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Giant Shapiro steps in Josephson-junction arrays

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Giant Shapiro steps have recently been observed in the *I-V* characteristics of Josephson-junction arrays, both with and without transverse magnetic fields. For magnetic fluxes per unit cell of the array $\Phi=(p/q)\Phi_0$, these steps occur at voltages per unit cell of $\hbar\omega/2eq$, where ω is the frequency of the rf field. I present a calculation of this effect, restricted to certain values of $f=p/q$ and certain current directions within the lattice. This calculation leads to results concerning the width of steps as a function of q and as a function of the amplitude of the driving field. It also leads to a prediction of subharmonic steps, at voltages of $(\hbar\omega/2e)(n/qm)$.

One of the most remarkable consequences of the ac Josephson effect is the appearance of Shapiro steps in the *I-V* characteristic of a Josephson junction exposed to a radio frequency (rf) field.[1,2] These steps are regions in which a range of currents is compatible with a single voltage. The voltages at which steps occur are related to the frequency ω of the rf field by

$$V_n=\frac{\hbar\omega}{2e}n\,, \tag{1}$$

for the voltage of the nth step. In junctions with an appreciable capacitance, it is possible to see subharmonic steps, with[3]

$$V_{n,m}=\frac{\hbar\omega}{2e}\frac{n}{m}\,. \tag{2}$$

Recently, Benz *et al.* have reported the observation of "giant" Shapiro steps (previously seen by other authors) in arrays of Josephson junctions.[4-6] For a square $N\times N$ array of junctions, these steps appear at voltages of

$$V_n=\frac{\hbar\omega}{2e}Nn\,, \tag{3}$$

corresponding to ordinary Shapiro steps with voltages $v_n=(\hbar\omega/2e)n$ across each unit cell of the junction. This is already somewhat striking, as one might expect the inevitable inhomogeneities of a real array to destroy the phase coherence necessary for the appearance of giant steps.

But Benz *et al.* also performed the same experiment in the presence of a magnetic field transverse to the plane of the array. For "rational" magnetic fields, for which the flux Φ per unit cell is a rational fraction $f=(p/q)$ of the superconducting flux quantum Φ_0, $\Phi=f\Phi_0$, they found steps at voltages

$$V_n=\frac{\hbar\omega}{2e}\frac{Nn}{q}\,, \tag{4}$$

corresponding to $1/q$ the voltage of the zero-field steps.

Although these results have been obtained also in numerical simulations, the detailed understanding of these steps may present considerable difficulty.[7,8] Even the zero-current ground states of an array are not completely understood as a function of field.[9] However, in this Rapid Communication I will show that for a special family of magnetic fields it is possible to obtain a detailed understanding of the step structure, at least for currents oriented in a particular direction. I will present a calculation leading to an implicit formula for the step widths. In the limit $q\gg 1$, this formula simplifies considerably, so that for low rf voltage the step widths may be obtained directly. The most significant result of this calculation is that even in the overdamped limit, in which the junction capacitance is negligible, subharmonic steps should appear, corresponding to voltages

$$v_{n,m}=\frac{\hbar\omega}{2e}\frac{n}{mq}\,, \tag{5}$$

across each junction of the array. Subharmonic steps should also appear in zero field.

The ground states of an array in the absence of quantum fluctuations are determined by requiring that the Hamiltonian,

$$H=-\sum_{\langle ij\rangle}\frac{\hbar i_0}{2e}\cos(\theta_i-\theta_j-A_{ij})\,, \tag{6}$$

should be minimized as a function of the superconducting phases θ_i of the sites of the array. The sum is over nearest-neighbor sites, and i_0 is the critical current of the junctions in the array. The bond terms A_{ij} are line integrals of the vector potential, which satisfy $\sum_P A_{ij}=2\pi f$, where the sum is a directed sum around a plaquette. It is easy to show that the transformations $f\to f+1$ and $f\to 1-f$ do not change any physical property of the model defined by Eq. (6).

The requirement that the Hamiltonian equation (6) be minimized is identical to the requirement that supercurrent be conserved at every site in the lattice.[10] The supercurrent across the $\langle ij\rangle$ bond is $i_0\sin(\theta_i-\theta_j-A_{ij})$. On a square lattice, a simple way of satisfying this constraint is for the supercurrent along the diagonal "staircases" into which the lattice may be separated to be constant (see Fig. 1). Indexing these staircases by k, and defining $\phi_k=\theta_i-\theta_j-A_{ij}$ on the kth staircase, we then find that there are locally stable states with

$$\phi_k=\pi fk+\alpha_0-\pi[fk+\alpha_0/\pi]_n\,, \tag{7}$$

where the brackets $[\,]_n$ indicate the nearest integer function. The requirement that the states be locally stable

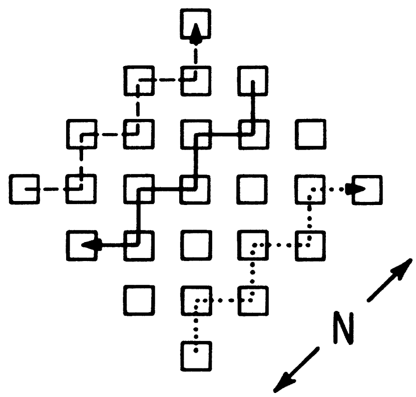

FIG. 1. The sample geometry considered in this paper, for the case $N=3$. The current is imagined to be constant along the "staircases" of the square lattice. Three such staircases are marked. This state will automatically satisfy current conservation at each superconducting island (marked by the squares).

constrains the ϕ_k to satisfy $-\pi/2 \le \phi_k \le \pi/2$, a constraint satisfied by the ϕ_k defined in Eq. (7). The constant α_0 is determined by minimizing the global energy. The result is that for odd q, $\alpha_0=0$, while for even q, $\alpha_0=\pi/2q$. The interested reader may consult Ref. 10 for details.

These staircase states are not the true ground states for all f. However, they are the true ground states for $f=\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{5}$, $\frac{3}{7}$, and $\frac{3}{8}$, and it has been conjectured that they are the true ground states for the "Fibonacci" sequence of values of f, $f=\frac{5}{13}$, $\frac{8}{21}$, $\frac{13}{34}, \dots$, which converges to one minus the golden mean Ω. ($f=1-\Omega$ is equivalent to $f=\Omega$ for this problem.[11])

If we wish now to find states that carry current along the [11] direction parallel to the staircases, we simply choose

$$\phi_k = \pi f k + \alpha + \alpha_0 - \pi[fk + (\alpha+\alpha_0)/\pi]_n , \tag{8}$$

with $\alpha \neq 0$. The current carried per staircase is then

$$\begin{aligned} \bar{i}(\alpha) &= \frac{1}{q}\sum_{k=1}^{q} i_0 \sin\phi_k(\alpha) \\ &= \frac{1}{q}\sum_{k=1}^{q} i_0 \cos\phi_k(\alpha_0)\sin(\alpha) \\ &= \frac{i_0 \csc(\pi/2q)}{q}\sin(\alpha) . \end{aligned} \tag{9}$$

Note that α has the range $-\pi/2q \le \alpha \le \pi/2q$. It follows that the maximum current for these states is $i_c = i_0/q$. These states are locally stable provided that α has the stated range.

Now let us turn to the problem of the resistive character of an array at finite voltage (ac or dc). I will ignore both quantum and thermal fluctuations in the following discussion. I will also assume that the junctions are strongly overdamped, so that it is possible to ignore capacitive effects. Then the current on a link is

$$i_{\langle ij\rangle} = \frac{\hbar}{2eR}\frac{d}{dt}(\theta_i - \theta_j) + i_0 \sin(\theta_i - \theta_j - A_{ij}) , \tag{10}$$

where R is the normal resistance. This current will be conserved at every site of the array.

I will make one further assumption, that the response frequency $\omega_R = 2ei_0R/\hbar$ for Eq. (10) is large compared to the frequency $\omega(v) = 2ev/\hbar$ corresponding to the voltage across a typical junction of the array. This is equivalent to assuming that the supercurrent is typically larger than the current in the normal channel.

Suppose that a time-dependent voltage $V(t) = V_0 - V_1 \times \cos(\omega t)$ is applied across an $N \times N$ diamond-shaped sample (Fig. 1) so that the current flows parallel to the staircases. The average voltage across an individual junction is then $v(t) = V(t)/2N$. (The factor of 2 arises because of the diagonal geometry.) We can satisfy the requirement of current conservation if we take a staircase form, as in Eq. (8), but make α a function of time, with

$$\dot{\alpha}(t) = \frac{2ev(t)}{\hbar} . \tag{11}$$

Then the normal currents and supercurrents will be separately conserved at each site.

If we make a small deviation in this state, and we are in the large resistance regime mentioned above, then the state will be dynamically stable provided that the static state for that α is locally stable. Thus we require $\alpha(t)$ to also satisfy $-\pi/2q \le \alpha(t) \le \pi/2q$. For any state with a finite dc voltage, this implies that $\alpha(t)$ will occasionally undergo a discontinuous change as a function of time. This is related to the process of vortex slippage in the array, and involves rapid time-scale behavior which I do not attempt to describe. This process of vortex slippage corresponds to a coherent movement of the vortex lattice across the direction of supercurrent flow, as suggested by Benz *et al.* and observed numerically by Lee, Stroud and Chung.[4,7] It is important to realize that the instability causing the vortex slippage is a local instability.[10] Thus it will not necessarily lead to the appearance of voltages at the boundaries, even if phases in the interior of the sample are varying rapidly. In the large resistance limit, the current passed during these events will be negligible.[12]

Thus we may calculate the current using

$$\alpha(t) = A(t) - (\pi/q)[qA(t)/\pi]_n , \tag{12a}$$

with

$$A(t) = \gamma_0 + \gamma_1 t - \beta \sin(\omega t) , \tag{12b}$$

where γ_0 is an initial phase, $\gamma_1 = eV_0/\hbar N$, and $\beta = eV_1/N\hbar\omega$. There will be a Shapiro step at a particular voltage if there is a range of possible net supercurrents at that voltage corresponding to different values of γ_0. We must thus calculate the average supercurrent corresponding to a particular $\alpha(t)$.

First we use a Fourier representation of the roundoff function equation (12a),

$$\alpha(t) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{jq} \sin[2jqA(t)] . \tag{13}$$

The supercurrent per staircase is given by Eq. (9), with $\alpha = \alpha(t)$. Consider $\exp[\iota\alpha(t)]$. We can write this as

$$\exp[\iota\alpha(t)] = \prod_{j=1}^{\infty}\left[\exp\left(\iota\frac{(-1)^{j+1}}{jq}\sin[2jqA(t)]\right)\right] . \tag{14}$$

Recognizing a generating function for a Bessel series, we can rewrite this as[13]

$$\exp[\imath\alpha(t)]=\prod_{j=1}^{\infty}\left[\sum_{l=-\infty}^{\infty}J_l((-1)^{j+1}/jq)\exp[2\imath ljqA(t)]\right]. \quad (15)$$

Clearly this leads to an expansion,

$$e^{\imath\alpha(t)}=g_0(q)+g_1(q)e^{2\imath qA(t)}+g_{-1}(q)e^{-2\imath qA(t)}+\cdots+g_m(q)e^{2\imath mqA(t)}+g_{-m}e^{-2\imath mqA(t)}+\cdots. \quad (16)$$

The $\{g_m\}$ are functions only of q, and can be related to sums of products of the Bessel functions $J_l((-1)^{j+1}/jq)$.

We must now average $\exp[\imath\alpha(t)]$ over time. This we can do order by order in the series equation (16). We write

$$\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}dt\exp[2\imath mqA(t)]=\exp(2\imath mq\gamma_0)\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}dt\cos[2mq\gamma_1 t-2mq\beta\sin(\omega t)]. \quad (17)$$

This is nonzero provided that $2mq\gamma_1=n\omega$, with n an integer. Then the integral gives a Bessel function,

$$\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}dt\exp[2\imath mqA(t)]=J_n(2mq\beta)\exp(2\imath mq\gamma_0). \quad (18)$$

Thus we expect to see steps in the I-V characeteristic for voltages V_0 satisfying

$$V_0=2N\frac{\hbar\omega}{2e}\frac{n}{mq}, \quad (19)$$

corresponding to a voltage of $(\hbar\omega/2e)(n/mq)$ across each junction. Again, $2N$ appears in Eq. (19) instead of N (as in the quantization observed by Benz *et al.*) because in the diagonal geometry the total number of junctions across an $N\times N$ sample is $2N$ (see Fig. 1).

It is simple to develop convergent numerical procedures for calculating the coefficients $g_m(q)$ of the series equation (16). Then a maximization (minimization) of the current

$$\bar{i}_s=\frac{i_0\csc(\pi/2q)}{q}\overline{\sin[\alpha(t)]}, \quad (20)$$

with respect to γ_0 will yield the width of the various steps. More insight may be gained, however, by studying the step width in a simple limit. The large-l terms in the product equation (15) will be suppressed for large q. We will thus approximate the full product by including only terms up to first order in q^{-1}. Then we have

$$\sin[\alpha(t)]\approx\left[\prod_{j=1}^{\infty}J_0\left(\frac{1}{jq}\right)\right]\sum_{m=1}^{\infty}2(-1)^{m+1}\left(\frac{J_1(1/mq)}{J_0(1/mq)}\right)\sin[2mqA(t)]. \quad (21)$$

If $mq\beta\ll 1$, then the current in each step will be dominated by the first term in this series for which the step voltage leads to a nonvanishing result upon integration. If we include only the effects of this first term, then a simple calculation yields the width $\Delta i_{n,m}$ of the step at the voltage $V_{n,m}=2N(\hbar\omega/2e)(n/mq)$. The result is

$$\Delta i_{n,m}=8N\frac{i_0\csc(\pi/2q)}{q}\left[\prod_{j=1}^{\infty}J_0(1/jq)\right]\left(\frac{J_1(1/mq)}{J_0(1/mq)}\right)J_n(2mq\beta). \quad (22)$$

Thus, provided that $mq\beta\ll 1$, we obtain

$$\Delta i_{n,m}\sim\frac{Ni_0}{mq}(mq\beta)^n. \quad (23)$$

For the large m steps, for which the requirement on $mq\beta$ does not hold, we must take into account the contribution of higher-order terms in the series.

The principal drawbacks of the above analysis are that it is restricted to particular values of f, and that it is restricted to current flowing in a particular direction with respect to the lattice. Nevertheless, one might hope that its main qualitative conclusions, that the subharmonic steps appear even for strongly overdamped junctions, and that the step widths scale according to Eq. (23) for small rf voltages, will hold in the more general case. The detailed predictions of this calculation can be checked by performing experiments upon diamond-shaped arrays. The reader will note that the results above apply also in the case $q=1$, corresponding to zero field or an integral number of flux quanta per cell.

This calculation also suggests directions for further study. The experimental and numerical results alluded to above seem not to show subharmonic steps (although there are some ambiguities, particularly in the experimental results). This may be due to boundary effects, to the difference between current and voltage driving, or to inhomogeneities (at least in the experimental arrays). All of these effects must be included in a full theory of the Shapiro steps.

While it is probably premature to hope for analytical results for general f soon, the calculation of currents for staircase-type states for other directions may prove tractable, although in these cases the normal currents and supercurrents will not be separately conserved.[10] Also, the form of $\alpha(t)$ chosen implies an occasional coherent shift of the vortices in the lattice to new positions. This shift will involve relatively rapid variation of the phases, the details of which may be quite interesting.

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