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Phase transitions in random-anisotropy magnets

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Recursion relations for m-component random-anisotropy magnets on hierarchical lattices are calculated to third order in $1/p$, where p is the number of parallel links at each level of the lattice. Setting $m=1$ gives the Ising spin glass, for which we find the usual ferromagnetic, spin-glass, and ferromagnet–spin-glass fixed points. The results for $p=4$ suggest that in three dimensions the ferromagnetic critical point is unstable for $m=2$, resulting in either a first-order transition or an infinite-susceptibility phase. For $m=3$ ferromagnetism may disappear altogether, at least in the strong anisotropy limit. The spin-glass critical exponents are functions of p, but independent of m.

I. INTRODUCTION

The random-anisotropy magnet[1,2] (RAM) and the Ising-spin-glass[3,4] (ISG) models have been discussed for over fifteen years. Both types of models can be studied in the infinite-range limit,[4,5] and for the Bethe lattice,[6,7] giving two kinds of mean-field theories. To describe more realistic lattices, we have high-temperature series,[8,9] computer simulations,[10,11] and renormalization-group[12,13] calculations. Nevertheless, a consensus on the behavior of these models for real lattices has been lacking. Some authors[14] concluded that the mean-field theory results have very little relation to the behavior on Bravais lattices with short-range interactions.

In this work we will treat the ISG and RAM models in a unified framework. Their behavior will be calculated for a particular class of hierarchical lattices,[15,16] which are somewhat more realistic than the infinite-range or Bethe-lattice models. In this way we will be able to calculate corrections to the mean-field theory in a systematic and straightforward manner. We will find that the mean-field theories do give useful information, but that some qualitatively new phenomena appear. Specifically, we will find that the ferromagnetic transition may change its character, or even disappear altogether, under conditions which should be realizable in experimental systems.

II. HIERARCHICAL LATTICES

The ($p=2$) hierarchical lattice was originally defined by Berker and Ostlund[15] to provide a toy model for which exact renormalization-group recursion relations could be calculated. It has since been generalized in many ways. For the calculations described here, the process of constructing the lattice is as follows. At each stage the bond between sites A and B is transformed into p parallel links, each of which consists of two bonds, with one site in the center. We imagine that this process is repeated over and over.

For a Hamiltonian which has a discrete spin variable on each site, it is straightforward to calculate the partition function for this lattice, by the iteration of dedecoration transformations.[17] The dedecoration transformation acts as the inverse of the process by which the lattice was constructed. Consequently, the critical behavior can be investigated via a set of recursion relations. Itzykson and Drouffe[18] claim that the fractal dimension, which is equal to $1+\log_2(p)$, should be used to compare this type of lattice to a Bravais lattice. These hierarchical lattices, however, do not contain any nonplanar subgraphs; that is one of the reasons why they are tractable. No Bravais lattice in more than two dimensions possesses this property. Therefore one should be wary of this comparison. The fractal dimension does not provide a complete description of the properties of a lattice.

Despite this *caveat*, the hierarchical lattices have a property which makes them clearly superior for our purposes to the Bethe lattices. That is, they contain a substantial density of closed loops. Therefore, competing interactions ("frustration"), which are the key to understanding the ISG and RAM models, are incorporated in a natural way. The only way of putting frustration into a Bethe lattice is by a careful imposition of boundary conditions.[19]

A class of hierarchical lattices which incorporate frustration without randomness was studied some time ago by McKay, Berker, and Kirkpatrick,[20] and others.[21,22] It seems unlikely that the detailed (and fascinating) structures found by those authors will survive the addition of randomness to the model.

III. RAM HAMILTONIAN

The RAM model was introduced by Harris, Plischke, and Zuckermann[1] (HPZ), who used the Hamiltonian

$$\mathbf{H}_{\rm HPZ}=-J\sum_{\langle ij\rangle}\mathbf{S}_i\cdot\mathbf{S}_j-D\sum_i[(\hat{\mathbf{n}}_i\cdot\mathbf{S}_i)^2-1]-H\sum_i S_i^1\ , \quad (1)$$

where $\langle ij\rangle$ indicates a sum over nearest neighbors, and the $\hat{\mathbf{n}}_i$ are uncorrelated random unit vectors which give the direction of the easy axis at each site (assuming $D>0$). While HPZ assumed that $\mathbf{S}_i$ and $\hat{\mathbf{n}}_i$ were three-component vectors, it is natural to consider the generalization to m-component vectors. We will assume that $\mathbf{S}_i$ is of unit length, for any m.

It is convenient to go to the strong anisotropy limit, $D/J \to \infty$, since each spin is then forced to lie parallel to its local anisotropy axis. We will also allow for some randomness in the exchange strength J, for two reasons. First, some fluctuations in J will be present in any real amorphous material of the type we are attempting to model. And, in any case, randomness in J will be generated by the random anisotropy under a renormalization-group transformation. Similarly, we allow for randomness in the strength of the external field. The Hamiltonian then has the form

$$\mathrm{H}_{\mathrm{RAM}} = -\sum_{\langle ij \rangle}[J(\hat{\mathbf{n}}_i \cdot \hat{\mathbf{n}}_j) + \delta_{Jij}]S_iS_j - \sum_i (H\hat{\mathbf{n}}_i^1 + \delta_{Hi})S_i \,. \tag{2}$$

Each S_i is now an Ising variable, which takes on only the values ± 1. Equation (2) has the convenient property that for $m=1$ it becomes the ISG model. This makes it easy to treat the ISG and the m-component RAM together, in a natural way.

IV. RECURSION RELATIONS

The thermodynamic behavior of Eq. (2) is determined, as usual, by the partition function,

$$Z = \operatorname*{Tr}_{S=\pm 1} \exp(-\mathrm{H}_{\mathrm{RAM}}) \,, \tag{3}$$

where the factor of $1/T$ has been absorbed into the coupling constants. In principle, the partial trace over the spins which have only two bonds can be performed exactly. If spin S_i is coupled to spin S_A by a bond J_{Ai}, and to S_B by J_{Bi}, then, when we trace over S_i, the spins S_A and S_B will have an effective coupling J_{ABi}, which is

$$J_{ABi} = \tfrac{1}{2}\{\ln[\cosh(J_{Ai}+J_{Bi})] - \ln[\cosh(J_{Ai}-J_{Bi})]\} \,. \tag{4}$$

The new effective bond J'_{AB} between spins S_A and S_B is obtained by summing over all of the parallel connections which join sites A and B:

$$J'_{AB} = \sum_{i=1}^{p} J_{ABi} \,. \tag{5}$$

When p is large we can invoke the central limit theorem, which describes the probability distribution for J'_{AB} in terms of its first and second moments. These moments depend on $\hat{\mathbf{n}}_A \cdot \hat{\mathbf{n}}_B$. Similarly, performing the trace over S_i will generate an effective external field at the A and B sites.

It is sufficient for our purposes to keep only the leading-order terms in $\hat{\mathbf{n}}_A \cdot \hat{\mathbf{n}}_B = \cos(\theta_A - \theta_B)$. This approximation will be adequate as long as the magnetization is small. It will break down for strong external fields or temperatures much less than T_c, conditions which will not be discussed in this work. So we can write J'_{AB} as

$$J'_{AB} = J'\cos(\theta_A - \theta_B) + \delta_{JAB} \,, \tag{6}$$

where the δ_J for different bonds are independent random variables with a Gaussian probability distribution

$$\mathrm{P}(\delta_J) = \left[\frac{2\pi}{m}\Delta_J\right]^{-1/2} \exp\left[-\frac{m}{2}\frac{\delta_J^2}{\Delta_J}\right] . \tag{7}$$

We describe the external field in an analogous fashion

$$H'_A = H'\cos(\theta_A) + \delta_{HA} \,, \tag{8}$$

with

$$\mathrm{P}(\delta_H) = \left[\frac{2\pi}{m}\Delta_H\right]^{-1/2} \exp\left[-\frac{m}{2}\frac{\delta_H^2}{\Delta_H}\right] . \tag{9}$$

Note that $\theta = 0$ is, by definition, the direction along which the (uniform) external field is applied.

The renormalization-group transformation is completed by rescaling all of the coupling constants by a factor of $1/p$. Due to the special nature of the hierarchical lattice, the reduced partition function has precisely the same structure as the original one. Thus we have generated a set of recursion relations for the new effective coupling constants, $\tilde{J}$, $\tilde{\Delta}_J$, $\tilde{H}$, and $\tilde{\Delta}_H$, in terms of the old ones. To fourth order, the recursion relations are

$$\tilde{J} = \frac{p}{m}J^2\left[1 - \frac{2}{m+2}J^2 - 2\Delta_J + \frac{8}{m}\Delta_J^2\right] , \tag{10a}$$

$$\tilde{\Delta}_J = \frac{p}{m}\left[\Delta_J^2 - \frac{4}{m}\Delta_J^3 + \frac{64}{3m^2}\Delta_J^4 + 2J^2\Delta_J - \frac{12}{m}J^2\Delta_J^2 + \frac{m-1}{m}J^4\right] , \tag{10b}$$

$$\tilde{H} = \frac{p}{m}HJ\left[1 - \frac{1}{m+2}J^2 - \frac{1}{m}\Delta_J + \frac{2}{m^2}\Delta_J^2\right] , \tag{10c}$$

$$\tilde{\Delta}_H = \frac{p}{m}\Delta_H\left[J^2 - \frac{4}{m}J^2\Delta_J + \Delta_J - \frac{2}{m}\Delta_J^2 + \frac{17}{3m^2}\Delta_J^3\right] . \tag{10d}$$

All terms nonlinear in the external field variables have been dropped. One should remember that we have ignored all of the higher moments of the probability distributions, which can only be formally justified when p is large. We expect, however, that in the usual fashion, the behavior of the recursion relations near their fixed points is "universal." So our results for the critical behavior should be qualitatively correct even when p is 4, which is a case of particular interest, since it has a fractal dimension of three.[18]

The critical behavior is described by the flows of the recursion relations near their fixed points. Setting the external fields to zero, Eqs. (10a) and (10b) have four fixed points. Each of the fixed points has four linear stability eigenvalues. The Liapunov exponents are the logarithms of these eigenvalues. It is convenient to define $g = 1/p$. Then the fixed points and their associated eigenvalues can be solved for as power series in g. From Eqs. (10), which go to fourth order in the coupling constants, we can locate the fixed points to order g^3, and the eigenvalues to order g^2. The results are displayed in Table I.

TABLE I. Properties of the fixed points of the renormalization-group recursion relations, Eqs. (10). J^* and Δ_J^* are the coordinates of the fixed point, and the λ's are the linear stability eigenvalues. $g=1/p$.

Paramagnetic fixed point	
$J^*=0$	$\Delta_J^*=0$
$\lambda_J=0$	$\lambda_{\Delta_J}=0$
$\lambda_H=0$	$\lambda_{\Delta_H}=0$
Ferromagnetic critical point	
$J^*=mg+\frac{2m^3}{m+2}g^3$	$\Delta_J^*=(m-1)m^2g^3$
$\lambda_J=2-\frac{4m^2}{m+2}g^2$	$\lambda_{\Delta_J}=2mg+2(m-1)mg^2$
$\lambda_H=1+\frac{m^2}{m+2}g^2$	$\lambda_{\Delta_H}=mg+(m-1)mg^2$
Spin-glass critical point	
$J^*=0$	$\Delta_J^*=m(g+4g^2+\frac{32}{3}g^3)$
$\lambda_J=0$	$\lambda_{\Delta_J}=2-4g+\frac{32}{3}g^2$
$\lambda_H=0$	$\lambda_{\Delta_H}=1+2g+\frac{1}{3}g^2$
FM-SG multicritical point	
$J^*=m\left[g+2g^2-\frac{2(m^2+2m-4)}{m+2}g^3\right]$	$\Delta_J^*=m[g+2(2-m)g^2-(m^2+11m-\frac{32}{3})g^3]$
$\lambda_J=2-\frac{4m^2}{m+2}g^2$	$\lambda_{\Delta_J}=2-2(m+2)g-2(m^2-5m-\frac{16}{3})g^2$
$\lambda_H=1+g-\frac{m(m+4)}{m+2}g^2$	$\lambda_{\Delta_H}=1-(m-2)g-(m^2+3m-\frac{1}{3})g^2$

V. DISCUSSION

It is useful to break up the discussion of the results into the three cases $m=1$, 2, and 3. The results for $m>3$ are qualitatively similar to the $m=3$ case. Before proceeding with the details, it should be noted that the results to lowest order in g for any m are identical to the Bethe-lattice results of Harris, Caflisch, and Banavar,[7] if we identify $g=1/\sigma$ in their work.

m = 1

This is the Ising spin glass, so it is no surprise that our results are qualitatively similar to those of Chen and Lubensky.[13] We find a ferromagnetic critical point, a spin-glass critical point, and a ferromagnet–spin-glass multicritical point, in addition to the trivial paramagnetic fixed point. The similarity is, in part, caused by the vanishing of the coefficient of the J^4 term in Eq. (10b) when $m=1$. This term was ignored by Chen and Lubensky, even for $m>1$. Since the J^4 term vanishes, the value of Δ_J^* at the ferromagnetic critical point is zero. We may note that the eigenvalue λ_{Δ_J} at this critical point reaches 1 when $g=\frac{1}{2}$. This is not significant here, since it occurs when $p=2$. If $p=2$, then the fractal dimension of the lattice is also two, so that T_c is actually zero. The power series in g should not be used when $T_c=0$.

m = 2

For $m=2$, the picture becomes more complex. The ferromagnetic critical point acquires a nonzero value of Δ_J^*. Now λ_{Δ_J} exceeds one at the ferromagnetic critical point when $g=\frac{1}{4}$, i.e., $p=4$, and a fractal dimension of three. The obvious interpretation of this is that the phase transition into the ferromagnetic state will be first order in three dimensions for $m=2$, since the fixed point is unstable.

We must be cautious about this conclusion. In the first place, as already noted, a hierarchical lattice with a fractal dimension of three differs in some very significant respects from a three-dimensional Bravais lattice. In the second place, while an unstable fixed point usually indicates a first-order transition for a translationally invariant system, this is less well understood for random systems. And in the third place, even if the transition really is first order, this may not be experimentally observable.[23]

The divergence which is seen in the high-temperature susceptibility series[9] for $m=2$ on three-dimensional Bravais lattices seems to resemble a spinodal point more than a normal critical point. Whether this should be considered evidence for a first-order transition is unclear. An alternative explanation is that, rather than a first-order transition, we have an intermediate phase, over a finite range of temperature, in which the magnetic susceptibility is infinite, but the magnetization is zero, as originally suggested by Aharony and Pytte.[24] This is similar to what is believed to occur in two dimensions[25,26] for the $m=2$ case. A similar phenomenon was suggested for different reasons by Derrida, Eckmann, and Erzan.[22]

The spin-glass critical point for $m=2$ remains qualitatively similar to that of the $m=1$ case. The only change from the $m=1$ case is that T_{sg} is smaller by a factor of $\sqrt{m}$, just as happens on the Bethe lattice.[7] This simplicity is due to the fact that $J^*=0$ at the spin-glass critical point, for any m.

$m=3$

The results for $m=3$ are, for the most part, similar to the $m=2$ results, but with one additional interesting feature. For $p=4$, the ferromagnet–spin-glass multicritical point apparently has a negative value of Δ_J^*. This is not physically accessible, since Δ_J is defined to be the second moment of a real probability distribution.

The ferromagnetic critical point is even more unstable than for $m=2$ and the same value of p. Given the behavior of the multicritical point, it seems likely that, in this case, we have no ferromagnetic state at all. This conclusion is supported by the high-temperature series[9] results, and a computer-annealing[27] study. All of these calculations, however, have been done in the limit $D/J \to \infty$. It may be that a ferromagnetic phase survives for small D when $m=3$, although this is widely believed not to happen.[28]

VI. SUMMARY

Renormalization-group recursion relations have been obtained for m-component magnets with strong random anisotropy on a class of hierarchical lattices. The fixed points and their stability eigenvalues have been calculated as a power series in $1/p$. The results for $p=4$, which yields a fractal dimension of three, are in good agreement with results obtained for three-dimensional Bravais lattices by other methods. For $m \geq 3$ and $p=4$, the ferromagnet–spin-glass multicritical point lies in the unphysical region, which probably means that there is no ferromagnetic phase, at least for strong anisotropy. For $m=2$ and $p=4$, the ferromagnetic critical point is unstable, which indicates that in this case we may have either a first-order transition or else an intermediate phase with an infinite susceptibility, but no magnetization.

Note added in proof. Hierarchical lattices have also been used to study the Heisenberg spin glass. See Banavar and Bray in Ref. 29.

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