X-ray-absorption and Kondo problems: Variations on the Abrikosov-Migdal field-theoretical renormalization-group approach

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The scaling equations for the x-ray and Kondo problems are rederived using the standard techniques employed in statistical-mechanical and field-theoretical problems. Specifically we obtain a Callan-Symanzik equation for the bare theory, the corresponding β functions, and anomalous dimensions for the involved fields being computed up to the three-loops order in the coupling constants. The different steps of the procedure are developed in some detail. In particular, we are able to identify the relevant Feynman diagrams for the vertex functions at the three-loops order. By exploiting this, we obtain for the Kondo model a β function in agreement with recent exact results. Moreover, when our series for β is naively extrapolated, the exact result emerges.

I. INTRODUCTION

The x-ray-absorption and Kondo problems are two prominent topics in both many-body physics and solid-state theory. In the past two decades many workers have produced theoretical and experimental results on these two phenomena. The x-ray problem was studied in the pioneer works by Mahan,[1] Roulet, Gavoret, and Nozières,[2,3] and Nozières and De Dominicis.[4] In Refs. 1–3 the main characteristics of perturbation theory have been established, while in Ref. 4 one has a one-body theory exact solution. This solution was also found by Schotte and Schotte[5,6] by using two different methods. Two interesting approaches are also that made by Combescot and Nozières[7] and the more recent one by Mahan[8] (see also the references therein). However, the history of the Kondo model is much more complex. In the review by Tsvelick and Wiegmann,[9] it is mentioned that there are more than 1500 papers on this problem. We only mention a small number of these works.

After the famous Kondo demonstration,[10] where the transition probability of conduction electrons in the presence of a magnetic impurity should exhibit a logarithmic divergence at zero temperature, early attempts to improve this result were made by Nagaoka,[11] Abrikosov,[12] Falk and Fowler,[13] Hamann,[14] Cheung and Mattuck,[15] and others. An important series of papers that appeared in the 1970's are those based in a renormalization-group approach. In the pioneer work by Abrikosov and Migdal,[16] a multiplicative renormalization scheme for the theoretical propagators and vertex parts is implemented. The idea is similar to the Gell-Mann–Low procedure for renormalizing quantum electrodynamics.[17,18] It leads to scaling equations analogous to the ones found in the theory of second-order phase transition.[19–22] These authors performed the theory up to the three-loops order in the exchange coupling. By using a slightly different procedure, namely, by employing a Lie differential equation, Fowler and Zawadowski[23] and Sólyom[24] also found the scaling equations up to the two-loops order.

Other important scaling approaches were made by Anderson *et al.*[25,26] via an equivalence of the Kondo problem with the thermodynamics of a classical one-dimensional Coulomb gas and by Anderson[27] in its pedestrian procedure known as "a poor man's derivation." This last method was improved by Sólyom and Zawadowski.[28] These scaling approaches have not been able, however, to give the correct physical behavior in the zero-temperature limit. The main reason for this failure is that when the temperature decreases the effective coupling *increases unboundedly*. Then there is a complete compensation of the impurity magnetic moment by the conduction electrons. It follows that at temperatures $T \ll T_K$, where T_K is the Kondo temperature, the physical magnitudes have a simple power-law behavior. This picture emerged from the works by Anderson *et al.*,[25–27] by Nozières[29] and especially from Wilson's important work.[30,31] Therefore, it is clear that these results cannot be obtained from any *finite-order* perturbation theory in the exchange coupling.

In the past few years the Kondo problem was solved exactly via a Bethe ansatz by Andrei,[32] Wiegmann[33,34] and Fateev and Wiegmann.[35] In spite of this, in a recent paper Barnes[36] revitalizes the diagrammatic methods to obtain scaling equations. He finds exact results for the Kondo model, in agreement with Wilson's numerical method. Its approach, although clear, is highly sophisticated and requires a great deal of work. Therefore, it should be desirable to find a simpler route to obtain the exact results.

This work is a first attempt in such a direction. We consider a generalized single-site impurity model with both exchange and potential scattering couplings. It includes the x-ray and the $S=\frac{1}{2}$-Kondo Hamiltonians as particular cases. The theoretical bare propagators and vertex parts are calculated up to the three-loops order in the coupling constants by keeping the relevant Feynman diagrams[37–39] and using the most popular zero-

temperature diagrammatic technique. We find the usual $(\omega/\Lambda)^n \ln^p|\Lambda/\omega|$ singularities (Λ is an energy cutoff and n and p are integers). Next a renormalized theory at the energy scale x is defined by using mass renormalization, pseudofermion wave renormalization, and constant coupling renormalization, exactly as in previous papers.[16,23,24] The renormalization conditions are an essential ingredient here.

To obtain the true dependence on Λ we now write a Callan-Symanzik-like equation[22,40,41] for the bare functions. The characteristic curves of this equation lead to the renormalization-group flow equations and to the physical behavior of the theory. The advantages of this procedure on the previous ones are mainly the following.

On one hand, the relation between the changes in temperature (or in frequencies) and the motions in the abstract flowing-coupling space here is more transparent: in the infrared (ultraviolet) limit, $T=0$ ($T \gg T_K$), the Hamiltonian is driven towards the strong- (weak-) coupling fixed point. This is exactly the behavior observed by Wilson.[30,31] Furthermore, scaling only holds in these limits, contrarily to the usual assumption involved in the Gell-Mann–Low formulation of the problem. On the other hand, and more remarkably, Callan-Symanzik implementation of the renormalization scheme leads to a natural selection of the relevant Feynman diagrams: by taking an adequate choice of the renormalization point, one concludes that only the simpler logarithmic singularities ($n=0$ and $p=1$) determine the β functions. As a consequence of this fact, our three-loops order term in the scaling equation turns out to be different from that reported in Ref. 16. Moreover, this (new) result now leads, by a simple extrapolation, to the exact Barnes' scaling law.[36]

The outline of this work is as follows. In Sec. II we briefly review the Hamiltonian and bare propagators. The vertex parts are calculated up to the two-loops order for the general case in which one has three arbitrary external frequencies here. The calculation in this section includes all the involved Feynman diagrams up to this order.

In Sec. III the renormalization of the theory is developed in some detail. First we obtain the two-loops β functions. It is explained how, by using appropriate renormalization conditions, one is led to physical results for these functions. By exploiting this idea in the next step we show what the relevant graphs involved at the three-loops order are. The Callan-Symanzik equation for the bare theory is then formulated. By making careful use of the renormalization-group invariants one now obtains the mentioned results for the three-loops β functions of the Kondo model.

In Sec. IV we analyze the question about how the scaling law arose and the specific results for both problems. In the x-ray absorption case we find the standard results[1-4] up to the lowest order in the corresponding coupling. For the Kondo model, on the other hand, the true flow equation is extracted as a conjecture from the finite-order perturbative series. The same method is not able, however, to give the strong-coupling limit for the anomalous dimension of the pseudofermion field.

Section V is devoted to concluding remarks. In particular, we indicate how the approach could be further improved. The plausible connection with Wilson's ideas[30,31] is also analyzed.

II. REVIEW OF THE HAMILTONIAN AND BARE PROPAGATORS

A standard expression for the Kondo Hamiltonian is[15]

$$H=\sum_{\mathbf{k},\alpha}\epsilon_{\mathbf{k}}a^{\dagger}_{\mathbf{k}\alpha}a_{\mathbf{k}\alpha}+E_0\sum_{\beta}b^{\dagger}_{\beta}b_{\beta}-\frac{J}{2N}\sum_{\mathbf{k}\mathbf{k}'}\sum_{\alpha\alpha',\beta\beta'}(\sigma_{\alpha'\alpha}\cdot\mathbf{S}_{\beta'\beta})b^{\dagger}_{\beta'}a^{\dagger}_{\mathbf{k}'\alpha'}a_{\mathbf{k}\alpha}b_{\beta}\ . \tag{2.1}$$

Here $a^{\dagger}_{\mathbf{k}\alpha}$ is a creation operator for a conduction s electron with momentum $\mathbf{k}$ and spin α, while $b^{\dagger}_{\beta}$ creates a deep d electron with spin β. Abrikosov's pseudofermion representation[12] of the impurity operator has been used. The components of the vector matrices $\sigma_{\alpha'\alpha}$ and $2\mathbf{S}_{\beta'\beta}=\tau_{\beta'\beta}$ are the usual Pauli matrices for s and d electrons. The bare s- and d-electron energies measured from the Fermi level are $\epsilon_{\mathbf{k}}$ and E_0, respectively. Finally, J is the s-d coupling constant ($J<0$ in the antiferromagnetic case) and N is the number of impurity atoms.

Instead of H_K, a more general Hamiltonian

$$H=H_K+\frac{V}{2N}\sum_{\mathbf{k}\mathbf{k}'}\sum_{\alpha\beta}b^{\dagger}_{\beta}a^{\dagger}_{\mathbf{k}'\alpha}a_{\mathbf{k}\alpha}b_{\beta} \tag{2.2}$$

will be considered. Note that when $J=0$ and $V\neq 0$ this equation resembles the x-ray-absorption Hamiltonian.[1-4] The only physical difference is that the deep level in Eq. (2.2) is twofold degenerate, while in the original Mahan–Nozières–De Dominicis model one has a nondegenerate level. The case where $E_0\cong 0$ (in addition to $J=0$, $V\neq 0$) also resembles the single-site approach to the Hamiltonian of a mixed valence metal in which the hybridization term has been neglected.[42,43]

We remark that the V term in Eq. (2.2) is written as a particle-particle repulsion ($V>0$), contrarily to the original picture of a particle-hole attraction.[1-4]

By using the decomposition

$$\sigma\cdot\tau=\sigma_z\tau_z+2(\sigma_+\tau_-+\sigma_-\tau_+)$$

we can write H in the form

$$H=\sum_{\mathbf{k},\sigma}\epsilon_{\mathbf{k}}a^{\dagger}_{\mathbf{k}\alpha}a_{\mathbf{k}\sigma}+E_0\sum_{\tau}b^{\dagger}_{\tau}b_{\tau}-\frac{J}{2N}\sum_{\mathbf{k}\mathbf{k}',\sigma}b^{\dagger}_{\sigma}a^{\dagger}_{\mathbf{k}',-\sigma}a_{\mathbf{k},\sigma}b_{-\sigma}+\frac{1}{2N}\sum_{\sigma\tau}\sum_{\mathbf{k}\mathbf{k}'}\left[V+(-1)^{\sigma+\tau}\frac{J}{2}\right]b^{\dagger}_{\tau}a^{\dagger}_{\mathbf{k}'\sigma}a_{\mathbf{k}\sigma}b_{\tau} \tag{2.3}$$

($\sigma,\tau=\pm\frac{1}{2}$). Hence there are three elementary processes associated with this Hamiltonian. They are shown in Fig. 1. Diagram 1(a) [1(b)] shows a spin–no-flip scattering with total spin $S_z=2\sigma$ ($S_z=0$). When $J\neq 0$ the strengths of these graphs are different from each other. On the other hand, diagram 1(c) represents the spin-flip scattering and only exists for $S_z=0$.

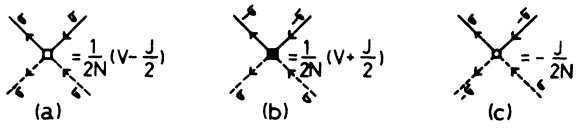

FIG. 1. The elementary processes associated with Eq. (2.3). Dashed solid lines correspond to the pseudofermion (conduction-electron) propagator. Note the spin structure of each vertex.

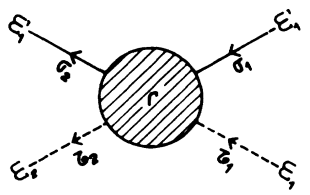

FIG. 2. The vertex part Γ. The energies $\varepsilon_1,\varepsilon_2$ ($\varepsilon_1',\varepsilon_2'$) are associated to the pseudofermion (conduction-electron) lines.

In the rest of this paper we make the rescaling $V/2N\to V$, $J/2N\to J$, in order to follow Nozières notation.[2-4] Now, we take a constant density of states (N_F) for the conduction band and perform the calculation of the Feynman diagrams up to the two-loops order in the coupling constants. In this section we only keep the *most divergent* contribution in each diagram (see also Sec. III).

For the pseudofermion (deep-electron) propagator $G_{d\sigma}(\omega)$ we obtain the result[44]

$$G_{d\sigma}^{-1}(\omega)=(\omega-E_{0R})\times\left[1+2N_F^2(V^2+\tfrac{3}{4}J^2)\ln\left|\frac{\Lambda}{\omega-E_{0R}}\right|\right], \tag{2.4}$$

where Λ is a cutoff for the momentum integration. The renormalized deep-electron energy E_{0R} arises from E_0 by adding the bubble diagrams up to the two-loops order. To get a simpler notation from now on we take $E_{0R}=0$. As is explained in Refs. 2 and 3, the conduction-electron propagator $G_\sigma(\mathbf{k},\mathbf{k}';\omega)$ does not have Λ divergences. The diagrammatic series break the translational invariance and therefore $G_\sigma(\mathbf{k},\mathbf{k}';\omega)$ is no longer diagonal in $\mathbf{k}$.

However, the density of states turns out to be multiplied by a *finite constant*. This is a crucial point. It implies that the renormalization properties of any function f coincide with those of $N_F f$. So, in the rest of this paper we make

$$N_F=1\ , \tag{2.5}$$

and the density of states disappears from our formulas. If needed, it can be restored by dimensional considerations.

Now we consider the vertex part

$$\Gamma_{\tau_1,\sigma_1;\tau_2,\sigma_2}(\varepsilon_1,\varepsilon_1';\varepsilon_2,\varepsilon_2')\ ,$$

which is the sum of all the diagrams of the form of Fig. 2 (without their external legs). This function can be decomposed into a scalar part Γ^0 and a vector part Γ^1, according to[15]

$$\Gamma_{\tau_1,\sigma_1;\tau_2,\sigma_2}(\{\omega\})=\Gamma^0(\{\omega\})\delta\sigma_1\sigma_2\delta\tau_1\tau_2+\tfrac{1}{2}\Gamma^1(\{\omega\})\boldsymbol{\sigma}_{\sigma_1\sigma_2}\cdot\boldsymbol{\tau}_{\tau_1\tau_2}\ . \tag{2.6}$$

From this it follows that

$$\begin{aligned}\Gamma^1(\{\omega\})&=\Gamma_{\sigma,-\sigma;-\sigma,\sigma}(\{\omega\})\ ,\\ \Gamma^0(\{\omega\})&=\Gamma_{\sigma,\sigma;\sigma,\sigma}(\{\omega\})-\tfrac{1}{2}\Gamma_{\sigma,-\sigma;-\sigma,\sigma}(\{\omega\})\\ &=\Gamma_{\sigma,-\sigma;\sigma,-\sigma}(\{\omega\})+\tfrac{1}{2}\Gamma_{\sigma,-\sigma;-\sigma,\sigma}(\{\omega\})\ .\end{aligned} \tag{2.7}$$

Now we calculate all the Feynman diagrams that contribute to Γ up to the two-loops order. Topologically the involved graphs are the same as one has in the x-ray similar problem.[1-4] The additional ingredient is that now a combinatorial factor arising from the three possible elementary processes of Fig. 1 must be computed. Finally one expresses the perturbative series for the two possible reaction channels defined by Eqs. (2.6) and (2.7). Keeping only the most singular terms of each diagram, one obtains

$$\begin{aligned}\Gamma^0(\varepsilon_1,\varepsilon_2;\Omega_1,\Omega_2)=V&-(V^2+\tfrac{3}{4}J^2)\left[\ln\left|\frac{\Lambda}{\Omega_1}\right|-\ln\left|\frac{\Lambda}{\Omega_2}\right|\right]\\ &+(V^3+\tfrac{9}{4}VJ^2+\tfrac{3}{4}J^3)\left[\ln^2\left|\frac{\Lambda}{\Omega_1}\right|-I(\Omega_2,\varepsilon_1')-I(\Omega_2,\varepsilon_2')\right]\\ &+(V^3+\tfrac{9}{4}VJ^2-\tfrac{3}{4}J^3)\left[\ln^2\left|\frac{\Lambda}{\Omega_2}\right|-I(\Omega_1,\varepsilon_1')-I(\Omega_1,\varepsilon_2')\right]\\ &+2(V^3+\tfrac{3}{4}VJ^2)\ln\left|\frac{\Lambda}{\varepsilon}\right|\end{aligned} \tag{2.8}$$

for the scalar part of Γ and

$$\Gamma^1(\varepsilon_1,\varepsilon_2;\Omega_1,\Omega_2)=-J+2JV\left[\ln\left|\frac{\Lambda}{\Omega_1}\right|-\ln\left|\frac{\Lambda}{\Omega_2}\right|\right]$$
$$+J^2\left[\ln\left|\frac{\Lambda}{\Omega_1}\right|+\ln\left|\frac{\Lambda}{\Omega_2}\right|\right]-J(3V^2+\tfrac{7}{4}J^2+3VJ)\ln^2\left|\frac{\Lambda}{\Omega_1}\right|$$
$$-J(3V^2+\tfrac{7}{4}J^2-3VJ)\ln^2\left|\frac{\Lambda}{\Omega_2}\right|+J\left[3V^2-\frac{J^2}{4}+VJ\right][I(\Omega_1,\varepsilon_1')+I(\Omega_1,\varepsilon_2')]$$
$$+J\left[3V^2-\frac{J^2}{4}-VJ\right][I(\Omega_2,\varepsilon_1')+I(\Omega_2,\varepsilon_2')]-2J\left[V^2-\frac{J^2}{4}\right]\ln\left|\frac{\Lambda}{\varepsilon}\right| \quad (2.9)$$

for the vector one. In these formulas

$$\Omega_1=\varepsilon_1+\varepsilon_1'=\varepsilon_2+\varepsilon_2' ,$$
$$\Omega_2=\varepsilon_1-\varepsilon_2'=\varepsilon_2-\varepsilon_1' \quad (2.10)$$

are the total energies associated with the particle-particle (Ω_1) and with the particle-hole (Ω_2) interaction channels.[45]

The first term in these expressions is the elementary interaction. The terms associated with the sums and differences of the $\ln\Lambda$'s represent the contribution of the one-loop graphs. The terms in $\ln^2\Lambda$ arise from the two-loops ladder graphs. The parquet (nonladder) diagrams[2,3] are represented by the terms in $I(\Omega_i,\varepsilon_j')$. The latter is a double-logarithmic function defined by

$$I(x,y)=\left[\ln\left|\frac{\Lambda}{x}\right|\ln\left|\frac{\Lambda}{y}\right|-\tfrac{1}{2}\ln^2\left|\frac{\Lambda}{y}\right|\right](1-\Delta_{y,0})$$
$$-\tfrac{1}{2}\ln^2\left|\frac{\Lambda}{x}\right|\Delta_{y,0} , \quad (2.11)$$

where the symbol $\Delta_{y,0}$ takes the value one (zero) for y equal to (different from) zero. Finally, the last term in these formulas is due to the irreducible[2,3] two-loops graph. It depends on $\varepsilon=\max(\varepsilon_1,\varepsilon_2)$.

The imaginary parts of the propagators have been neglected in this calculation. Note that Eqs. (2.8) and (2.9) are general expressions for Γ^0 and Γ^1. No hypothesis on the external frequencies has been made. The only restriction is the conservation law (2.10), and therefore these functions depend on three independent variables.[46]

III. RENORMALIZATION OF THE THEORY

Expressions (2.4), (2.8), and (2.9) for the Green's functions and for the vertex parts are not satisfactory. These formulas contain a strong dependence with the cutoff, which is of the form $\ln\Lambda$ for G_d and $\ln^2\Lambda$ for Γ. Therefore, the expressions diverge in both limits, $\Lambda\to 0$ and $\Lambda\to\infty$. However, this double divergence is hard to accept in a physical framework. The latter is especially clear in the case $\omega/\Lambda\to\infty$, because electrons with energies far away from the Fermi level should not produce any type of singularity in the physical magnitudes. The main reason for which this type of behavior arises has been clearly explained in Wilson's numerical works about Kondo[30,31] and Anderson[47] models. The point is that in these problems one can distinguish an infinite number of energy scales determining the phenomena.

To obtain a finite theory the various scales must be separated in some way. Wilson's method provides a possible manner for doing this. Starting, for example, with an energy scale Λ_0, one considers in a second step the scale $\Lambda_0/2$, then $\Lambda_0/(2^2)$, then $\Lambda_0/(2^3)$, and so on. When the problem is attacked by using a continuous perturbation theory such as the one explained in Sec. II, we are simultaneously considering all the possible scales involved in the phenomena. No separation has been made, and one finds divergent results.

Here we propose another way to separate the various scales, by using a multiplicative renormalization method. Although this type of method has been considered some years ago,[16,23–28] we present in this work a different procedure to obtain the renormalization-group flow equations (RGFE's). The idea is to employ for the Kondo problem the formal machinery that is used in field theory to renormalize, for example, a classical ϕ^4 theory,[22] or a nonlinear σ model,[48] etc. Essential ingredients of these methods are the renormalization conditions and the Callan-Symanzik equation.

In this case, the first step of the procedure has been given already when the bare energy E_0 was replaced by E_{0R}, absorbing the bubble diagrams. This is similar to the bare mass renormalization in field theory.[22] Exactly as in that case, one must postulate here that the observable quantity is E_{0R} rather than E_0. Moreover, when the theory is worked out to a *finite order,* some bubble diagrams turns out to be divergent for $\Lambda\to\infty$. So, if E_{0R} is a finite quantity, E_0 is not.

Now we consider the renormalization of G_d^{-1}. A renormalized theory at the energy scale x is defined by requiring that

$$G_{d\sigma,R}^{-1}(\omega)\cong\omega, \quad |\omega|\cong x \quad (3.1)$$

(recall that E_{0R} has been taken as zero). Condition (3.1) means that the renormalized function has the same form as the bare one (hereafter renormalized quantities are denoted with a suffix R). The relation between the renormalized and the bare propagators is of the form

$$G_{d\sigma;R}^{-1}(\omega)=Z_d G_{d\sigma}^{-1}(\omega) . \quad (3.2)$$

Taking into account Eq. (2.4) one obtains

$$Z_d = 1 - 2(V^2 + \tfrac{3}{4}J^2)\ln\frac{\Lambda}{x} , \tag{3.3}$$

up to the two-loops order. Strictly speaking, Eq. (3.1) only holds at $|\omega| = x$. However, $G_{d\sigma,R}$ turns out to be independent of Λ at any frequency, and then it is finite in both limits $\Lambda \to 0$ and $\Lambda \to \infty$. The constant $Z_d^{1/2}$ gives the wave-function renormalization of the deep-electron field. The effect of Z_d is to cancel the two-loops graphs of G_d^{-1}, which has the logarithmic divergence.

Now we consider the renormalization of the vertex parts. In an intermediate step we define the vertices $\tilde{\Gamma}^0$ and $\tilde{\Gamma}^1$ by

$$\tilde{\Gamma}^{0,1}(\{\omega\}) = Z_d \Gamma^{0,1}(\{\omega\}) . \tag{3.4}$$

The multiplication of Γ^0 by Z_d cancels the corresponding irreducible diagram. This is not the case for Γ^1 [see Eqs. (2.8) and (2.9)]. We note, however, that in *both* vertices there remain Λ-depending terms, in particular the divergences of the type $\ln^2\Lambda$. To avoid this problem in the next step renormalized couplings V_R and J_R are introduced. Putting Eq. (3.4) in the form

$$\begin{aligned} \tilde{\Gamma}^0(\{\omega\}) &= V Z_V(\{\omega\}) , \\ \tilde{\Gamma}^1(\{\omega\}) &= -J Z_J(\{\omega\}) \end{aligned} \tag{3.5}$$

(the bare couplings have been factored out) and introducing a renormalization point

$$P = \{\bar{\omega}\} = (\bar{\varepsilon}_1, \bar{\varepsilon}_2; \bar{\Omega}_1, \bar{\Omega}_2)$$

to be specified, we define the renormalized couplings via

$$V_R = V Z_V|_P , \quad J_R = J Z_J|_P . \tag{3.6}$$

When V and J are eliminated in favor of V_R and J_R and replaced in Eq. (3.5), the resulting expressions becomes *asymptotically* free of Λ singularities (i.e., for $\Lambda \to 0$ or $\Lambda \to \infty$). However, we must note that while the $\ln\Lambda$ terms are canceled out in the differences of the type

$$\ln|\Lambda/\Omega| - \ln|\Lambda/\bar{\Omega}| = \ln|\bar{\Omega}/\Omega| , \tag{3.7a}$$

another thing happens with the $\ln^2\Lambda$ terms. For this case, one has differences like

$$\ln^2|\Lambda/\Omega| - \ln^2|\Lambda/\bar{\Omega}| \cong 2\ln|\Lambda|\bar{\Omega}|\ln|\bar{\Omega}/\Omega| , \tag{3.7b}$$

and cancellation only occurs at frequencies $\Omega \cong \bar{\Omega}$. In other words, the theory becomes finite *only* at the vicinity of the renormalization point.

Now we take a scale factor x and define the point P by imposing

$$|\bar{\Omega}_1| = |\bar{\Omega}_2| = |\bar{\varepsilon}'_1 \bar{\varepsilon}'_2|^{1/2} = x . \tag{3.8}$$

Taking into account Eq. (2.10), these relations can be fulfilled, for example, if the pseudofermions are placed in their "mass shell" ($\bar{\varepsilon}_1 = \bar{\varepsilon}_2 = 0$) while the conduction electrons are endowed with an energy x above (or below) the Fermi level. The renormalized functions are then related to the bare ones as follows:

$$\begin{aligned} \Gamma_R^{0,1}(\{\omega\}, V_R, J_R, x) &= Z_d \Gamma^{0,1}(\{\omega\}, V, J, \Lambda) , \\ \Sigma_{d,R}(\omega, V_R, J_R, x) &= Z_d \Sigma_d(\omega, V, J, \Lambda) , \\ G_{d,R}(\omega, V_R, J_R, x) &= Z_d^{-1} G_d(\omega, V, J, \Lambda) . \end{aligned} \tag{3.9}$$

(Σ_d is the pseudofermion self-energy.) More generally we suppose

$$\Gamma_R^{(n,m)}(\{\omega\}, V_R, J_R, x) = (Z_d)^n \Gamma^{(n,m)}(\{\omega\}, V, J, \Lambda) \tag{3.10}$$

for a vertex with $2n$ ($2m$) external legs of pseudofermions (conduction electrons). This assumption is based on the fact that the conduction-electron field does not undergo renormalization.

Now we can take two different routes. A possibility is to study the different renormalized theories that correspond to a given bare one. The second path is on the contrary, to vary the parameters of the bare theory keeping as fixed the renormalized ones.

We adopt the latter procedure, which is known in the analogous field-theory problem[22] as renormalization of the bare theory. To do this one imposes on the left-hand side of Eq. (3.10) the condition

$$\Lambda \frac{\partial}{\partial \Lambda}(\Gamma_R^{(n,m)})|_R = 0 , \tag{3.11}$$

where the subscript means that the renormalized magnitudes (x, V_R and J_R) remain fixed. We obtain the condition

$$\left[\Lambda\frac{\partial}{\partial\Lambda} - \beta_V(V,J,\Lambda/x)\frac{\partial}{\partial V} - \beta_J(V,J,\Lambda/x)\frac{\partial}{\partial J} + n y_d(V,J,\Lambda/x)\right]\Gamma^{(n,m)}(\{\omega\},V,J,\Lambda) = 0 , \tag{3.12}$$

where

$$\begin{aligned} \beta_V(V,J,\Lambda/x) &= -\left[\Lambda\frac{\partial}{\partial\Lambda}V\right]_R , \\ y_d(V,J,\Lambda/x) &= \left[\Lambda\frac{\partial}{\partial\Lambda}\ln Z_d\right]_R , \\ \beta_J(V,J,\Lambda/x) &= -\left[\Lambda\frac{\partial}{\partial\Lambda}J\right]_R . \end{aligned} \tag{3.13}$$

Equation (3.12) is a Callan-Symanzik-like equation. We must deduce explicit expressions for the derivatives in Eq. (3.13). From Eq. (3.6) it follows that

$$0 = \Lambda\frac{\partial}{\partial\Lambda}\ln J + \Lambda\frac{D}{D\Lambda}\ln Z_J , \tag{3.14}$$

and a similar relation for V. Expanding the total derivatives of the factors Z_J and Z_V we obtain

$$\beta_J\left[1+\frac{J}{Z_J}\frac{\partial Z_J}{\partial J}\right]+\beta_V\frac{J}{Z_J}\frac{\partial Z_J}{\partial V}=\frac{J}{Z_J}\left[\Lambda\frac{\partial}{\partial\Lambda}Z_J\right], \tag{3.15a}$$

$$\beta_J\frac{V}{Z_V}\frac{\partial Z_V}{\partial J}+\beta_V\left[1+\frac{V}{Z_V}\frac{\partial Z_V}{\partial V}\right]=\frac{V}{Z_V}\left[\Lambda\frac{\partial}{\partial\Lambda}Z_V\right]. \tag{3.15b}$$

From this system one can calculate β_J and β_V. Note that all the involved derivatives must be taken at the renormalization point. Finally, the quantity y_d is equal to

$$y_d=\Lambda\frac{\partial}{\partial\Lambda}\ln Z_d-\beta_J\frac{\partial}{\partial J}\ln Z_d-\beta_V\frac{\partial}{\partial V}\ln Z_d\ . \tag{3.16}$$

The β functions define the RGFE's of the problem while y_d determines the anomalous dimension of the deep-electron field.

Now we consider both the two-loops and the three-loops approximations of the theory.

(i) *Two-loops approximation.* The derivatives $(\Lambda(\partial/\partial\Lambda)Z_J)_P$ and $(\Lambda(\partial/\partial\Lambda)Z_V)_P$ can be calculated from Eqs. (2.8), (2.9), (3.4), and (3.5). For an *arbitrary* point P these derivatives contain terms of the form

$$\ln|\bar{\Omega}_1/\bar{\Omega}_2|,\ \ln|\Lambda^2/(\bar{\varepsilon}'_1\bar{\varepsilon}'_2)|,\ \text{and}\ \ln|\Lambda^2/(\bar{\Omega}_1\bar{\Omega}_2)|\ . \tag{3.17}$$

By taking the renormalization point as in Eq. (3.8), the terms in $\ln|\bar{\Omega}_1/\bar{\Omega}_2|$ are eliminated: one then has a symmetry between the input and the output of each diagram. The remaining two terms behave as $\ln(\Lambda/x)$. The physical magnitudes cannot depend, however, on arbitrary calculations parameters. Then one is forced to take the renormalization point satisfying

$$x=\Lambda\ . \tag{3.18}$$

This choice is equivalent to neglecting in the perturbative series all the terms of the form $\ln^p\Lambda$, with $p\geq 2$. To obtain physical results, only the $\ln\Lambda$ singularities would be relevant. The effective Z functions are, then, up to the two-loops order,

$$Z_J^{\text{eff}}=1-2J(1+J)\ln\frac{\Lambda}{x},\ Z_V^{\text{eff}}=1\ , \tag{3.19}$$

while Z_d^{eff} is given by Eq. (3.3). Note that at the renormalization point

$$Z_J^{\text{eff}}|_P=Z_V^{\text{eff}}|_P=Z_d^{\text{eff}}|_P=1\ . \tag{3.20}$$

From Eq. (3.10) one obtains

$$\left[\Lambda\frac{\partial}{\partial\Lambda}Z_J\right]_P=-2J(1+J)\ ,\qquad \left[\Lambda\frac{\partial}{\partial\Lambda}Z_Y\right]_P=0\ , \tag{3.21}$$

and from Eq. (3.15) it follows that

$$\beta_J=-2J^2(1+J)\ ,\qquad \beta_V=0 \tag{3.22}$$

(although Z_J has a one-loop term, it vanishes at the point P). To calculate y_d, only the first term of Eq. (3.16) is needed because the first correction to Z_d is two-loops order. We obtain

$$y_d=-2(V^2+\tfrac{3}{4}J^2)\ . \tag{3.23}$$

The Eqs. (3.22) and (3.23) are our results in the two-loops approximation. Note that the choice (3.18) implies that the functions β_J, β_V, and y_d are no longer dependent on the ratio Λ/x, and they only depend on the physical constants V and J.

(ii) *Three-loops approximation.* In this paper all the involved Feynman diagrams have been calculated assuming that the external frequencies ω_i are small in comparison with the cutoff Λ, i.e., $|\omega_i|\ll\Lambda$. In other words one uses the notion that Λ is *a priori* a genuine cutoff for the momentum integrals. Then in any step of the calculation the following approximation has been made:

$$\ln\left|\frac{\Lambda+\omega_i}{\omega_i}\right|\cong\ln\left|\frac{\Lambda}{\omega_i}\right|\ . \tag{3.24}$$

This is particularly useful in the computation of diagrams with two or more loops because the analytic work is then simplified.

In Sec. II only the leading contribution of each diagram has been kept. However, to perform the theory up to the three-loops order is now necessary to keep also the subleading contributions of the two-loops diagrams. Then the Λ-independent terms are required. Although the parquet graphs have terms of this type, they depend on ratios of external frequencies. We neglect these contributions because we are interested in quantities independent of the renormalization conditions. On the other hand, the two-loops diagram for Σ_d and the corresponding irreducible graph of Γ produce relevant constant terms.

As results from expanding the corresponding logarithmic integral, we must make the change

$$\ln\left|\frac{\Lambda}{\omega}\right|\rightarrow\ln\left|\frac{\Lambda}{\omega}\right|+1 \tag{3.25}$$

in the last term of the Eqs. (2.4), (2.8), and (2.9).

Now we look for the three-loops order terms. For Σ_d these contributions are represented by the two diagrams[16] of Fig. 3. The total contribution of these graphs is

$$\Sigma_d^{(3)}(\omega)=3J^3\omega\left[\ln^2\left|\frac{\Lambda}{\omega}\right|+2\ln\left|\frac{\Lambda}{\omega}\right|+\cdots\right] \tag{3.26}$$

(the ellipsis indicates nonsingular terms). For $J=0$ there is an exact cancellation between the two diagrams. For Γ there are many more three-loops diagrams. Some of them are shown in Fig. 4.

Let $I_G=I_G(\Omega_1,\varepsilon_1,\varepsilon_2)$ be the loop integral that corresponds to a given graph G of this figure. Together with G there is a symmetrical diagram $\bar{G}$ (not shown) that is ob-

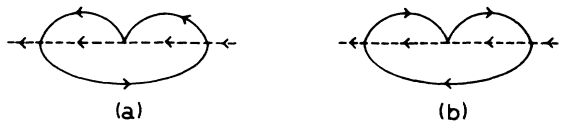

FIG. 3. Diagrams contributing to Σ_d at the three-loops order. Only the topological structure of the graphs has been indicated. With each vertex one must associate one of the elementary processes shown in Fig. 1, whenever the total-spin conservation law is fulfilled. The allowed vertex configurations then produce a topological factor affecting the loop integral.

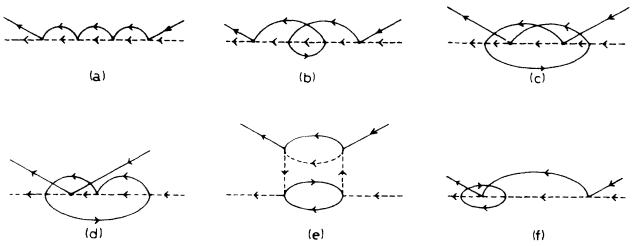

FIG. 4. Some of the three-loops diagrams for Γ. Again we have only indicated the loop structure of the graphs. The associated $\bar{G}$ diagrams (see the text) are not displayed. The loop integral and the topological factors for these graphs are given in Table I. Only diagrams (d) and (f) are relevant for the β functions.

tained from G by reversing all the internal conduction lines. Its contribution is $I_{\bar{G}}=-I_G(\Omega_2,\varepsilon_1,\varepsilon_2)$, because the signs of the three elementary loops are changed.[2,3]

If f_G^α ($f_{\bar{G}}^\alpha$) is the topological factor[49] of the diagram G ($\bar{G}$) contributing to the channel α of Γ, for the combined contribution of G and $\bar{G}$ to Γ^α it follows that

$$\begin{aligned}\Gamma^\alpha_{G+\bar{G}}(\{\omega\})&=f_G^\alpha I_G(\Omega_1)-f_{\bar{G}}^\alpha I_G(\Omega_2)\\&\equiv\tfrac{1}{2}C_+^\alpha[I_G(\Omega_1)+I_G(\Omega_2)]\\&\quad+\tfrac{1}{2}C_-^\alpha[I_G(\Omega_1)-I_G(\Omega_2)]\end{aligned}\tag{3.27}$$

($\alpha=0,1$). Clearly only the factor C_+^α matters because the difference of two logarithmic integrals does not contribute to the β functions. If I_G has a simple $\ln\Lambda$ term and f_G^α is different from $f_{\bar{G}}^\alpha$, an effective contribution to Γ^α arises. As a reference we give in Table I the loop integrals for the graphs of Fig. 4, together with the corresponding C_+ factors for the two channels of Γ.

The function $\phi(x)$ of Table I is defined by

$$\phi(x)=-\int_0^x\frac{\ln(1-u)}{u}du=\sum_{n=1}^{\infty}\frac{1}{n^2}x^n\,,\tag{3.28a}$$

and it satisfies

$$\phi(1)=\zeta(2)=\pi^2/6\,,\tag{3.28b}$$

ζ being the Riemann zeta function.[50] The leading term in these graphs is of the form $\ln^p\Lambda$, with $1\le p\le 3$. For the ladder graph 4(a) is $p=3$ and there is no correction term. In a typical parquet graph such as 4(b) there are, on the other hand, subleading terms of the form

$$\Gamma_{\text{subl}}\cong\left[\sum_{n\ge 1}b_n\left[\frac{\Omega}{\Lambda}\right]^n\right]\ln\left|\frac{\Lambda}{\Omega}\right|\,.\tag{3.29}$$

Note that the summation runs over $n\ge 1$. Then, con-

TABLE I. Loop integral and topological factors for the diagrams of Fig. (4). The formulas displayed in the first column of this table represent only the singular contributions to the loop integrals. In the case of diagram (c) the calculation has been made only for the choice $\varepsilon_1=\varepsilon_2=0$ of the external frequencies. In all the remaining diagrams the formulas are general ($\varepsilon\cong\varepsilon_1\cong\varepsilon_2$ and $\varepsilon'\cong\varepsilon_1'\cong\varepsilon_2'$). To obtain these formulas it is convenient to perform first the frequency integrals. The topological factors for the vertices Γ^0 and Γ^1 (C_+^0 and C_+^1, respectively) are defined by Eq. (3.27), being shown only for the relevant graphs. In the cases of diagrams (d) and (f) these coefficients include an extra factor 2, due to the two symmetrical possibilities one can draw such graphs.

Graph	Logarithmic integral $I(\Omega_1)$	C_+^0	C_+^1
(a)	$-\ln^3\left\|\frac{\Lambda}{\Omega_1}\right\|$		
(b)	$\frac{1}{3}\ln^3\left\|\frac{\Lambda}{\Omega_1}\right\|-2\left[\ln 2\ln\left[1-\frac{\Omega_1}{2\Lambda}\right]+\phi(-\Omega_1/2\Lambda)\right]\ln\left\|\frac{\Lambda}{\Omega_1}\right\|+\cdots$		
(c)	$8\zeta(2)\ln\left\|\frac{\Lambda}{\Omega_1}\right\|+2\left[\phi(\Omega_1/\Lambda)-\phi(-\Omega_1/2\Lambda)-\ln 2\ln\left[1-\frac{\Omega_1}{2\Lambda}\right]\right]\ln\left\|\frac{\Lambda}{\Omega_1}\right\|+\cdots$	0	0
(d)	$-\frac{1}{2}\ln^2\left\|\frac{\Lambda}{\varepsilon}\right\|-\ln\left\|\frac{\Lambda}{\varepsilon}\right\|+\cdots$	$6VJ^3$	$2J^4$
(e)	$-\ln\left\|\frac{\Lambda}{\varepsilon}\right\|\ln\left\|\frac{\Lambda}{\varepsilon'}\right\|+\frac{1}{2}\ln^2\left\|\frac{\Lambda}{\Omega_1}\right\|+\cdots$	0	$-4J^2\left[V^2-\frac{J^2}{4}\right]$
(f)	$-\frac{1}{2}\ln^2\left\|\frac{\Lambda}{\Omega_1}\right\|-\ln\left\|\frac{\Lambda}{\Omega_1}\right\|+\cdots$	0	$-4J^2\left[V^2-\frac{J^2}{4}\right]$

sistently with Eq. (3.24), these terms must be neglected because they are *a priori* small (when one assumes that $|\Omega| \ll \Lambda$). The so-called fundamental graphs[2,3] [Fig. 4(c)] are the only ones with a logarithmic leading term ($p=1$). However, as $C^0_+ = C^1_+ = 0$, these diagrams do not contribute to the Z factors. Diagrams 4(d) and 4(e) are vertex corrections to the irreducible part[2,3] of the interaction. Graph 4(d) is a one-loop correction to the vertex I (or II) of Fig. 5(a). Its leading part contributes with $p=2$. Its subleading part is, on the other hand, a simple logarithmic correction with a *constant coefficient*. Graph 4(e), which represents a correction to the vertex III, has not, on the other hand, a subleading $\ln\Lambda$ term and it contributes with $p=2$. Finally, Graph 4(f) represents a two-loops irreducible-vertex correction to the one-loop diagram; see Fig. 5(b). It has $p=2$ and also contributes with a subleading $\ln\Lambda$ part.

To calculate the β functions we have neglected above the terms in $\ln^p\Lambda$, with $p \geq 2$. From Eq. (3.24) we now argue that the terms of the form $(\Omega/\Lambda)^n \ln|\Lambda/\Omega|$ ($n \geq 1$) must also be neglected. It follows that only the *simple* logarithmic corrections arising from the skeleton diagrams of Fig. 5 contribute to this order. (This is an important difference from the calculation made in Ref. 16; see the following.)

Now, from Eqs. (2.4) and (3.26) we obtain the effective propagator

$$(G^{\text{eff}}_{d,\sigma})^{-1} = \omega \left[1 + 2(V^2 + \tfrac{3}{4}J^2) \left[\ln\frac{\Lambda}{x} + 1 \right] - 6J^3 \ln\frac{\Lambda}{x} \right] , \tag{3.30}$$

up to the three-loops order. From Eqs. (2.8), (2.9), and (3.25), and the results of Table I, one finds for the effective vertices

$$\Gamma^1_{\text{eff}} = -J \left[1 - 2J \ln\frac{\Lambda}{x} + 2 \left[V^2 - \frac{J^2}{4} \right] \left[\ln\frac{\Lambda}{x} + 1 \right] + 4J(J^2 - 2V^2) \ln\frac{\Lambda}{x} \right] , \tag{3.31a}$$

$$\Gamma^0_{\text{eff}} = V \left[1 + 2(V^2 + \tfrac{3}{4}J^2) \left[\ln\frac{\Lambda}{x} + 1 \right] - 6J^3 \ln\frac{\Lambda}{x} \right] . \tag{3.31b}$$

From Eqs. (3.2) and (3.30) it follows that

$$Z^{\text{eff}}_d = 1 - 2(V^2 + \tfrac{3}{4}J^2) \left[\ln\frac{\Lambda}{x} + 1 \right] + 6J^3 \ln\frac{\Lambda}{x} + O(L^4) , \tag{3.32}$$

and from Eqs. (3.4), (3.6), (3.31), and (3.32) we now find

$$Z^{\text{eff}}_J = 1 - 2J \ln\frac{\Lambda}{x} - 2J^2 \left[1 + \ln\frac{\Lambda}{x} \right] + 2(5J^3 - 4V^2 J) \ln\frac{\Lambda}{x} + O(L^4) , \tag{3.33a}$$

$$Z^{\text{eff}}_V = 1 + O(L^4) , \tag{3.33b}$$

for the relevant Z factors (L is the loop expansion parameter). Note that at the renormalization point these functions satisfy

$$Z^{\text{eff}}_d|_p = 1 - 2(V^2 + \tfrac{3}{4}J^2) + O(L^3) ,$$
$$Z^{\text{eff}}_J|_p = 1 - 2J^2 + O(L^3) . \tag{3.34}$$

Taking into account everything, the Eq. (3.15a) can be written as

$$\beta_J[1 - 4J^2 + O(L^4)] = -2J^2[1 + J - 3J^2 + 4V^2 + O(L^3)]$$

[from Eq. (3.34) the terms in Z_J^{-1} now do contribute]. It follows that

$$\beta_J = -2J^2(1 + J + J^2 + 4V^2) + O(L^4) , \tag{3.35a}$$

$$\beta_V = O(L^4) . \tag{3.35b}$$

Finally, from Eqs. (3.16), (3.32), (3.34), and (3.35) one finds

$$y_d = -2(V^2 + \tfrac{3}{4}J^2) + O(L^4) . \tag{3.36}$$

Equations (3.35) and (3.36) are the main results of this work. They are discussed in the next section, together with the RGFE's and scaling.

IV. FLOW EQUATIONS AND SCALING

Starting from the general solution of the first-order partial differential equation (3.12), which can be constructed from the associated characteristic curves[51], we can deduce an important property of the vertex functions $\Gamma^{(n,m)}$. If the cutoff Λ undergoes an enlargement $\Lambda \to \rho^{-1}\Lambda$, one has

$$\Gamma^{(n,m)}(\{\omega\}, V, J, \rho^{-1}\Lambda) = \exp\left[-\int_J^{J(\rho)} \frac{n y_d(V,J')}{\beta_J(V,J')} dJ' \right] \Gamma^{(n,m)}(\{\omega\}, V, J(\rho), \Lambda) , \tag{4.1}$$

where $J(\rho)$ is a flowing coupling which satisfies the RGFE's,

$$\frac{dJ(s)}{ds} = -\beta_J(J(s)), \quad s = \ln\rho . \tag{4.2a}$$

That is, except by a complicated prefactor, the vertex at $(J, \rho^{-1}\Lambda)$ coincides with the one at $(J(\rho), \Lambda)$. The flow in this problem is only one dimensional, at least up to the three-loops order. The corresponding equation for V is

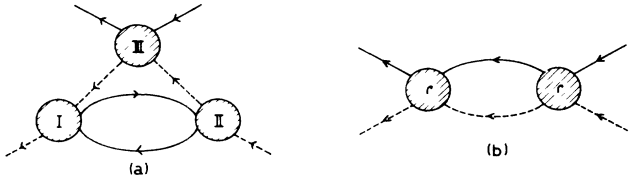

FIG. 5. Skeleton diagrams for Γ. The graph (a) gives origin to the diagrams of Figs. 4(d) and 4(e), while graph (b) produces the correction of Fig. 4(f).

$$\frac{dV(s)}{ds}=0 , \tag{4.2b}$$

and therefore this is a fixed parameter.

From dimensional considerations one more property is obtained. If $d_{n,m}$ is the canonical dimension of $\Gamma^{(n,m)}$,

$$[\Gamma^{(n,m)}]=\Lambda^{d_{n,m}} ,$$

one has

$$\Gamma^{(n,m)}(\{\rho\omega\},V,J,\Lambda)=\rho^{d_{n,m}}\Gamma^{(n,m)}(\{\omega\},V,J,\rho^{-1}\Lambda) . \tag{4.3}$$

By combining Eqs. (4.1) and (4.3) one obtains, after performing the change $\rho\omega_i\rightarrow\omega_i$,

$$\Gamma^{(n,m)}(\{\omega\},V,J_0,\Lambda)=\rho^{d_{n,m}}[\psi_V(J(\rho))]^n \times\Gamma^{(n,m)}(\{\rho^{-1}\omega\},V,J(\rho),\Lambda) . \tag{4.4}$$

Here

$$\psi_V(J)\equiv\exp\left[-\int_{J_0}^{J}\frac{y_d(V,J')}{\beta_J(V,J')}dJ'\right] \tag{4.5a}$$

is the prefactor of the Eq. (4.1) for $n=1$. The function $J(\rho)$ is defined from Eq. (4.2a) as follows:

$$\Phi_V(J(\rho))\equiv-\int_{J_0}^{J(\rho)}\frac{dJ'}{\beta_J(V,J')}=\ln\rho \tag{4.5b}$$

$[J(\rho=1)=J_0]$. Equations (4.4) and (4.5) are similar to the standard Gell-Mann–Low (GML) equations.[9,16,17] There is, however, a difference. On the right-hand side of Eq. (4.4) the function Γ depends on ρ via the flowing coupling $J(\rho)$. This fact prevents one from separating, in a general case, a factor with the external physical variable (frequency or temperature, see the following). In the usual GML formulation one supposes that this separation is always possible.[52]

In two important cases Eq. (4.4) reduces to the GML hypothesis. In the first possibility one is at the beginning at a fixed point of the RGFE. Then $J_0=J_*$ and

$$\beta_J(V,J_*)=0 . \tag{4.6}$$

In such a case from Eq. (4.2a), J remains equal to J_* for any "time" s. Equation (4.4) must be replaced by

$$\Gamma^{(n,m)}(\{\omega\},V,J_*,\Lambda)=\rho^{d_{n,m}+ny_*}\Gamma^{(n,m)}(\{\rho^{-1}\omega\},V,J_*,\Lambda) , \tag{4.7}$$
$$y_*=y_d(V,J_*) .$$

So, there is a rigorous scaling for any ρ, and y_* is indeed the anomalous dimension of the pseudofermion field.

In the second situation one starts from an arbitrary J_0 and takes the infrared (ultraviolet) limit for ρ: $\rho\rightarrow 0$ ($\rho\rightarrow\infty$). The flow iterates, in general, towards a fixed point J_{in}^* (J_{ul}^*) which is called an infrared-stable (IS) [ultraviolet-stable (US)] fixed point. Now Eq. (4.4) leads to

$$\Gamma^{(n,m)}(\{\omega\},V,J_0,\Lambda)=[c_{\text{in,ul}}(\rho)]^n\rho^{d_{n,m}+ny^*_{\text{in,ul}}}\Gamma^{(n,m)}(\{\rho^{-1}\omega\},V,J^*_{\text{in,ul}},\Lambda),\quad \rho\rightarrow 0,\infty , \tag{4.8}$$

where $c(\rho)$ is, in general, a smooth function of ρ. [If J^* is a double zero of β_J this factor is of the form $(\ln\rho)^\nu$, $\nu\propto(dy_d/dJ)_{J=J^*}$.] In this case one would have only *asymptotic scaling* (i.e., for $\rho\rightarrow 0,\infty$).

By considering that all the external frequencies are of the same order [which is the case specified by the renormalization condition (3.8)], $\omega_i\cong\omega$, one can take ρ as follows:

$$\rho=\frac{\omega}{\Lambda} . \tag{4.9a}$$

On the other hand, if the absolute temperature (T) of the system is not zero, then the frequencies of the external lines are of the form $\omega_i=(2n_i+1)T$, as results from the Matsubara diagrammatic technique for fermions.[38,39] Then one chooses ρ to be equal to

$$\rho=\frac{T}{\Lambda} . \tag{4.9b}$$

In any case, Eq. (4.7) gives

$$\Gamma^{(n,m)}(\varepsilon,V,J^*,\Lambda)=\left[\frac{\varepsilon}{\Lambda}\right]^{d_{n,m}+ny^*}\times\text{const} \tag{4.10}$$

($\varepsilon=\omega$ or T).

In the general flowing case, one must specify what the desired limit is. If one is interested in studying low- (high-) energy scattering processes or low- (high-) temperature behavior, according to Eq. (4.9) the IS- (US-) fixed point is then required. One obtains, from Eq. (4.8)

$$\Gamma^{(n,m)}(\varepsilon,V,J_0,\Lambda) =\left[c_{\text{in,ul}}\left[\frac{\varepsilon}{\Lambda}\right]\right]^n\left[\frac{\epsilon}{\Lambda}\right]^{d_{n,m}+ny^*_{\text{in,ul}}}\times\text{const} ,$$
$$\varepsilon\rightarrow 0,\infty . \tag{4.11}$$

Now we apply these ideas to the two approximations we have made.

(i) *Two-loops approximation.* In this case the relevant functions are given by Eqs. (3.22) and (3.23). The RGFE is then

$$\frac{dJ}{ds}=2J^2(1+J) . \tag{4.12}$$

There are two fixed points: $J_*=0$ and $J_*=-1$. The first one is a double zero of β_J and it is IS (US) for $J_0>0$ ($J_0<0$). The $J_*=-1$ is, on the other hand, an IS fixed point. The anomalous dimension calculated from Eq. (3.23) is in the two cases

$$y_* = \begin{cases} -2V^2, & J_*=0 \\ -2(v^2+\frac{3}{4}), & J_*=-1 \end{cases} . \tag{4.13a}$$

(4.13b)

When $J_0=0$, one is already at the start in a fixed point. The corresponding Hamiltonian is our slightly generalized version of the x-ray absorption problem. It is interesting to analyze now some particular cases of Eq. (4.10). When $m=0$ and $n=1$ the vertex reduces to the pseudofermion self-energy $\Sigma_d(\omega)$, being $d_{1,0}=1$. So,

$$\Sigma_d(\omega)=\left[\frac{\omega}{\Lambda}\right]^{1-2V^2}, \quad J_0=0 , \tag{4.14}$$

This is the well-known Mahan–Nozières–De Dominicis (MND) result[1–4] at leading order in V; see Eq. (2.5). Analogously, in the case $n=m=1$ the vertex is Γ^0 and $d_{1,1}=0$. The exponent of ρ reduces now to Eq. (4.13a), in agreement with the corresponding MND result.

Now we discuss what happens when $J_0\neq 0$. Specifically we are interested in the case $J_0<0$, $|J_0|\ll 1$. The initial behavior is dictated by

$$dJ/ds \cong 2J^2 ,$$

which is integrated to give

$$J(\rho)\cong \tfrac{1}{2}\left[\ln\frac{\rho_k}{\rho}\right]^{-1}, \quad \rho \gtrsim \rho_k . \tag{4.15a}$$

Here

$$\rho_k = \exp\left[-\frac{1}{2|J_0|}\right] \tag{4.15b}$$

is the Kondo dimensionless temperature.[10–12] By computing the function $\psi_V(J)$, Eq. (4.5a), we can write Eq. (4.11) in the ultraviolet limit as follows:[53]

$$\Gamma^{1,0}(\varepsilon,V,J_0,\Lambda) \underset{(\epsilon\gg\rho_k)}{=} -J_0 \times \begin{cases} b(J_0)\left[1-\frac{3}{8}\left[\ln\frac{\varepsilon}{\rho_k}\right]^{-1}+\cdots\right], & V=0 \quad (4.16a) \\ \tilde{b}(J_0,V)\left[\frac{\varepsilon}{\rho_k}\right]^{-2V^2}, & V\neq 0 \quad (4.16b) \end{cases}$$

(b and $\tilde{b}$ are functions of J_0 and V). Therefore, the initial behavior is controlled by the ($J_*=0$)-US fixed point. Equation (4.16a) resembles the naive extrapolation one makes from perturbation theory.[10] On the other hand, according to Eq. (4.16b), for $V\neq 0$ the vertices should vanish in the ultraviolet limit with a critical index equal to the associated MND's one.

Now we look for the infrared limit. Equation (4.12) shows that $J(\rho)\to -1$ for $\rho\gg\rho_k$ (J is at the vicinity of $J_*=-1$ already for $\rho\gtrsim\rho_k$). From Eqs. (4.11) and (4.13b) one obtains

$$\Sigma_d(\omega)\cong\left[\frac{\omega}{\Lambda}\right]^{-1/2-2V^2}, \quad \omega\ll\rho_k , \tag{4.17}$$

from which the associated propagator would be not singular. For the vertex functions in the pure Kondo model ($V=0$) the behavior should be

$$\Gamma^{1,0}\cong\left[\frac{\varepsilon}{\Lambda}\right]^{-3/2}, \quad \varepsilon\ll\rho_k . \tag{4.18}$$

This, however, cannot be the case. In fact, from Eq. (4.18) one concludes that the electrical resistance (which is related to $|\Gamma|^2$) should have a strong divergence at $T=0$. This magnitude has a known finite limit.[9,29] To see what the problem is here, we must note that $J_*=-1$ is in fact a *finite* value for the coupling constant. We are not justified, therefore, to neglect the higher-order terms in the RGFE: all the powers of J^n could be important to determine the true flow at $\rho\ll\rho_k$. Equations (4.17) and (4.18) must then be checked.

(ii) *Three-loops approximation.* In the next improved version the relevant functions are given by Eqs. (3.35) and (3.36). The RGFE is now

$$\frac{dJ}{ds}=2J^2(1+J+J^2+4V^2) . \tag{4.19}$$

Consider first the case $V=0$. It is clear that β_J does not have a nonzero fixed point for real J. So, starting from an arbitrarily small negative J the flow now reaches the strong-coupling limit ($J=-\infty$). However, here there is a serious problem. Indeed, for $J\ll -1$ the solution of Eq. (4.19) can be written as

$$J(\rho)=\left[6\ln\frac{\rho_\tau}{\rho}\right]^{-1/3}, \quad \rho\gtrsim\rho_\tau\cong|J_0|\rho_k . \tag{4.20}$$

Then the $J=-\infty$ limit is reached for a *finite* change of scale $\rho=\rho_\tau$ (or, roughly speaking, in a finite "time" $1-\ln\rho_\tau$). Now, if one attempts to extrapolate Eq. (4.20) for the scales $\rho\ll\rho_\tau$ an impossible physical behavior is obtained: the coupling becomes ferromagnetic and iterates ultimately towards the trivial fixed point. It follows that the infrared limit cannot be studied from Eq. (4.20).

The one-loop approximation also has this same problem. However, by comparing Eq. (4.15) with (4.20) we note that a slight improvement has been made: for small J_0 one has $\rho_\tau\ll\rho_k$, and therefore smaller scales could be considered. With this in mind, we now observe a rather

crucial feature of Eq. (4.19): the coefficient a_4 of the J^4 term is also 2.[54] This particular value strongly *suggests* that the generalization of the RGFE to *all* the orders in J is, for $V=0$,

$$dJ/ds = 2J^2(1+J+J^2+J^3+\cdots) = 2J^2/(1-J) . \quad (4.21)$$

Now, as it has been recently proved by Barnes,[36] this equation is just the exact RGFE for the Kondo problem. Equation (4.21) now gives a correct crossover from the weak-coupling to the strong-coupling region.

In a finite form Eq. (4.21) is

$$-\frac{1}{J}-\ln|J| = 2(s-\ln\rho_\tau) , \quad (4.22)$$

where $\rho_\tau = |J_0|\rho_k$. In the infrared limit, it reduces to

$$J(\rho) \cong -\rho_\tau^2 \exp(-2s) = -\left[\frac{\rho_\tau}{\rho}\right]^2 . \quad (4.23)$$

In other words, the point $J=-\infty$ should be the IS fixed point for the Kondo model with negative initial coupling.

To find the scaling law one should calculate, according to Eq. (4.11), the anomalous dimension at the strong-coupling regime. In particular, if $y_d(J\to-\infty)$ turns out to be a positive quantity, the vertex functions will exhibit a smooth behavior for $\epsilon\to 0$. Such a calculation, however, goes beyond from the possibilities of any finite-order *weak-coupling* perturbation theory, and therefore it will not be attempted here. Note that only the two-loops contribution to y_d is known. So, for this quantity not even a simple extrapolation like Eq. (4.21) is available.

To understand the reason for the absence of three-loops corrections in y_d, we can look for the pure potential-scattering case ($J=0$). Then, according to the exact solution of Ref. 4, the anomalous dimension is

$$y_d(g) = -\frac{2}{\pi^2}[\delta(g)]^2, \quad g = N_F V , \quad (4.24)$$

δ being the scattering phase shift at the Fermi surface. Taking into account Eqs. (39) and (44) of Ref. 4, one finds the expansion (N_F has been taken as a constant)

$$y_d(g) = -2(g^2 - \tfrac{2}{3}\pi^2 g^4 + \cdots) , \quad (4.25)$$

and there are no three-loops corrections.

To close this section we make a comment about the general case in which both couplings J and V are nonzero. At least up to the three-loops calculation here performed, the coupling V is an invariant, see Eq. (3.35b). The exact Eq. (4.24) gives a critical index that exhibits a dependence on the constant V. In the statistical mechanical models this happens only when the corresponding coupling is a marginal variable.[55,56] So, one must conclude here that the line $J=0$ is, as a whole, a fixed one. Taking into account this and our results, it is natural to suppose that V should also be an invariant when $J\neq 0$. If this is the case, it is clear that the infrared properties of the model cannot depend on V: for $\rho\to 0$ the general Hamiltonian (2.3) should exhibit an universal Kondo behavior. On the other hand, the ultraviolet properties are dictated by the potential-scattering coupling, see Eq. (4.16). The form of the singularities for $\rho\to\infty$ is again independent of the initial J coupling.

In the renormalization-group language, one describes this situation saying that for $\rho\to 0$ the coupling J is relevant (irrelevant) at the US (IS) fixed point. The opposite happens for $\rho\to\infty$.

V. CONCLUDING REMARKS

The original purpose of this work was to find the RGFE and scaling laws for the x-ray and Kondo problems borrowing the formal renormalization-group machinery of field theory. Items such as renormalization conditions, the relation between the bare and renormalized theories, Callan-Symanzik equation, fixed points, asymptotic scaling, and critical indices have been developed in some detail. Although here one has a quantum problem, the procedure works exactly as in, for example, a classical ϕ^4 theory in field theory.

Finally, let me recapitulate the main factors about our treatment, comparing this with other approaches. (i) In the papers by Fowler and Zawadowski[23] and by Sólyom[24] the RGFE were derived via a Lie differential equation. Although elegant, the underlying idea is perhaps somewhat opaque for readers not familiarized with the theory of the Lie algebras. In addition, their procedure must solve equations self-consistently. (ii) At the two-loops approximation the present approach gives scaling equations in agreement with those obtained by Anderson[27] and by Sólyom and Zawadowski[28] via the so-called "poor man's renormalization-group method."

(iii) Our approach is very similar to the one made by Abrikosov and Migdal in their classical paper. As already mentioned, there are, however, two important differences. The first one refers to the scaling law. The Callan-Symanzik equation gives only asymptotic scaling, in disagreement with the ideas of Gell-Mann–Low from which the dependence on the temperature of a physical magnitude could always be factorized out. The second difference is our neglect in the diagrammatic series of terms of the form $(\Omega/\Lambda)^p \ln|\Lambda/\Omega|$, $p\geq 1$. They would induce, in the β functions, contributions of the form

$$(\bar{\Omega}/\Lambda)(1-p\ln|\Lambda/\bar{\Omega}|) ,$$

which explicitly depend on the renormalization point ($\bar{\Omega}$). We have neglected such terms, although in the end the choice $\bar{\Omega}=\Lambda$ was made. Note that by the same reasoning the nonsingular terms of the form $(\Omega/\Lambda)^p$, $p\geq 1$, have also been neglected. Proceeding in this way we found Eq. (3.35) for β_J up to the three-loops order, which is different from the corresponding result of Ref. (16). In addition, we now know the contribution of the potential scattering at this order. More interestingly, by making a naive extrapolation one is led to Eq. (4.21), in agreement with Barnes' work.[36]

(iv) Another relevant point is the connection with Wilson's ideas. Indeed, as has been explained earlier, by taking the external frequencies near to the renormalization point, $\Omega_i \cong x$, one can construct a renormalized

theory free of Λ dependence. The Callan-Symanzik equation is just the expression of this fact. The choice $x=\Lambda$, Eq. (3.18), for the renormalization point implies the effective cancellation of the stronger singularities in the perturbative series. The β functions turn out to be (Λ/x) independent and therefore *finite*. Suppose one is interested in knowing the true behavior of a response function when the system undergoes a proof at the external frequency Ω. By combining the two previous conditions it is clear that the physical answer emerges when one takes Λ of the order of Ω, $\Lambda\cong\Omega$. Now, taking into account the meaning of Λ, this is seen to imply that only the scattering processes in which *all the involved electrons* have energies smaller than Ω are relevant to determine the response at the frequency Ω. In other words, we have effectively separated the energy scales x in two sets: $x\leq\Omega$ and $x>\Omega$. Only the first set is involved in the physical behavior. This is in a spirit similar to Wilson's separation of the scales.[30,31,47] Another point in which the results of those works can be related to ours is the asymptotic limit. Indeed, from Eqs. (4.9), (4.11), and (4.23) one concludes that when the limit $T\to 0$ is taken the system iterates towards the strong-coupling regime, no matter what may be the negative initial coupling.

(v) We emphasize that at the level of this paper the Eq. (4.21) for the Kondo model is only a conjecture. To rigorously establish this there are probably two routes. One could intend to perform *weak-coupling* perturbation theory at all the orders in J, summing up the resulting series.

Although such a calculation could be an extremely difficult task, the true infrared limit should be available from this method. In fact, in a remarkable recent work Berkovich and Lowenstein[57] have proved that the Kondo problem is Borel summable, showing that the crossover between weak- and strong-coupling regimes is *essentially perturbative* in nature. So, in principle, the strong-coupling regime is accessible to perturbative methods based in weak-coupling series. A more rational procedure to obtain the infrared limit is perhaps to construct a *strong-coupling* perturbation theory (i.e., power series in $1/J$). In this way one would avoid the technical problems involved in summing up infinite series. Both expansions, the weak- and the strong-coupling series would be matched at $J\cong -1$ as a final step. Work in such a direction is in progress.

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[46]Taking $\varepsilon_1'=\varepsilon_2'=0$ and $\Omega_1=\Omega_2=\varepsilon_1=\varepsilon_2=\omega$ our formulas reduce to Eqs. (4.26) and (4.27) of Ref. 24, if the rescaling is taken into account.
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$$f^{\alpha}_{G(\bar{G})} = \sum_{(n)} \delta_{n_a+n_b+n_c,3} \nu^{\alpha}_{G(\bar{G})}(\{n\}) \times (\nu+J/2)^{n_\alpha}(\nu-J/2)^{n_b}(-J)^{n_c} ,$$

where ν is a combinatorial factor counting the number of ways one can distribute n_a,n_b,n_c elementary processes of type a,b,c (see Fig. 1) between the three vertices of the diagram $G(\bar{G})$. ν depends, in general, on α, and for $J\neq 0$ one can have $\nu^{\alpha}_{G}(\{n\})\neq\nu^{\alpha}_{\bar{G}}(\{n\})$. The configurations violating the total-spin conservation law in at least one vertex give a zero contribution to f.

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[52]See especially Eq. (2.2.11) of Ref. 9 and Eq. (14) of Ref. 16.
[53]Although $J_*=0$ is a double zero of β_J, there are no logarithmic corrections because $(dy_d/dJ)_{J_*=0}=0$.
[54]Abrikosov and Migdal have obtained a contribution to a_4 proportional to π^2. The reason this number appears is that these authors keep the terms of the form $\phi(x)\ln|x|$ in the diagrams; see Table I and Eq. (3.28).
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