

Chiral Luttinger liquid and the edge excitations in the fractional quantum Hall states

X. G. Wen

School of Natural Sciences, Institute for Advanced Study, Princeton, New Jersey 08540

(Received 25 January 1990)

The low-energy effective theory of the edge excitations in the fractional quantum Hall (FQH) states is derived. The edge excitations are shown to form a new kind of state which is called the chiral Luttinger liquid (χ LL). The effective theory is exactly soluble. This enables us to easily calculate all the low-energy properties of the edge excitations. We calculate the electron propagator and the spectral function, which clearly demonstrate the non-Fermi-liquid behaviors of the χ LL. We also calculate the interference effects between excitations on different edges. We demonstrate that the properties of the edge excitations are closely related to the properties of the FQH states on compactified spaces. Thus the properties of the edge excitations can be used to characterize the topological orders in the FQH states. We also show that the FQH states with filling fractions $\nu \neq 1/l$ must have at least two branches of edge excitations.

I. INTRODUCTION

In the last few years many people have studied the low-energy dynamical properties of the quantum Hall (QH) states.¹ The experiments clearly observed gapless excitations in finite QH systems. It is generally believed that the gapless excitations are localized at the edges of the systems. This is because the QH states are incompressible and contain no bulk gapless excitations. Using Laughlin's arguments,² one can easily prove the existence of the gapless excitations in a finite QH system. The real nontrivial issue is to understand the dynamics of the edge excitations. For integral QH states,³ the dynamical properties of the edge excitations are shown to be described by one-dimensional (1D) Fermi-liquid theory. While for the FQH states,⁴ they are described by the U(1) Kac-Moody (KM) algebras. Some static properties (e.g., dc transport properties) of the edges states in the FQH regime are studied in Ref. 5.

In general, the gapless edge excitations may have many branches.³⁻⁵ The dynamics of the edge excitations is generally described by several U(1) KM algebras in the low-energy limit.⁴ This is equivalent to say that the charge-zero sector of the edge excitations is described by the charge-zero sector of a Fermi-liquid theory (in the low-energy limit). Such a Fermi-liquid theory contains many branches of fermions. The charges of the fermions are shown to satisfy a sum rule⁴

$$\sum_I \frac{v_I}{|v_I|} q_I^2 = \nu e^2. \quad (1.1)$$

In (1.1) v_I and q_I are the velocities and the charges of the fermions in the I th branch and ν is the filling fraction of the FQH state. In general, the charges q_I can be irrational numbers.⁶ The relation between the edge excitations and the Fermi liquid can be used to calculate many properties of the edge excitations. The responses of the edge states to external electromagnetic fields are calculated in Ref. 6, which lead to a practical way to experimentally

measure the charges q_I carried by the fermions.

However, as emphasized in Refs. 4 and 6, although the charge-zero sector of the edge excitations are described by a Fermi-liquid theory, the charged edge states may *not* be described by Fermi-liquid theories. The charges of the charged edge states especially may not be multiples of q_I . Therefore, to be accurate, we will call q_I the optical charges of the edge excitations. This is because q_I are measured only through the current correlation functions and do not correspond to the charges of the charged edge states. Strictly speaking, the edge states in the FQH regime are not Fermi liquids. In this paper, we will derive an effective theory which describes both the charged and the neutral excited edge states. We will concentrate on the non-Fermi-liquid behaviors of the edge excitations. Although the edge states are not Fermi liquids, the effective theory of the edge excitations is still exactly soluble. One can easily obtain all the low-energy properties of the edge excitations from the effective theory.

II. THE EDGE EXCITATIONS ON A DISC

Consider a FQH state on a disc with filling fraction ν . Let us assume that the edge excitations have only one branch. This implies that the charge-zero sector of the edge excitations are described by a single U(1) KM algebra:^{4,6}

$$\begin{aligned} [j_k^+, j_k^+] &= e^2 \frac{\nu}{2\pi} k \delta_{k+k'}, \\ [j_k^+, j_k^-] &= [j_k^-, j_k^-] = 0, \\ [H, j_k^\pm] &= \nu k j_k^\pm, \end{aligned} \quad (2.1)$$

where

$$j_I^\pm = \frac{1}{2}(j^0 \pm \frac{1}{v} j^\sigma), \quad j_k^\alpha = \int d\sigma \frac{1}{\sqrt{L}} e^{i\sigma k} j^\alpha(\sigma),$$

and L is the length of the edge. The (optical) charge of the fermions in the corresponding Fermi-liquid theory is

given by $q = \sqrt{v}$.

The charged excited states arise from adding (or subtracting) electrons to (from) the edge. Therefore, those charged states are generated by electron creation (or annihilation) operators $\psi^\dagger$ (or ψ). The electron operator ψ carries a unit charge

$$[Q, \psi] = e\psi, \quad (2.2)$$

where $Q = \int d\sigma e^{i\sigma k} j^0(\sigma)$. Because all the low-lying excitations have the same velocity v , ψ also satisfies

$$[H, \psi_k] = vk\psi_k. \quad (2.3)$$

If ψ_k had a velocity different from v , the current operator $j_k^0 = \sum \psi_{k'}(k+k')\psi_{k+k'}$ would also have that velocity. This would contradict (2.1).

The total Hilbert space of the edge excitations is generated not only by the current operator j^+ but also by the charged operator ψ . Therefore the Hilbert space of the edge excitations forms a representation of the algebra (2.1)–(2.3). To understand the properties of the charged edge states, we first need to find the representation of the algebra (2.1)–(2.3).

The structure of the Hilbert space of the edge excitations can be understood even without any calculations. First, the charge-zero sector of the edge states forms an irreducible representation of the $U(1)$ KM algebra. The Hilbert space of such a representation is denoted $\mathcal{H}_{\text{KM}}$. The charge- e excited states are obtained by adding an electron to the system. The system with one more electron is essentially identical to the original system. Thus the charge- e sector also forms the irreducible representation of the KM algebra. A similar result can be obtained for a general charge- Ie sector. From the above discussions, we see that the total Hilbert space of the edge excitations is given by

$$\mathcal{H}_{\text{disc}} = \bigoplus_I \mathcal{H}_{\text{KM}}^{(I)} = \mathcal{H}_{\text{KM}} \otimes \mathcal{H}_p,$$

where $\mathcal{H}_p$ is spanned by states $|I\rangle$. The state $|I\rangle$ has a charge Ie and $\mathcal{H}_{\text{KM}}^{(I)} = \mathcal{H}_{\text{KM}} \otimes \{|I\rangle\}$ corresponds to the charge- Ie sector.

In the following we are going to show that the representation of the algebra (2.1)–(2.3) can be constructed from chiral boson theories. For convenience we will assume that the disc has a unit radius (i.e., $L = 2\pi$) and set $e = v = 1$. Chiral boson theory is defined by the Lagrangian⁷

$$\mathcal{L} = \frac{1}{8\pi} [(\partial_0 \phi)^2 - (\partial_\sigma \phi)^2], \quad (2.4)$$

where the real scalar field ϕ satisfies the “chiral” constraint

$$(\partial_0 - \partial_\sigma)\phi = 0. \quad (2.5)$$

In the following, we will follow Ref. 8 to quantize the chiral boson theory (2.4) and (2.5). The operator ϕ satisfies the equation of motion

$$(\partial_0 - \partial_\sigma)(\partial_0 + \partial_\sigma)\phi = 0. \quad (2.6)$$

The solutions of (2.6) take the form

$$\begin{aligned} \phi &= \phi_0 + \tilde{\phi}_0 + p_\phi(t + \sigma) + \tilde{p}_\phi(t - \sigma) \\ &+ i \sum_{n(\neq 0)} \frac{1}{n} (\alpha_n e^{-in(t+\sigma)} + \tilde{\alpha}_n e^{-in(t-\sigma)}). \end{aligned} \quad (2.7)$$

The canonical momentum of ϕ is given by $\pi = (1/4\pi)\partial_0\phi$:

$$\pi = \frac{1}{4\pi} (p_\phi + \tilde{p}_\phi) + \frac{1}{4\pi} \sum_{n(\neq 0)} (\alpha_n e^{-in(t+\sigma)} + \tilde{\alpha}_n e^{-in(t-\sigma)}), \quad (2.8)$$

p_ϕ and α_n describe the left-moving excitations, while $\tilde{p}_\phi$ and $\tilde{\alpha}_n$ describe the right-moving ones. From the commutator between ϕ and π we find that $\tilde{\phi}_0$, $\tilde{p}_\phi$, and $\tilde{\alpha}_n$ satisfy the algebra

$$\begin{aligned} [\tilde{\alpha}_n, \tilde{\alpha}_m] &= n\delta_{n+m}, \\ [\tilde{\phi}_0, \tilde{p}_\phi] &= i, \\ \text{others} &= 0, \end{aligned} \quad (2.9)$$

and ϕ_0 , p_ϕ , and α_n satisfy

$$\begin{aligned} [\alpha_n, \alpha_m] &= n\delta_{n+m}, \\ [\phi_0, p_\phi] &= i, \\ \text{others} &= 0. \end{aligned} \quad (2.10)$$

At this stage we may impose the constraint (2.5) by dropping $\tilde{\phi}_0$, $\tilde{p}_\phi$, and $\tilde{\alpha}_n$. A more systematic and careful treatment of the chiral boson theory can be found in Ref. 7. Notice that algebra (2.10) just describes many independent oscillators. The Hilbert space of the chiral boson theory is defined as the Fock space of the oscillator algebra (2.10). The operators α_n generate the irreducible representation of the KM algebra $\mathcal{H}_{\text{KM}}$. The space generated by the “zero modes” ϕ_0 and p_ϕ needs more careful treatment and will be discussed later. The Hamiltonian of the chiral boson theory is given by

$$H = \frac{1}{2} p_\phi^2 + \sum_{n(>0)} \alpha_n \alpha_{-n}. \quad (2.11)$$

The electrical current in the chiral boson theory is identified as

$$j^\alpha = \frac{\sqrt{v}}{2\pi} \epsilon^{\alpha\beta} \partial_\beta \phi_L, \quad (2.12)$$

where

$$\phi_L(t, \sigma) = \phi_0 + p_\phi(t + \sigma) + i \sum_{n(\neq 0)} \frac{1}{n} \alpha_n e^{-in(t+\sigma)}. \quad (2.13)$$

The total charge operator is

$$Q = \sqrt{v} p_\phi = -\sqrt{v} i \partial_{\phi_0}. \quad (2.14)$$

Using (2.10) one can explicitly check that the current in (2.12) satisfies the KM algebra (2.1). Therefore the Hilbert space of the chiral boson theory forms a representation of the KM algebra (2.1).

The charged operators in the chiral boson theory have a form $\Psi = e^{i\gamma\phi_L}$. Because we want to identify the charged operator Ψ as an electron operator, Ψ must satisfy

fy the anticommutation relation

$$\Psi(\sigma)\Psi(\sigma') = -\Psi(\sigma')\Psi(\sigma), \quad \sigma' \neq \sigma. \quad (2.15)$$

Using the formula

$$e^A e^B = e^{[A, B]} e^B e^A, \quad (2.16)$$

we find that⁹

$$\begin{aligned} \Psi(\sigma)\Psi(\sigma') &= e^{i\gamma^2(\sigma+\sigma')/2} (e^{-i\sigma} - e^{-i\sigma'}) \gamma^2 \\ &\times :e^{i\gamma[\phi_L(\sigma)+\phi_L(\sigma')]}: . \end{aligned} \quad (2.17)$$

Therefore Ψ is a fermionic operator if $\gamma^2 \equiv l$ is an odd integer.

In order to identify the operator Ψ as an electron operator, we not only require Ψ to be a fermionic operator, we also require Ψ to carry a unit charge. From (2.14) we see that the charge of Ψ is given by $\gamma\sqrt{l}$. This implies that only when the filling fraction satisfies $\nu=1/l$, can the operator Ψ be identified as an electron operator:

$$\psi^\dagger(t, \sigma) = \eta \Psi = \eta e^{i\sqrt{l}\phi_L(t, \sigma)}, \quad (2.18)$$

where η is a constant which may depend on the cutoff. As a direct consequence of the above result, a FQH state with filling fraction $\nu \neq 1/l$ must have more than one branch of edge excitations.

The physical Hilbert space of the chiral boson theory (2.4) and (2.5) is generated by the operators j_k^+ and ψ , or equivalently by α_n and $e^{i\sqrt{l}\phi_0}$. The operator $e^{i\sqrt{l}\phi_0}$ generates the charged excited states. Because α_n and $e^{i\sqrt{l}\phi_0}$ commute, the Hilbert space can be written as

$$\mathcal{H}_{\text{KM}} \otimes \mathcal{H}_p, \quad (2.19)$$

where $\mathcal{H}_p$ is spanned by the states $|I\rangle$. The state $|I\rangle$ carries a charge $Q=Ie$ and $p_\phi = I\sqrt{l}$. Thus the Hilbert space of the chiral boson theory is identical to the Hilbert space $\mathcal{H}_{\text{disc}}$ that we obtained before.

The commutation relation (2.3) can easily be derived. First, notice that

$$[H, \psi(t, \sigma)] = -i\partial_0 \psi(t, \sigma). \quad (2.20)$$

From (2.13) we see that $\psi(t, \sigma)$ depend on t and σ only through the combination $t + \sigma$. This implies that

$$\partial_0 \psi(t, \sigma) = \partial_\sigma \psi(t, \sigma); \quad (2.21)$$

(2.3) can be easily obtained from (2.20) and (2.21). We find that the chiral boson theory (2.4) and (2.5) together with the quantization condition

$$p_\phi = \sqrt{l} \times \text{integers} \quad (2.22)$$

form a representation of the algebra (2.1)–(2.3)

Let us summarize our results. Consider a FQH state on a disc with a filling fraction ν . Assume that the edge excitations only have one branch and assume that it costs infinitely small energy to add a single electron to the FQH state. Under those assumptions we show that the edge excitations of such a FQH state are described by the chiral boson theory (2.4) and (2.5) and (2.22). The chiral boson theory contains the charge- e fermion operator only

when $\nu=1/l$, where l is an odd integer. Such an operator is identified as the electron operator. Therefore, a FQH state with filling fraction $\nu \neq 1/l$ must have more than one branch of edge excitations. For the $\nu=1/l$ FQH states, Haldane suggested that the edge excitations have only one branch.¹⁰ In this case, the edge excitations are described by the chiral boson theory (2.4) and (2.5) and (2.22). The Hilbert space of the chiral boson theory is generated by the operators α_n and ψ .

The electron Green's function can be calculated using (2.16). We find that

$$\langle \psi^\dagger(t, \sigma) \psi(0, 0) \rangle = \eta^2 e^{-il(t+\sigma)/2} (e^{-i(t+\sigma)} - 1)^{-l}. \quad (2.23)$$

In the thermodynamic limit, $\sigma \ll L=2\pi$ and $t \ll L/v=2\pi$, and we have

$$\langle \psi^\dagger(t, \sigma) \psi(0, 0) \rangle = \eta^2 \left[\frac{i}{t + \sigma} \right]^l. \quad (2.24)$$

In the momentum space (2.24) becomes

$$\langle \psi_k^\dagger \psi_k \rangle \propto \frac{(\omega - k)^{l-1}}{\omega + k - i\delta}. \quad (2.25)$$

If electrons are described by the Fermi-liquid theory then the Green's function should be

$$\langle \psi^\dagger(t, \sigma) \psi(0, 0) \rangle = \frac{i}{t + \sigma}. \quad (2.26)$$

The anomalous exponent in the propagator (2.24) implies that the electrons on the edge of the FQH states do not form a Fermi liquid. The electrons are strongly correlated and form new kind of states. Those states resemble the Luttinger liquid,¹¹ in which the electron propagator also has an anomalous exponent. Because the excitations in our states only move in one direction, we will call such states chiral Luttinger liquids (χ LL).

The Luttinger liquids contain both right-moving and left-moving excitations, while the χ LL contain only left (or right) moving excitations. Because the chiral property of the χ LL, the exponent in the electron propagator is expected to be a topological invariant. In the chiral boson theory considered here, the exponent is given by $1/\nu$, which is quantized as an odd integer. The exponent remains unchanged no matter how we perturb the Hamiltonian. In contrast, the exponent of the electron propagator in the Luttinger liquid can take arbitrary real values. The exponent depends on the interactions between electrons and is not a topological invariant. From the above discussion we see that the χ LL and the Luttinger liquid have some fundamental distinctions.

However, as pointed out in Refs. 4 and 6, the edge states of FQH systems are closely related to a Fermi liquid of charge- $q=(1/\sqrt{l})$ fermions. Or more precisely, the charge-zero sector of the χ LL is described by the charge-zero sector of the charge- $q=(1/\sqrt{l})$ Fermi-liquid theory. If we were only interested in the processes that conserve the total charge of the system, then the χ LL could be regarded as a Fermi liquid. But the charged excited states are not described by the charge- q Fermi-liquid theory. In particular the charges of the edge states are quantized as integers instead of as multi-

plets of $1/\sqrt{l}$.

As we increase the size of the system, the constraint on the total charge becomes less and less important. In fact, the χ LL and the charge- q Fermi liquid have the same thermodynamic properties. We may effectively treat the edge states as a charge- $q=(1/\sqrt{l})$ Fermi liquid if we were only interested in those thermodynamic properties.

To further demonstrate the similarity between the χ LL and the Fermi liquid, we would like to calculate the "edge capacity" of the FQH states. Assume that a $\nu=1/l$ FQH state is in equilibrium with a charge reservoir of voltage V . The total charge of the FQH state is a function of V , $Q=Q(V)$. The edge capacity is defined as

$$C = \frac{dQ}{dV}. \quad (2.27)$$

The edge capacity can also be obtained from

$$\frac{1}{C} = \frac{d^2 E(Q)}{d^2 Q}, \quad (2.28)$$

where $E(Q)$ is the total energy of the FQH state. Comparing (2.28) and (2.11) and (2.14), we find that the capacity of the χ LL is given by

$$C = \nu = \frac{1}{l} \frac{e^2 L}{2\pi v}. \quad (2.29)$$

The capacity of a charge- q Fermi liquid is given by $q^2 N_0$, where $N_0=1$ is the density of states of the Fermi liquid. We find that (2.29) is also the capacity of the charge- $q=(1/\sqrt{l})$ Fermi liquid. Therefore, despite the charge, quantization conditions are different, the capacity of the χ LL and the capacity of the charge- $q=(1/\sqrt{l})$ Fermi liquid are identical.

To observe the non-Fermi-liquid behaviors of the χ LL we need to use the processes that change the total charge of the system and probe the microscopic structures in the states. Electron tunneling and photoemission are two such experiments. Those experiments measure the electron spectral function

$$n_{\omega,k} = \sum_n |\langle n | \psi_k | 0 \rangle|^2 \delta(\omega - \omega_n), \quad (2.30)$$

where ω_n is the energy of the state $|n\rangle$. From the electron propagator we find that the spectral function and the electron "density of states" in the χ LL are given by

$$n_{\omega,k} \propto \omega^{l-1} \delta(\omega + k) \theta(-\omega), \quad (2.31)$$

$$N(\omega) = \int \frac{dk}{2\pi} n_{\omega,k} \propto \omega^{l-1} \theta(-\omega).$$

Measuring the spectral function allows us to determine the anomalous exponent.

The χ LL are characterized by the following properties:

(1) The χ LL contain a conserved current which forms a U(1) KM algebra with a central charge $q^2/2\pi$. q is called the optical charge of the χ LL.

(2) The χ LL contain a local charged operator. The U(1) charge of the operator is given by q_0 , which may not be equal to q .

(3) All the excited states have the same velocity $v = \varepsilon/k$, where ε and k are the total energy and the total

momentum of the excited states.

When $q = q_0$ the χ LL is just a charge- q Fermi liquid. When $q \neq q_0$ the χ LL is different from Fermi liquid. But the charge-zero sector of the χ LL is still described by the charge-zero sector of the charge- q Fermi liquid.

Before ending this section we would like to discuss the relation between the chiral boson theory and the microscopic theory of the FQH states.¹⁰ Consider a FQH system confined in a circular potential well. The filling fraction of the FQH states is $\nu=1/l$. The ground state has an angular momentum M_0 and is given by the Laughlin wave function

$$\Phi_0(z_i) = \left[\prod_{i < j} (z_i - z_j)^l \right] \exp \left[-\frac{1}{4} \sum_i |z_i|^2 \right]. \quad (2.32)$$

For such simple FQH states, we may assume that the total energy of the system is a single-valued smooth function of the total angular momentum

$$E = E(M). \quad (2.33)$$

Haldane¹⁰ pointed out that the charge-zero edge excitations in such a system are generated by multiplying a symmetric polynomial to the ground-state wave function

$$|n_1, n_2, \dots\rangle = P_{(n_1, n_2, \dots)}(z_i) \Phi_0(z_i), \quad (2.34)$$

where

$$P_{(n_1, n_2, \dots)}(z_i) = \sum_{\{i_1, \dots, i_{n_1}; j_1, \dots\}} (z_{i_1} \cdots z_{i_{n_1}}) (z_{j_1}^2 \cdots z_{j_{n_2}}^2) \cdots. \quad (2.35)$$

The excited state $|n_1, n_2, \dots\rangle$ has angular momentum $K + M_0$ and energy $K dE/dM$ [assume $E(M_0)=0$], where $K = \sum j n_j$. Such a state is called the K th-excited state. The number of states at the K th level is given by

$$N_K = \sum_{n_1, n_2, \dots} \delta \left[\sum j n_j - K \right]. \quad (2.36)$$

In the chiral boson theory the excited states are generated by the operators α_n , $n > 0$:

$$|n_1, n_2, \dots\rangle = \alpha_1^{n_1} \alpha_2^{n_2} \cdots |0\rangle. \quad (2.37)$$

The energy of the state $|n_1, n_2, \dots\rangle$ is given by $Kv \times (2\pi/L)$, where $K = \sum j n_j$. We will again call such a state the K th-level state. The number of the K th-level states in the chiral boson theory is given by the same formula (2.36). Because dE/dM is the angular velocity of the edge excitations $dE/dM = v(2\pi/L)$, the K th-level states in the FQH states and the K th-level states in the chiral boson theory have the same energy. Adding m electrons to the system increases the angular momentum by $\ln(m+1)/2$ and the energy by

$$\Delta E = \frac{m(m+1)}{2} l \frac{dE}{dM} = m\mu,$$

where μ is the electron chemical potential. If we choose $\mu = (l/2)(dE/dM)$, we find that $\Delta E = (m^2/2)lv(2\pi/L)$.

This again agrees with the result in the chiral boson theory that the operator $e^{im\sqrt{l}\phi_0}$ creates m electrons and increases the total energy by $\frac{1}{2}m^2lv(2\pi/L)$ [See (2.11)]. From the above discussions we find that the microscopic FQH theory and the chiral boson theory give rise to the same Hilbert space and the same Hamiltonian for the low-lying edge excitations. In this way we show that the chiral boson theory (2.4) and (2.5) and (2.22) describe all of the dynamical properties of the edge excitations in the $\nu = 1/l$ FQH states.

We would like to remark that comparing to the microscopic theory discussed above, the chiral boson theory is more general. The chiral boson theory remains to be valid even when the edge potential is not a smooth function and when the electron interaction is modified near the edge. This is because the KM algebra (2.1) is a consequence of the gauge symmetry. The validity of the KM algebra is independent of the detailed structures of the edge configuration. The chiral boson theory also applies to the hierarchy FQH states, which have many branches of edge excitations. The chiral boson theory in this case contains several boson fields, one boson field for each branch. For the $\nu \neq 1/l$ FQH states, the total energy is not a smooth single-valued function of M . In this case the symmetric polynomials do not generate all the low-lying excitations.

III. EDGE EXCITATIONS ON A CYLINDER

In this section we will discuss the edge excitations on a cylinder. We will assume the FQH state on a cylinder to have a filling fraction $\nu = 1/l$. Since the cylinder has two edges, one may naively expect that the Hilbert space of the edge excitations on the cylinder is a direct product of the Hilbert spaces on the edges of two discs, $\mathcal{H}_{\text{disc}} \otimes \mathcal{H}_{\text{disc}}$, where $\mathcal{H}_{\text{disc}} = \mathcal{H}_{\text{KM}} \otimes \mathcal{H}_p$ is constructed in the last section. However, this naive expectation is incorrect. There are new kinds of excitations on the edges of the cylinder. Those excitations are not contained in $\mathcal{H}_{\text{disc}} \otimes \mathcal{H}_{\text{disc}}$. Such new excitations transfer multiples of a fractional charge e/l from one edge to the other. The new excitations can be induced by adiabatic turning on the unit flux going through the cylinder (Fig. 1). This adiabatic operation transfer e/l charge from one edge to the other. Because the charges in $\mathcal{H}_{\text{disc}} \otimes \mathcal{H}_{\text{disc}}$ are quantized as integers, such an excited state is not in $\mathcal{H}_{\text{disc}} \otimes \mathcal{H}_{\text{disc}}$. Turning on l unit flux transfers one electron between the edges. This excitation adds an electron to one edge and subtracts an electron from the other edge. Such an excitation lies within the Hilbert space $\mathcal{H}_{\text{disc}} \otimes \mathcal{H}_{\text{disc}}$. From the above considerations we conclude that the edge excitations on the cylinder contain l sectors. Each sector is given by $\mathcal{H}_{\text{disc}} \otimes \mathcal{H}_{\text{disc}}$, since different sectors are related by adiabatic turning on the unit flux. The total Hilbert space of the edge excitations on the cylinder is given by

$$\begin{aligned} \mathcal{H}_{\text{cyl}} &= \bigoplus_{M=1}^l (\mathcal{H}_{\text{disc}} \otimes \mathcal{H}_{\text{disc}})^{(M)} \\ &= \mathcal{H}_{\text{disc}} \otimes \mathcal{H}_{\text{disc}} \otimes \mathcal{H}_{\text{glo}}, \end{aligned} \quad (3.1)$$

where $\mathcal{H}_{\text{glo}}$ contains l states $|M\rangle$, $M=1, 2, \dots, l$. Those

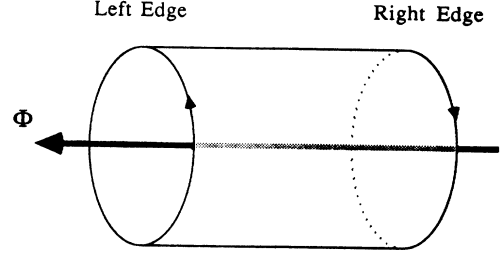


FIG. 1. A FQH state on a cylinder with magnetic flux Φ .

states are generated by an operator T , which transfers the e/l charge from one edge to the other:

$$|M+1\rangle = T^M |0\rangle. \quad (3.2)$$

The operator T is induced by adiabatic turning on the unit flux. Such an operation induces transitions between sectors. But turning on l unit flux does not change the sectors. From the above discussions, we also see that the allowed values of the charges on the two edges are labeled by three integers I_1, I_2 , and $M=1, 2, \dots, l$:

$$Q^{(R)} = \left[I_1 + \frac{1}{m} M \right], \quad Q^{(L)} = \left[I_2 - \frac{1}{m} M \right], \quad (3.3)$$

where the superscript R and L denote the right edge and the left edge (Fig. 1).

In the following we are going to show that the Hilbert space $\mathcal{H}_{\text{cyl}}$ can be constructed from a (nonchiral) boson theory

$$\mathcal{L} = \frac{1}{8\pi} [(\partial_0 \phi)^2 - (\partial_\sigma \phi)^2]. \quad (3.4)$$

Repeating the discussions in Sec. II, we may write the ϕ field as

$$\begin{aligned} \phi(t, \sigma) &= \phi_0 + \tilde{\phi}_0 + p_\phi(t + \sigma) + \tilde{p}_\phi(t - \sigma) \\ &\quad + i \sum_{n \neq 0} \frac{1}{n} (\alpha_n e^{-in(t+\sigma)} + \tilde{\alpha}_n e^{-in(t-\sigma)}). \end{aligned}$$

The operators $\phi_0, \tilde{\phi}_0, p_\phi, \tilde{p}_\phi, \alpha_n$, and $\tilde{\alpha}_n$ satisfy the following algebra:

$$\begin{aligned} [\alpha_n, \alpha_m] &= n \delta_{n+m}, \\ [\tilde{\alpha}_n, \tilde{\alpha}_m] &= n \delta_{n+m}, \\ [\phi_0, p_\phi] &= [\tilde{\phi}_0, \tilde{p}_\phi] = i, \\ \text{others} &= 0. \end{aligned} \quad (3.5)$$

The operators ϕ_0, p_ϕ , and α_n describe the excitations on the right edge, while $\tilde{\phi}_0, \tilde{p}_\phi$, and $\tilde{\alpha}_n$ describe the excitations on the left edge (see Fig. 1). From the preceding section we see that the total charge operators on the two edges are given by

$$Q^{(R)} = \frac{p_\phi}{\sqrt{l}}, \quad Q^{(L)} = -\frac{\tilde{p}_\phi}{\sqrt{l}}. \quad (3.6)$$

Equations (3.3) and (3.6) imply that p_ϕ and $\tilde{p}_\phi$ are quan-

tized as

$$p_\phi = \left[I_1 + \frac{1}{l} M \right] \sqrt{l}, \quad \tilde{p}_\phi = \left[-I_2 + \frac{1}{l} M \right] \sqrt{l}. \quad (3.7)$$

The charged operators in the boson theory have a form $:e^{i\gamma\phi_L(t+\sigma)+i\tilde{\gamma}\phi_R(t-\sigma)}:$. In order for the charged operators to consist of the quantization condition (3.7), γ and $\tilde{\gamma}$ must be quantized. The allowed values of γ and $\tilde{\gamma}$ are given by

$$\gamma = \left[n_1 + \frac{1}{l} n_3 \right] \sqrt{l}, \quad \tilde{\gamma} = \left[-n_2 + \frac{1}{l} n_3 \right] \sqrt{l}, \quad (3.8)$$

where n_i , $i=1,2,3$ are three integers. The operator $:e^{i\sqrt{l}\phi_L(t+\sigma)}:$ ($:e^{i\sqrt{l}\phi_R(t-\sigma)}:$) create an electron on the left (right) edge, while the operator $:e^{i\phi(t,\sigma)/\sqrt{l}}:$ transfers $1/l$ charge from one edge to the other.

The Hilbert space that satisfies the quantization condition (3.7) is generated by the operators α_n , $\tilde{\alpha}_n$, $e^{i\sqrt{l}\phi_0}$, $e^{i\sqrt{l}\tilde{\phi}_0}$, and $e^{i(\phi_0+\tilde{\phi}_0)/\sqrt{l}}$. The operators α_n and $\tilde{\alpha}_n$ generate the irreducible representation of the two KM algebras, $\mathcal{H}_{\text{KM}}^{(R)} \otimes \mathcal{H}_{\text{KM}}^{(L)}$. The total Hilbert space is given by $\mathcal{H}_{\text{KM}}^{(R)} \otimes \mathcal{H}_{\text{KM}}^{(L)} \otimes \mathcal{H}_{pp}$, where $\mathcal{H}_{pp}$ is spanned by the states $|I_1, I_2, M\rangle$, with I_1, I_2 denoting the integers and $M=1,2,\dots,l$. The charges of the state $|I_1, I_2, M\rangle$ are given in (3.3). At this stage it is not difficult to see that the Hilbert space constructed above is identical to $\mathcal{H}_{\text{cyl}} = \mathcal{H}_{\text{disc}} \otimes \mathcal{H}_{\text{disc}} \otimes \mathcal{H}_{\text{glo}}$. We conclude that the edge excitations on the two edges of the cylinder are described by the Lagrangian (3.4) and the quantization condition (3.7).

Using the effective theory (3.4), we are able to study the quantum-interference effects when the two edges are brought together to form a torus. Notice that only electrons can tunnel between two edges. The operators that transfer an electron from one edge to the other are given by $e^{\pm i\sqrt{l}\phi}$. After including the electron tunneling between two edges, the system is described by the following low-energy effective Lagrangian:

$$\mathcal{L} = \frac{1}{8\pi} [(\partial_0\phi)^2 - (\partial_\sigma\phi)^2 + g \cos(\sqrt{l}\phi)], \quad (3.9)$$

where g measures the strength of the electron tunneling. Equation (3.9) is the standard sine-Gordon theory (or the clock model).¹² The charged operator $e^{i\gamma\phi}$ has a dimension γ^2 . Therefore, the operator $\cos(\gamma\phi)$ is relevant if $\gamma < \sqrt{2}$ and irrelevant if $\gamma > \sqrt{2}$. The operator $\cos(\sqrt{l}\phi)$ is relevant only when $l=1$ (i.e., for $\nu=1$ quantum Hall states). In this case, an arbitrary small electron tunneling will open a finite-energy gap to edge excitations. But for $l \geq 3$ the operator $\cos(\sqrt{l}\phi)$ is irrelevant. It can open an energy gap only when g is greater than a finite critical value. This is a very nontrivial result. It would be interesting to observe this gap-opening phase transition in the type of experiments discussed in Ref. 13.

The system (3.9) has many degenerate ground states $|m\rangle$ after the energy gap is opened. Those ground states are characterized by $\langle m | e^{i\phi/\sqrt{l}} | m \rangle = e^{i2\pi(m/l)}$, which correspond to the minimums of the potential $\cos(\sqrt{l}\phi)$. Different ground states are related by the operator

$$U = e^{i2\pi p_\phi/\sqrt{l}} = e^{i2\pi \tilde{p}_\phi/\sqrt{l}} \quad [\text{see (3.7)}]:$$

$$U|m\rangle = |m+1\rangle.$$

From the quantization condition (3.7) we find that $|m\rangle = |m+l\rangle$. Therefore the ground states in our system are l -fold degenerate. This is consistent with the well-known result that the $\nu=1/l$ FQH state has l degenerate ground states on a torus.^{14,15} The solitons (kinks) in the sine-Gordon theory (2.9) carry charge e/l and correspond to the quasiparticles in the FQH states. As a soliton propagates all the way around the circle, it will transform the ground state $|M\rangle$ into $|M+1\rangle$. This also agrees with the results in Ref. 15.

From the above example we see that the properties of the edge excitations and the properties of the FQH states on compacted space are closely related. We emphasize that this relation is very important. We know that the properties of the FQH states on compacted spaces can be used to characterize the hierarchy structures, or more precisely, the topological orders in the FQH states.^{16,15} Because of the above relation, the properties of the edge excitations can also be used to characterize the topological orders in the FQH states. The dynamical properties of the edge excitations provide a practical way to experimentally measure the topological orders in the FQH states. Using this relation, one should also be able to determine the properties of the edge excitations from the properties of the FQH states on compacted spaces.

IV. CONCLUSIONS

In this paper we derive the effective theory of the edge excitations in the FQH states. In particular, we discuss the properties of the charged excited states. The edge excitations are shown to form new kinds of states which are not described by Fermi-liquid theories. Such new states are called chiral Luttinger liquids. The χ LL are closely related to Fermi liquids. Actually it can be shown that the charge zero sector of the χ LL is identical to the charge zero sector of a charge- q Fermi liquid. Here q is the optical charge of the χ LL.

The χ LL is described by the chiral boson theory (2.4) and (2.5) and (2.22). The chiral boson theory is exactly soluble. Using this effective theory we can easily obtain all the low-energy properties of the edge excitations. We calculated the electron propagator and the spectral function. The electron tunneling and the photoemission experiments can be used to demonstrate the non-Fermi-liquid behaviors of the χ LL.

Using the effective theory, we studied the interference effects between excitations on different edges. We demonstrated that the properties of the edge excitations are closely related to the properties of the FQH states in compacted spaces. The properties of the edge excitations can be used to characterize the hierarchy structures, or the topological orders in the FQH states. Using this relation we can also derive the properties of the edge excitations from the properties of the FQH states in compacted space.

Another nontrivial result obtained from the chiral boson theory is that the FQH states with $\nu \neq 1/l$ must contain more than one branch of edge excitations. This result is very general. It is independent of the edge potentials, electron interactions, etc.

The dynamical properties of edge excitations contain very rich structures which reflect the rich topological orders in the FQH states. Experimental and theoretical studies of edge excitations may lead to a much deeper understanding of the FQH states and may open a new era in the FQH theory.

Note added in proof. Recently the propagator (2.23) at equal time has been confirmed by numerical calculation¹⁷ for the $\nu = \frac{1}{2}$ Laughlin state.

ACKNOWLEDGMENTS

I would like to thank F. D. M. Haldane, D. Tsui, and Y. S. Wu for many discussions. Especially, I would like to thank Y. S. Wu for his careful reading of the manuscript and many helpful comments. This work was supported in part by U. S. Department of Energy Grant No. DE-AC02-76ER02220.

-
- ¹S. J. Allen, Jr., H. L. Störmer, and J. C. M. Hwang, *Phys. Rev. B* **28**, 4875 (1983); D. B. Mast, A. J. Dahm, and A. L. Fetter, *Phys. Rev. Lett.* **54**, 1706 (1985); D. C. Glatli *et al.*, *ibid.* **54**, 1710 (1985); S. A. Govorkov *et al.*, *Pis'ma Zh. Eksp. Teor. Fiz.* **45**, 252 (1987) [*JETP Lett.* **45**, 316 (1987)]; **94**, 226 (1988) [**67**, 342 (1988)]; I. E. Batove *et al.* (unpublished).
²R. Laughlin, *Phys. Rev. B* **23**, 5632 (1981); B. I. Halperin, *ibid.* **25**, 2185 (1982); R. Tao and Y. S. Wu, *ibid.* **30**, 1097 (1984).
³B. I. Halperin, *Phys. Rev. B* **25**, 2185 (1982).
⁴X. G. Wen (unpublished).
⁵C. W. J. Beenakker, *Phys. Rev. Lett.* **64**, 216 (1990); A. H. MacDonald, *ibid.* **64**, 220 (1990).
⁶X. G. Wen, *Phys. Rev. Lett.* **64**, 2206, 1990.
⁷See, for example, D. Gross *et al.*, *Nucl. Phys. B* **256**, 253 (1985); R. Floreanini and R. Jackiw, *Phys. Rev. Lett.* **59**, 1873 (1988).
⁸See, for example, M. Green, J. Schwartz, and E. Witten, *Superstring Theory Vol. 1* (Cambridge University Press, Cambridge, 1986), p. 323; D. Gross *et al.*, *Phys. Rev. Lett.* **54**, 502 (1985).
⁹M. Green, J. Schwartz, and E. Witten, *Superstring Theory Vol.*

- I* (Cambridge University Press, Cambridge, 1986), p. 332.
¹⁰F. D. M. Haldane, Lecture in Fractional Statistics and High- T_c Superconductivity Workshop, University of Minnesota, 1989 (unpublished).
¹¹J. M. Luttinger, *J. Math. Phys.* **15**, 609 (1963); D. C. Mattis and E. H. Lieb, *ibid.* **6**, 304 (1965); F. D. M. Haldane, *J. Phys. C* **14**, 2585 (1981).
¹²See, for example, S. Coleman, *Phys. Rev. D* **11**, 2088 (1985); J. V. José, L. P. Kadanoff, S. Kirkpatrick, and D. R. Nelson, *Phys. Rev. B* **16**, 12 (1977); L. P. Kadanoff, *Ann. Phys. (N.Y.)* **120**, 39 (1979).
¹³R. J. Haug *et al.*, *Phys. Rev. Lett.* **61**, 2797 (1988); S. Washburn *et al.*, *ibid.* **61**, 2801 (1988).
¹⁴P. A. Maksym, *J. Phys. C* **18**, L433 (1985); F. D. M. Haldane, *Phys. Rev. Lett.* **55**, 2095 (1985); Q. Niu, D. J. Thouless, and Y. S. Wu, *Phys. Rev. B* **31**, 3372 (1985).
¹⁵X. G. Wen and Q. Niu, *Phys. Rev. B* **41**, 9377 (1990).
¹⁶X. G. Wen, *Phys. Rev. B* **40**, 7387 (1989); X. G. Wen, *Int. J. Mod. Phys. B* **4**, 239 (1990).
¹⁷X. G. Wen (unpublished).

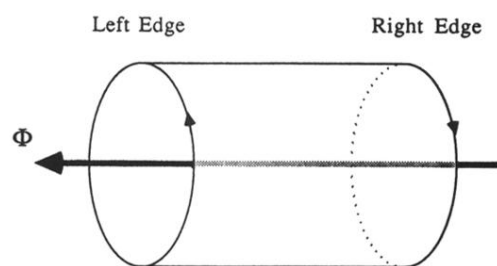


FIG. 1. A FQH state on a cylinder with magnetic flux Φ .