

$$U(R,e)=\mu+\frac{\rho\Gamma^2}{4}\int_\tau^R\frac{dR'}{\epsilon(R')}\left[(2+e)\ln\frac{R'}{\tau}+2\left[C+1+e+\frac{eC}{2}\right]\right].$$

We have already seen that for any loop the "polarizability" is given by $q=-4\pi\beta\rho m^2/3$ which for our ellipse means

$$q=-\frac{\pi}{12}\beta\rho\Gamma^2R^4(1+2e).$$

Next, to compute the "magnetic susceptibility" $V^{-1}\sum_i q_i n_i$ the sum over states is replaced as follows:

$$\sum_i\to\frac{V}{\tau^6}\int_{\substack{\tau\le R\le\infty\\0\le e\le E}}R^2dR\,d\Theta\frac{\pi Re}{\tau}\frac{R\,de}{\tau}.$$

This is so because there are $Re\,d\psi/\tau$ ellipses with eccentricity e with orientation between ψ and $\psi+d\psi$ and, for a given orientation, $R\,de/\tau$ different ellipses with eccentricity between e and $e+de$. Hence

$$\chi(R,E)=\frac{4\pi^2}{\tau^8}\int_\tau^R R'^4dR'\int_0^E e\,de\,q(R',e)\times\exp[-\beta U(R',e)].$$

The integrals over e can be done explicitly and we get the following set of scaling equations for a dilute gas of elliptical vortex rings:

$$\frac{dU_1}{dR}=\frac{\rho\Gamma^2}{4\epsilon(R)}\ln\frac{R}{\tau},$$
$$\frac{dU_2}{dR}=\frac{\rho\Gamma^2}{2\epsilon(R)},$$
$$V=U_1+(1+C/2)U_2,$$
$$U=\mu+2U_1+(1+C)U_2, \tag{11}$$
$$\frac{d\epsilon}{dR}=\frac{4\pi^4}{3}\beta\rho\Gamma^2\left[\frac{R}{\tau}\right]^8e^{-\beta U}\times\left[\frac{1}{(\beta V)^2}-\frac{(1+\beta EV)}{(\beta V)^2}e^{-\beta EV}+\frac{4}{(\beta V)^3}-2[2+2\beta EV+(\beta EV)^2]\frac{e^{-\beta EV}}{(\beta V)^3}\right],$$

which have to be solved numerically with the initial conditions $\epsilon(\tau)=1, U_1(\tau)=U_2(\tau)=0$. We have done this for three values of E: 0.1, 0.3, and 0.5. In all three cases there is a phase transition and we compute the critical exponent ν in the same manner that was used for circular rings. The linear least-squares fit using the reduced temperature interval $t\in[10^{-6},10^{-3}]$ gives $\nu=0.51$, 0.50, and 0.53 for $E=0.1$, 0.3, and 0.5, respectively.

To obtain the specific heat, we proceed as was done previously; namely, write down the equations in terms of dimensionless variables $x=R/\tau$, ϵ, $u_1=\beta U_1, u_2=\beta U_2$, etc. Defining also $\omega=\beta\Omega$, and the dimensionless parameter $p=\beta\rho\tau\Gamma^2$, as before. Because of the enlarged phase space, the potential Ω (per unit volume) is now

$$\Omega=-\frac{4\pi^2}{\beta\tau^3}\int_1^\infty x^4e^{-\beta U}\left[\frac{1}{(\beta V)^2}-\frac{(1+\beta EV)}{(\beta V)^2}e^{-\beta EV}\right]dx.$$

The same reasoning that led to (9) in the circular case now gives

$$\frac{d\epsilon}{dx}=\frac{4\pi^4}{3}x^8e^{-u}pF,$$
$$\frac{d\epsilon_p}{dx}=\frac{4\pi^4}{3}x^8e^{-u}(F-pFu_p+pF_p),$$
$$\frac{d\epsilon_{pp}}{dx}=\frac{4\pi^4}{3}x^8e^{-u}[pF_{pp}+2F_p(1-pu_p)+pF(u_p^2-u_{pp})-2Fu_p],$$
$$\frac{du_1}{dx}=\frac{p}{4\epsilon}\ln(x),$$
$$\frac{du_{1p}}{dx}=\frac{\ln(x)}{4\epsilon^2}(\epsilon-p\epsilon_p),$$
$$\frac{du_{1pp}}{dx}=\frac{\ln(x)}{4\epsilon^3}(2p\epsilon_p^2-p\epsilon\epsilon_{pp}-2\epsilon\epsilon_p),$$
$$\frac{du_2}{dx}=\frac{p}{2\epsilon},$$
$$\frac{du_{2p}}{dx}=\frac{1}{2\epsilon^2}(\epsilon-p\epsilon_p),$$

TABLE I. Critical behavior of a dilute gas of vortex loops for different values of the maximum allowed eccentricity E: $\rho_s \sim t^{\nu}, c_v \sim t^{-\alpha}$.

E	ν	α
0 (circles)	0.57	0.64
0.1	0.51	0.96
0.3	0.50	0.94
0.5	0.53	0.99

$$\frac{du_{2pp}}{dx} = \frac{1}{2\epsilon^3}(2p\epsilon_p^2 - p\epsilon\epsilon_{pp} - 2\epsilon\epsilon_p) ,$$

$$\frac{dc_v}{dx} = \frac{x^4 e^{-u}}{p^3}[p^2(G_{pp} - 2u_p G_p) + p^2(u_p^2 - u_{pp})G + 2p(u_p G - G_p) + 2G] ,$$

where u, v, F, and G are given by

$$u = \beta\mu + 2u_1 + (1+C)u_2 ,$$

$$v = u_1 + (1 + C/2)u_2 ,$$

$$F = \frac{1}{v^2} - \frac{(1+Ev)}{v^2}e^{-Ev} + \frac{4}{v^3} - 2[2 + 2Ev + (Ev)^2]\frac{e^{-Ev}}{v^3} ,$$

$$G = \frac{1}{v^2} - \frac{(1+Ev)}{v^2}e^{-Ev} ,$$

and the initial conditions are now (since the factor $\beta\mu = 1.39p$ has been included in the equations), $\epsilon(1) = 1$, $u_1(1) = u_2(1) = \omega(1) = \ldots = \omega_{pp}(1) = c_v(1) = 0$, where $c_v(\infty)$ is the specific heat of the model. These equations are again solved numerically for $E = 0.1$, 0.3, and 0.5, and the singular behavior of the specific heat gives $\alpha = 0.96$, 0.94, and 0.99, respectively. The results for the critical exponents are summarized in Table I.

DISCUSSION

Our main motivation in this work has been to study the possibility that, as sometimes stated in the literature in the past,[12] vortex loops may be the driving mechanism responsible for the λ transition in liquid helium. Of course, there are detailed numerical computations of the critical exponents of the 3D XY model that are in very good agreement with experiment.[13] They, however, offer little insight into the physical mechanism responsible for the transition. We have carried out computations of ν and α using a scale-dependent mean-field theory approach in the spirit of Kosterlitz and Thouless,[1] with a phase space drastically cut to six (circles) or eight (ellipses) dimensions. This is an uncontrolled approximation in the sense that there is no quantitative estimate of what is being neglected, although one would intuitively expect circles to be a reasonable first approximation and ellipses to provide corrections. It has been established that a dilute gas of vortex rings does indeed have a second-order phase transition in this context, and a first computation, that of ν for circular rings, gives $\nu = 0.57$, which is closer to the experimental $\nu = 0.67$ than the $\nu = 0.50$ obtained in a $2+\epsilon$ expansion.[14] The specific-heat exponent, however, is $\alpha = 0.64$ which is not as close to the experimental value of $\alpha \sim -0.01$. Enlarging the dimension of phase space to include elliptical shapes gives $0.50 < \nu < 0.53$ and $0.94 < \alpha < 0.99$, which are values still farther from the experimental ones. The conclusions are then that either vortices are not responsible for the λ transition in liquid helium, or that, if they are, the interaction cannot be modeled by a scale-dependent mean-field approximation, or the infinite dimensionality of the phase space of vortex rings has to be taken fully into account, or both.

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