Classical $O(N)$ Heisenberg model: Extended high-temperature series for two, three, and four dimensions

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We present simple tables of integers from which it is possible to reconstruct the high-temperature series coefficients through β^{14} for the susceptibility, for the second correlation moment, and for the second field derivative of the susceptibility of the $O(N)$ classical Heisenberg model on a simple (hyper)cubic lattice in dimension $d=2$, 3, and 4 and for any N. To construct the tables we have used the recent extension of the high-temperature series by M. Luscher and P. Weisz and some analytic properties in N that we have derived from the Schwinger-Dyson equations of the $O(N)$ model. We also present a numerical study of these series in the $d=2$ case. The main results are: (a) the extended series give further support to the Cardy-Hamber-Nienhuis exact formulas for the critical exponents when $-2<N<2$; (b) for $N\geq 3$ there are no indications of any critical point at finite β; (c) the series are consistent with the low-temperature asymptotic forms predicted by the perturbative renormalization group.

I. INTRODUCTION

High-temperature (HT) expansions have recently been computed[1] through 14th order for the $O(N)$ symmetric $P(\boldsymbol{\phi}^2)$ field theory on a simple (hyper)cubic lattice of dimension $d=2$, 3, and 4. As a particularly interesting special case of this calculation, the HT series for the zero field susceptibility $\chi(N,\beta)$, for the second moment of the correlation functions $m^{(2)}(N,\beta)$, and for the second derivative of the susceptibility with respect to the external magnetic field H at zero field $\chi^{(4)}(N,\beta) \equiv d^2\chi/dH^2|_{H=0}$ have also been obtained for the $O(N)$-symmetric classical Heisenberg model[2] (sometimes also called the N-vector model). These series will be the subject of our consideration. Although for some special cases significantly longer expansions are available, notably for the $N=0$, 1, and 2 cases,[3-5] in general this is quite a valuable extension. Previously, HT series valid for all N had been computed,[6] through β^9 for $d=2$, 3, and 4 and later through β^{11} for $d=2$ only.[7]

The original results of M. Luscher and P. Weisz, kindly made available to us by the authors, come as a large numerical file from which all results may be reconstructed. We have shown[7] that, for the $O(N)$ Heisenberg model, it is possible to cast the coefficients of the HT series into an explicit analytic form valid (at fixed d) for any N. This follows from the fact that all correlation functions are solutions of the Schwinger-Dyson equations of the theory.[7] More precisely we can express each HT coefficient as a simple rational function of N with integer coefficients.

In Sec. II we give tables for HT coefficients in the case of $d=2$, 3, and 4 dimensions, thus making a large amount of information readily available in a conveniently unified format. In Sec. III we present various numerical tests we have performed on these series for the $d=2$ case and for any N. For $-2<N<2$, we have compared the critical exponent γ of the susceptibility and the exponent ν of the correlation length with the exact formulas proposed some time ago.[8] For $N>2$, we have confirmed and made more precise our previous results[7] on the location of the nearest (unphysical) singularities in the complex inverse temperature plane. For $N\geq 3$, no indication of a critical point at finite β has emerged from our numerical analysis, whereas we show that the HT series for the susceptibility and the correlation length are entirely consistent with the asymptotic low-temperature behavior predicted by the renormalization group.[9]

II. THE HT SERIES

Let us start by defining our notation. The variables of the model are N-component classical spins of unit length,

$$v(x)=(v_1(x),v_2(x),\ldots,v_N(x))\;;\qquad v^2(x)=v(x)\cdot v(x)=1\;, \tag{2.1}$$

arranged on the sites x of a d-dimensional (hyper)cubic lattice. The Hamiltonian H is:

$$H\{v\}=-\sum_{x}\sum_{\mu=1,\ldots,d} v(x)\cdot v(x+e_\mu)\;. \tag{2.2}$$

The sum over x extends over all lattice sites and $e_1,e_2,\ldots,e_d$ are the elementary lattice vectors.

As shown in Ref. 7 the HT expansion coefficients of any correlation function of this $O(N)$ model have a simple analytic dependence on N and can be expressed in terms of a table of a small set of integers. In this section we present our tables and illustrate how to construct from them the HT coefficients through β^{14} for the following thermodynamic quantities: the susceptibility

$$\chi(N,\beta)=\sum_x \langle v(0)\cdot v(x)\rangle = 1+\sum_{r=1}^{\infty}\chi_r(N)\beta^r\ ; \tag{2.3}$$

the second correlation moment

$$m^{(2)}(N,\beta)=\sum_x x^2\langle v(0)\cdot v(x)\rangle = \sum_{r=1}^{\infty} b_r(N)\beta^r\ ; \tag{2.4}$$

the second field derivative of the susceptibility

$$\chi^{(4)}(N,\beta)=\frac{3}{N(N+2)}\sum_{x,y,z}\langle [v(0)\cdot v(x)][v(y)\cdot v(z)]\rangle_{\text{conn}}=\frac{3}{N(N+2)}\left[-2+\sum_{r=1}^{\infty} d_r(N)\beta^r\right] . \tag{2.5}$$

Our tables for the HT expansion coefficients are organized as follows: (i) Tables I, II, and III for the coefficients $\chi_r(N)$ of the susceptibility in dimension $d=2$, $d=3$, and $d=4$, respectively; (ii) Tables IV, V, and VI for the coefficients $b_r(N)$ of the second correlation moment in dimension $d=2$, $d=3$, and $d=4$, respectively; (iii) Tables VII, VIII, and IX for the quantities $d_r(N)$ of the coefficients of the second field derivative of the susceptibility in dimension $d=2$, $d=3$, and $d=4$, respectively. From the Schwinger-Dyson equations[7] we can see that

TABLE I. The structure in N of HT expansion coefficients $\chi_r(N)$ in (2.3) for the susceptibility in $d=2$ dimensions. In each section of the table we give the information necessary to reconstruct the coefficients $\chi_r(N)$ following Eqs. (2.6), (2.7), and (2.8). Each section is headed by the order r of the coefficient to which it refers. Then we give the set of exponents $k_1(r), k_2(r), \ldots$ appearing in Eq. (2.8) for the denominator polynomial $Q_r(N)$. Since the expansions presented here do not extend beyond order $r=14$, at most the first seven exponents are nonvanishing. Finally we report the numerator $P_r(N)=\sum_{j=0}^{D(r)} p_j(r)N^j$. Examples are described in Sec. II.

	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$
$k_1,k_2,\ldots$	0	0	1	1	1,1	2,1
$p_0(r)$	4	12	72	200	2272	12 480
$p_1(r)N$			$32N$	$76N$	$1176N$	$11\,504N$
$p_2(r)N^2$					$160N^2$	$3216N^2$
$p_3(r)N^3$						$304N^3$

	$r=7$	$r=8$	$r=9$	$r=10$
$k_1,k_2,\ldots$	3,1,1	3,1,1	3,1,1,1	4,2,1,1
$p_0(r)$	417 024	1 135 872	24 987 648	541 900 800
$p_1(r)N$	$609\,216N$	$1\,534\,272N$	$33\,490\,688N$	$1\,068\,128\,256N$
$p_2(r)N^2$	$331\,872N^2$	$747\,648N^2$	$16\,697\,024N^2$	$851\,428\,352N^2$
$p_3(r)N^3$	$84\,480N^3$	$164\,000N^3$	$3\,937\,760N^3$	$354\,026\,368N^3$
$p_4(r)N^4$	$10\,480N^4$	$17\,568N^4$	$499\,088N^4$	$83\,381\,952N^4$
$p_5(r)N^5$	$512N^5$	$748N^5$	$33\,288N^5$	$11\,509\,280N^5$
$p_6(r)N^6$			$928N^6$	$940\,624N^6$
$p_7(r)N^7$				$42\,944N^7$
$p_8(r)N^8$				$880N^8$

	$r=11$	$r=12$	$r=13$	$r=14$
$k_1,k_2,\ldots$	5,3,1,1,1	5,2,1,1,1	5,3,1,1,1,1	6,3,2,1,1,1
$p_0(r)$	118 251 847 680	79 855 288 320	10 398 597 120 000	336 120 973 885 440
$p_1(r)N$	$317\,228\,351\,488N$	$184\,191\,475\,712N$	$25\,994\,174\,791\,680N$	$1\,020\,323\,217\,014\,784N$
$p_2(r)N^2$	$365\,993\,254\,912N^2$	$175\,887\,863\,808N^2$	$27\,646\,878\,220\,288N^2$	$1\,350\,405\,061\,607\,424N^2$
$p_3(r)N^3$	$237\,682\,079\,744N^3$	$89\,758\,213\,632N^3$	$16\,274\,644\,795\,392N^3$	$1\,020\,603\,615\,150\,080N^3$
$p_4(r)N^4$	$95\,679\,606\,272N^4$	$26\,392\,951\,552N^4$	$5\,778\,453\,815\,296N^4$	$483\,920\,864\,215\,040N^4$
$p_5(r)N^5$	$24\,859\,140\,608N^5$	$4\,510\,598\,528N^5$	$1\,262\,144\,117\,248N^5$	$148\,402\,198\,919\,168N^5$
$p_6(r)N^6$	$4\,228\,162\,304N^6$	$433\,283\,904N^6$	$163\,961\,916\,416N^6$	$28\,971\,880\,307\,712N^6$
$p_7(r)N^7$	$470\,195\,072N^7$	$21\,353\,088N^7$	$10\,904\,109\,952N^7$	$3\,268\,548\,640\,256N^7$
$p_8(r)N^8$	$33\,693\,152N^8$	$357\,888N^8$	$77\,551\,488N^8$	$122\,418\,059\,008N^8$
$p_9(r)N^9$	$1\,501\,536N^9$	$-3184N^9$	$-39\,454\,656N^9$	$-18\,160\,144\,896N^9$
$p_{10}(r)N^{10}$	$39\,200N^{10}$	$80N^{10}$	$-2\,408\,000N^{10}$	$-2\,838\,681\,600N^{10}$
$p_{11}(r)N^{11}$	$512N^{11}$		$-40\,640N^{11}$	$-157\,779\,456N^{11}$
$p_{12}(r)N^{12}$			$256N^{12}$	$-2\,584\,864N^{12}$
$p_{13}(r)N^{13}$				$84\,432N^{13}$
$p_{14}(r)N^{14}$				$2752N^{14}$

for any correlation function

$$F(\beta,N)=\sum_{r=0}^{\infty} f_r(N)\beta^r ,$$

the N dependence of the HT expansion coefficient $f_r(N)$ has the following structure:

$$f_r(N)=\frac{P_r(N)}{N^s Q_r(N)} , \tag{2.6}$$

where $P_r(N)$ and $Q_r(N)$ are polynomials in the variable N of the form

$$P_r(N)=\sum_{j=0}^{D(r)} p_j(r)N^j , \tag{2.7}$$

$$Q_r(N)=\prod_{l=1}^{L(r)} (N+2l)^{k_l(r)} . \tag{2.8}$$

The coefficients $p_j(r)$ in (2.7) are integers and the exponents $k_l(r)$ in (2.8) are positive integers. The values of the exponent s in (2.6), and of $D(r)$ and $L(r)$ in (2.7) and (2.8), respectively, depend on the function $F(\beta,N)$ under consideration.[10,11] The HT coefficients of $\chi(N,\beta)$, $m^{(2)}(N,\beta)$, and $\chi^{(4)}(N,\beta)$ also have this structure and for them we always have $s=r$ in (2.6).

The sets of integers $p_j(r)$, $k_l(r)$, $D(r)$, and $L(r)$ for the HT expansion coefficients $\chi_r(N)$, $b_r(N)$, and $d_r(N)$, are reported in the tables as follows. The tables are divided into 14 sections. Each section is headed by the order r of the coefficient to which it refers and contains the information necessary to reconstruct the coefficient following Eqs. (2.6), (2.7), and (2.8). First we give the set of exponents $k_1(r),k_2(r),\ldots,k_{L(r)}(r)$ appearing in Eq. (2.8) for the denominator $Q_r(N)$. Since the expansions do not extend beyond the order $r=14$, at most the first seven exponents are nonvanishing. Finally, we give the coefficients $p_j(r)$ for the numerator $P_r(N)$ in (2.7).

As an example, let us construct in $d=2$ dimensions the HT coefficient $\chi_5(N)$ of (2.3) and the coefficient $d_5(N)$ of (2.5). For $\chi_5(N)$ we use Table I, and for $d_5(N)$ we use

TABLE II. The structure in N of HT expansion coefficients $\chi_r(N)$ in (2.3) for the susceptibility in $d=3$ dimensions. For the use of the table see the caption of Table I.

	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$
$k_1,k_2,\ldots$	0	0	1	1	1,1	2,1
$p_0(r)$	6	30	300	1452	28 272	270 816
$p_1(r)N$			144N	666N	19 116N	310 296N
$p_2(r)N^2$					3024N^2	114 312N^2
$p_3(r)N^3$						13 476N^3

	$r=7$	$r=8$	$r=9$	$r=10$
$k_1,k_2,\ldots$	3,1,1	3,1,1	3,1,1,1	4,2,1,1
$p_0(r)$	15 626 880	74 489 472	2 847 568 896	108 255 780 864
$p_1(r)N$	27 729 888N	129 538 464N	5 193 056 640N	274 456 495 104N
$p_2(r)N^2$	19 020 144N^2	87 007 488N^2	3 778 300 896N^2	296 103 049 728N^2
$p_3(r)N^3$	6 271 680N^3	28 118 784N^3	1 399 800 432N^3	177 344 398 656N^3
$p_4(r)N^4$	989 304N^4	4 362 672N^4	278 225 208N^4	64 445 676 384N^4
$p_5(r)N^5$	59 328N^5	258 354N^5	28 031 388N^5	14 541 739 920N^5
$p_6(r)N^6$			1 115 856N^6	1 987 352 808N^6
$p_7(r)N^7$				150 093 888N^7
$p_8(r)N^8$				4 784 508N^8

	$r=11$	$r=12$	$r=13$	$r=14$
$k_1,k_2,\ldots$	5,3,1,1,1	5,3,1,1,1	5,3,1,1,1,1	6,3,2,1,1,1
$p_0(r)$	41 222 946 816 000	195 470 369 095 680	11 135 576 427 724 800	632 609 126 775 521 280
$p_1(r)N$	137 827 824 353 280N	645 668 315 578 368N	37 234 096 307 896 320N	2 510 657 938 044 223 488N
$p_2(r)N^2$	204 681 013 383 168N^2	947 108 563 906 560N^2	55 589 350 635 700 224N^2	4 524 841 591 275 257 856N^2
$p_3(r)N^3$	178 043 163 816 960N^3	813 739 409 599 488N^3	48 958 061 664 927 744N^3	4 905 292 095 121 784 832N^3
$p_4(r)N^4$	100 715 199 380 736N^4	454 747 923 263 232N^4	28 310 386 659 250 176N^4	3 571 783 932 314 320 896N^4
$p_5(r)N^5$	38 875 384 707 840N^5	173 480 151 209 088N^5	11 317 910 548 800 768N^5	1 847 293 475 548 572 672N^5
$p_6(r)N^6$	10 440 501 635 712N^6	46 077 593 061 312N^6	3 205 796 898 092 544N^6	699 626 042 228 688 384N^6
$p_7(r)N^7$	1 949 080 239 936N^7	8 514 940 815 264N^7	647 734 280 214 336N^7	197 061 627 772 250 880N^7
$p_8(r)N^8$	247 560 454 704N^8	1 071 698 638 656N^8	92 551 274 785 344N^8	41 467 574 110 306 944N^8
$p_9(r)N^9$	20 342 269 008N^9	87 360 454 320N^9	9 107 063 617 056N^9	6 484 097 497 687 296N^9
$p_{10}(r)N^{10}$	971 521 080N^{10}	4 143 560 904N^{10}	584 874 211 968N^{10}	741 110 387 409 216N^{10}
$p_{11}(r)N^{11}$	20 393 856N^{11}	86 473 548N^{11}	21 978 467 784N^{11}	59 988 858 732 000N^{11}
$p_{12}(r)N^{12}$			365 034 816N^{12}	3 247 646 112 432N^{12}
$p_{13}(r)N^{13}$				105 152 764 464N^{13}
$p_{14}(r)N^{14}$				1 534 827 960N^{14}

Table VII. In both cases we have $s=r=5$. Moreover, since in both cases only two nonzero exponents $k_l(r)$ are reported, we have $L(5)=2$. In the case of $\chi_5(N)$ we read that $k_1(5)=1$ and $k_2(5)=1$, so that $\chi_5(N)=P_5(N)/N^5Q_5(N)$ with

$$P_5(N)=2272+1176N+160N^2$$

and

$$Q_5(N)=(N+2)(N+4) .$$

In the case of $d_5(N)$ we have $k_1(5)=2$ and $k_2(5)=1$; therefore $d_5(N)=P_5(N)/N^5Q_5(N)$ with

$$P_5(N)=-548\,352-746\,624N-322\,112N^2-41\,984N^3$$

and

$$Q_5(N)=(N+2)^2(N+4) .$$

III. ANALYSIS OF THE HT SERIES AND SCALING BEHAVIOR IN $d=2$

In this section we briefly discuss some of the information that can be extracted from the HT series for general N in $d=2$ dimensions. In particular we address the following three topics: (i) evaluating the critical exponents in the interval $-2\le N<2$; (ii) how the phase structure of the model changes as N varies through $N=2$; (iii) the possibility of observing the scaling behavior for $N\ge 3$. Throughout this section, for convenience, we shall use the variable $\tilde{\beta}=\beta/N$.

A. The region $-2<N<2$

Computing Padé approximants (PA's) to the logarithmic derivative of the susceptibility is the simplest way to study how the location of the nearest singularities in the complex $\tilde{\beta}$ plane depends on N. For most PA's of sufficiently high order that can be constructed with the

TABLE III. The structure in N of HT expansion coefficients $\chi_r(N)$ in (2.3) for the susceptibility in $d=4$ dimensions. For the use of the table see the caption of Table I.

	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$
$k_1,k_2,\ldots$	0	0	1	1	1,1	2,1
$p_0(r)$	8	56	784	5392	148 672	2 034 560
$p_1(r)N$			$384N$	$2584N$	$106\,224N$	$2\,442\,336N$
$p_2(r)N^2$					$17\,280N^2$	$941\,728N^2$
$p_3(r)N^3$						$114\,688N^3$

	$r=7$	$r=8$	$r=9$	$r=10$
$k_1,k_2,\ldots$	3,1,1	3,1,1	3,1,1,1	4,2,1,1
$p_0(r)$	167 281 152	1 141 670 400	62 382 821 376	3 400 710 586 368
$p_1(r)N$	$309\,591\,424N$	$2\,094\,884\,992N$	$121\,195\,140\,608N$	$9\,100\,261\,273\,600N$
$p_2(r)N^2$	$221\,412\,288N^2$	$1\,485\,805\,312N^2$	$93\,884\,139\,392N^2$	$10\,369\,670\,408\,192N^2$
$p_3(r)N^3$	$75\,893\,248N^3$	$505\,455\,808N^3$	$36\,861\,548\,736N^3$	$6\,554\,375\,483\,136N^3$
$p_4(r)N^4$	$12\,355\,552N^4$	$81\,788\,224N^4$	$7\,686\,364\,960N^4$	$2\,506\,705\,466\,240N^4$
$p_5(r)N^5$	$758\,784N^5$	$4\,999\,832N^5$	$803\,568\,400N^5$	$592\,435\,053\,888N^5$
$p_6(r)N^6$			$32\,881\,536N^6$	$84\,285\,035\,808N^6$
$p_7(r)N^7$				$6\,583\,610\,112N^7$
$p_8(r)N^8$				$215\,712\,064N^8$

	$r=11$	$r=12$	$r=13$	$r=14$
$k_1,k_2,\ldots$	5,3,1,1,1	5,3,1,1,1	5,3,1,1,1,1	6,3,2,1,1,1
$p_0(r)$	1 854 784 339 968 000	12 625 624 388 075 520	1 031 680 005 117 050 880	84 209 341 360 672 604 160
$p_1(r)N$	$6\,507\,206\,561\,300\,480N$	$44\,075\,529\,131\,589\,632N$	$3\,668\,974\,340\,257\,087\,488N$	$354\,117\,002\,807\,089\,299\,456N$
$p_2(r)N^2$	$10\,144\,726\,549\,250\,048N^2$	$68\,381\,820\,807\,036\,928N^2$	$5\,829\,688\,582\,007\,029\,760N^2$	$676\,833\,701\,251\,717\,791\,744N^2$
$p_3(r)N^3$	$9\,262\,176\,095\,203\,328N^3$	$62\,142\,501\,325\,361\,152N^3$	$5\,463\,212\,398\,573\,780\,992N^3$	$778\,434\,448\,512\,363\,003\,904N^3$
$p_4(r)N^4$	$5\,493\,689\,774\,611\,456N^4$	$36\,696\,046\,625\,317\,888N^4$	$3\,357\,344\,071\,618\,248\,704N^4$	$601\,158\,227\,890\,935\,693\,312N^4$
$p_5(r)N^5$	$2\,219\,048\,397\,100\,032N^5$	$14\,761\,517\,329\,086\,976N^5$	$1\,422\,882\,886\,390\,367\,232N^5$	$329\,388\,213\,282\,312\,007\,680N^5$
$p_6(r)N^6$	$621\,825\,382\,745\,600N^6$	$4\,120\,910\,196\,661\,504N^6$	$425\,692\,335\,370\,364\,928N^6$	$131\,902\,761\,502\,374\,287\,360N^6$
$p_7(r)N^7$	$120\,677\,956\,329\,216N^7$	$797\,038\,775\,468\,160N^7$	$90\,424\,684\,925\,898\,496N^7$	$39\,172\,983\,215\,344\,258\,048N^7$
$p_8(r)N^8$	$15\,867\,628\,423\,360N^8$	$104\,487\,262\,760\,192N^8$	$13\,511\,340\,590\,377\,728N^8$	$8\,660\,734\,711\,633\,452\,544N^8$
$p_9(r)N^9$	$1\,343\,887\,412\,672N^9$	$8\,826\,447\,348\,224N^9$	$1\,382\,638\,894\,146\,176N^9$	$1\,417\,064\,748\,813\,474\,816N^9$
$p_{10}(r)N^{10}$	$65\,868\,866\,304N^{10}$	$431\,657\,516\,320N^{10}$	$91\,840\,736\,637\,056N^{10}$	$168\,739\,537\,213\,635\,072N^{10}$
$p_{11}(r)N^{11}$	$1\,413\,310\,464N^{11}$	$9\,244\,527\,744N^{11}$	$3\,551\,309\,757\,888N^{11}$	$14\,166\,455\,804\,010\,752N^{11}$
$p_{12}(r)N^{12}$			$60\,413\,709\,312N^{12}$	$791\,991\,379\,982\,784N^{12}$
$p_{13}(r)N^{13}$				$26\,371\,601\,439\,200N^{13}$
$p_{14}(r)N^{14}$				$394\,346\,385\,408N^{14}$

available HT coefficients, we find that the smallest positive pole (which we may call the *critical pole* from now on) accurately reproduces the known critical singularities for $N=0$, 1, and 2, and smoothly interpolates[7] among them as N varies continuously between 0 and 2.

It has been argued[8] that for $-2 \le N < 2$ the critical exponent ν of the correlation length and the critical exponent η of the spin-spin correlation function are, respectively,

$$\nu = \frac{1}{4-2t} \tag{3.1}$$

and

$$\eta = 2 - \frac{3}{2t} - \frac{t}{2} \tag{3.2}$$

with

$$N = -2\cos\left[\frac{2\pi}{t}\right] \tag{3.3}$$

and $1 \le t \le 2$. From the scaling relation $\gamma=(2-\eta)\nu$ it follows that the exponent of the susceptibility is

$$\gamma = \frac{3+t^2}{4t(2-t)} . \tag{3.4}$$

This latter formula compares well[7] to a calculation of γ as the residue at the critical pole of PA's of the log-derivative of the susceptibility. If, analogously, we compute ν starting with the second moment definition of the correlation length

$$\xi^2(N,\tilde{\beta}) = \frac{m^{(2)}(N,\tilde{\beta})}{2d\chi(N,\tilde{\beta})} ,$$

we find that the overall approximation is not as good.

We find, however, that a better agreement with the exact formulas for both γ and ν can be obtained by a different method which involves $\chi^{(4)}(N,\beta)$. It is useful to recall that $\chi^{(4)}(N,\beta) \simeq (\beta_c - \beta)^{-\gamma-2\Delta}$. Assuming the validity of hyperscaling we have $2\Delta - d\nu - \gamma = 0$. Then we can perform a critical point renormalization in order to

TABLE IV. The structure in N of HT expansion coefficients $b_r(N)$ in (2.4) for the second correlation moment in $d=2$ dimensions. For the use of the table see the caption of Table I.

	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$
$k_1,k_2,\ldots$	0	0	1	1	1,1	2,1
$p_0(r)$	4	32	328	1408	21 728	156 928
$p_1(r)N$			$160N$	$640N$	$14\,232N$	$171\,072N$
$p_2(r)N^2$					$2208N^2$	$59\,840N^2$
$p_3(r)N^3$						$6784N^3$

	$r=7$	$r=8$	$r=9$	$r=10$
$k_1,k_2,\ldots$	3,1,1	3,1,1	3,1,1,1	4,2,1,1
$p_0(r)$	6 487 296	21 596 160	560 007 168	14 220 853 248
$p_1(r)N$	$10\,904\,512N$	$34\,468\,352N$	$912\,207\,616N$	$32\,437\,182\,464N$
$p_2(r)N^2$	$7\,052\,384N^2$	$21\,040\,128N^2$	$585\,628\,864N^2$	$31\,140\,450\,304N^2$
$p_3(r)N^3$	$2\,192\,384N^3$	$6\,162\,816N^3$	$190\,951\,904N^3$	$16\,449\,182\,208N^3$
$p_4(r)N^4$	$328\,688N^4$	$878\,208N^4$	$33\,905\,168N^4$	$5\,251\,439\,360N^4$
$p_5(r)N^5$	$18\,944N^5$	$48\,640N^5$	$3\,117\,448N^5$	$1\,045\,045\,888N^5$
$p_6(r)N^6$			$115\,616N^6$	$127\,390\,272N^6$
$p_7(r)N^7$				$8\,712\,896N^7$
$p_8(r)N^8$				$255\,616N^8$

	$r=11$	$r=12$	$r=13$	$r=14$
$k_1,k_2,\ldots$	5,3,1,1,1	5,3,1,1,1	5,3,1,1,1,1	6,3,2,1,1,1
$p_0(r)$	3 549 643 407 360	10 920 048 721 920	398 311 403 028 480	14 381 419 069 440 000
$p_1(r)N$	$10\,748\,529\,770\,496N$	$31\,897\,824\,264\,192N$	$1\,152\,646\,176\,964\,608N$	$49\,571\,261\,401\,006\,080N$
$p_2(r)N^2$	$14\,330\,317\,561\,856N^2$	$40\,878\,320\,844\,800N^2$	$1\,467\,863\,723\,606\,016N^2$	$76\,559\,245\,123\,780\,608N^2$
$p_3(r)N^3$	$11\,099\,228\,633\,088N^3$	$30\,320\,450\,846\,720N^3$	$1\,086\,884\,435\,984\,384N^3$	$70\,130\,377\,503\,539\,200N^3$
$p_4(r)N^4$	$5\,553\,033\,387\,520N^4$	$14\,475\,616\,055\,296N^4$	$521\,655\,212\,892\,160N^4$	$42\,548\,875\,902\,058\,496N^4$
$p_5(r)N^5$	$1\,888\,211\,571\,200N^5$	$4\,684\,012\,402\,688N^5$	$171\,508\,785\,347\,072N^5$	$18\,101\,157\,266\,890\,752N^5$
$p_6(r)N^6$	$446\,700\,183\,296N^6$	$1\,053\,225\,265\,152N^6$	$39\,822\,249\,767\,936N^6$	$5\,581\,472\,628\,355\,072N^6$
$p_7(r)N^7$	$73\,788\,019\,072N^7$	$165\,529\,253\,888N^7$	$6\,624\,256\,364\,416N^7$	$1\,272\,436\,773\,582\,848N^7$
$p_8(r)N^8$	$8\,364\,424\,672N^8$	$17\,912\,071\,424N^8$	$788\,536\,224\,640N^8$	$216\,726\,437\,780\,480N^8$
$p_9(r)N^9$	$620\,464\,608N^9$	$1\,274\,573\,696N^9$	$65\,778\,403\,648N^9$	$27\,616\,170\,103\,808N^9$
$p_{10}(r)N^{10}$	$27\,094\,944N^{10}$	$53\,696\,896N^{10}$	$3\,655\,303\,744N^{10}$	$2\,605\,954\,492\,416N^{10}$
$p_{11}(r)N^{11}$	$526\,848N^{11}$	$1\,013\,248N^{11}$	$121\,407\,936N^{11}$	$177\,259\,542\,528N^{11}$
$p_{12}(r)N^{12}$			$1\,818\,880N^{12}$	$8\,230\,177\,408N^{12}$
$p_{13}(r)N^{13}$				$233\,423\,296N^{13}$
$p_{14}(r)N^{14}$				$3\,045\,888N^{14}$

estimate ν. To this purpose, we form the HT series $[\chi(N,\tilde{\beta})]^2=\sum_{r=0} t_r(N)\tilde{\beta}^r$ and compute PA's to the function

$$(1-x)D\ln\left[\sum_{r=0}\frac{d_r(N)}{t_r(N)}x^r\right]$$

at $x=1$.

Analogously, we can estimate γ by computing PA's to the function

$$(1-x)D\ln\left[\sum_{r=0}\frac{d_r(N)}{b_{r+1}(N)}x^r\right]$$

at $x=1$. The results of this calculation with [5/5] and [6/6] PA's are shown in Fig. 1. The convergence appears to be very good over most of the interval $-2<N<2$ except near its ends, where it is somewhat slower, due to the fact that the behavior of the series is less regular. Notice that we have not insisted on computing the best possible estimate of γ and ν for each value of N, but, for simplicity, we have relied on a single numerical procedure for the whole range of N. However, we can conclude that the validity of 3.1 and 3.4 is clearly supported by our calculation and the agreement is rapidly improving as more HT coefficients are used.

B. The region $N>2$

As is well known,[9] for $N\geq 3$ the $O(N)$ model is asymptotically free and no critical singularities are expected to occur at any real finite $\tilde{\beta}$. This is well confirmed by our study of the log-derivative of $\chi(N,\tilde{\beta})$. In a systematic study of all PA's for $N\geq 3$ no convincing indication has emerged of the presence of a critical point at a finite real $\tilde{\beta}$. Rather, as N varies between 2 and 3, most PA's consistently indicate that the critical pole collides with another pole producing a pair of complex conjugate poles that slowly move into the complex plane and, as $N\rightarrow\infty$, reach the limiting points $\tilde{\beta}_{\pm}\simeq 0.33(1\pm i)$. This is illustrated in Fig. 2, where we have plotted the real and imag-

TABLE V. The structure in N of HT expansion coefficients $b_r(N)$ in (2.4) for the second correlation moment in $d=3$ dimensions. For the use of the table see the caption of Table I.

	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$
$k_1,k_2,\ldots$	0	0	1	1	1,1	2,1
$p_0(r)$	6	72	1164	8064	204 528	2 456 448
$p_1(r)N$			576N	3888N	146 124N	2 929 248N
$p_2(r)N^2$					23 760N^2	1 122 528N^2
$p_3(r)N^3$						136 080N^3

	$r=7$	$r=8$	$r=9$	$r=10$
$k_1,k_2,\ldots$	3,1,1	3,1,1	3,1,1,1	4,2,1,1
$p_0(r)$	170 289 792	956 823 552	42 088 707 072	1 820 564 029 440
$p_1(r)N$	312 319 968N	1 728 273 408N	80 027 621 760N	4 769 433 145 344N
$p_2(r)N^2$	221 383 920N^2	1 206 868 608N^2	60 705 869 280N^2	5 321 186 764 800N^2
$p_3(r)N^3$	75 264 960N^3	404 721 600N^3	23 384 805 744N^3	3 295 384 254 720N^3
$p_4(r)N^4$	12 170 904N^4	64 724 352N^4	4 800 525 624N^4	1 236 516 916 608N^4
$p_5(r)N^5$	743 616N^5	3 921 696N^5	495 870 684N^5	287 290 885 440N^5
$p_6(r)N^6$			20 110 032N^6	40 275 314 208N^6
$p_7(r)N^7$				3 107 523 840N^7
$p_8(r)N^8$				100 804 176N^8

	$r=11$	$r=12$	$r=13$	$r=14$
$k_1,k_2,\ldots$	5,3,1,1,1	5,3,1,1,1	5,3,1,1,1,1	6,3,2,1,1,1
$p_0(r)$	776 696 905 728 000	4 095 660 376 719 360	256 698 768 799 825 920	15 959 759 305 121 464 320
$p_1(r)N$	2 669 258 856 775 680N	13 934 832 721 330 176N	885 687 727 893 184 512N	65 126 313 374 826 627 072N
$p_2(r)N^2$	4 076 555 067 445 248N^2	21 070 456 869 617 664N^2	1 365 396 429 834 780 672N^2	120 776 409 770 039 967 744N^2
$p_3(r)N^3$	3 647 307 112 190 976N^3	18 668 449 163 280 384N^3	1 242 082 941 562 257 408N^3	134 799 493 109 438 742 528N^3
$p_4(r)N^4$	2 121 429 738 632 448N^4	10 756 490 095 534 080N^4	741 684 670 798 319 616N^4	101 075 961 535 383 207 936N^4
$p_5(r)N^5$	841 182 276 069 120N^5	4 227 254 991 461 376N^5	305 895 801 259 154 688N^5	53 821 049 308 716 331 008N^5
$p_6(r)N^6$	231 709 799 815 296N^6	1 154 847 354 120 192N^6	89 237 579 848 134 144N^6	20 972 091 994 847 557 632N^6
$p_7(r)N^7$	44 274 759 493 440N^7	219 020 166 457 344N^7	18 527 188 676 264 256N^7	6 070 560 956 664 671 232N^7
$p_8(r)N^8$	5 741 865 229 488N^8	28 215 891 209 472N^8	2 712 711 410 020 416N^8	1 310 640 195 233 880 576N^8
$p_9(r)N^9$	480 498 609 168N^9	2 347 567 763 904N^9	272 727 691 809 312N^9	209 854 631 867 283 456N^9
$p_{10}(r)N^{10}$	23 310 434 136N^{10}	113 324 238 144N^{10}	17 842 958 870 016N^{10}	24 507 398 880 206 592N^{10}
$p_{11}(r)N^{11}$	495 849 600N^{11}	2 400 521 184N^{11}	681 157 898 376N^{11}	2 022 286 243 464 576N^{11}
$p_{12}(r)N^{12}$			11 463 937 344N^{12}	111 357 392 644 800N^{12}
$p_{13}(r)N^{13}$				3 659 422 931 136N^{13}
$p_{14}(r)N^{14}$				54 103 051 872N^{14}

inary parts of the nearest pole in the first quadrant of the complex $\tilde{\beta}$ plane, versus the variable $z = 1 - 1/N$ chosen for convenience. The plot has been obtained as follows: we have considered a set of equally spaced values of z given by $z_l = l/30$ with $0 \le l \le 30$ and, for each z_l, we have computed all possible $[m/n]$ PA's to $D \ln[\chi(N,\tilde{\beta})]$ with both m and $n > 3$ and $m + n > 9$ (there are 18 such approximants). At $z=0$ (i.e., $N=1$), for each PA, we have chosen the pole $\tilde{\beta}^{[m/n]}$ nearest to the known value of $\tilde{\beta}_c$, namely $\tilde{\beta}_c \simeq 0.44\ldots$. We have then simply computed the mean value $\tilde{\beta}_0$ over all approximants and taken the rms deviation as a rough measure of the error. The evolution with N of the critical pole is then computed with the following iterative procedure: when $z = z_l$ with $l > 0$, for each available PA we have chosen the pole $\tilde{\beta}^{[m/n]}$ nearest to $\tilde{\beta}_{l-1}$ and computed the mean value $\tilde{\beta}_l$ and the rms deviation.

Precisely the same procedure for estimating the critical singularity can be carried out for $D \ln[m^{(2)}(N,\tilde{\beta})]$ and $D \ln[\chi^{(4)}(N,\tilde{\beta})]$ yielding results which are perfectly consistent with the ones previously obtained for the susceptibility. These sets of "independent measurements" might be combined resulting in a significant reduction of the error bars. It should also be noticed that, for $N < 2$ the nearest singularity (which is the critical point) is algebraic and therefore it can be located with high precision by PA's of the log-derivative, whereas for $N \ge 2$ the nearest singularity changes its nature and although it is still detectable by the same numerical procedure, it can only be located with a somewhat greater uncertainty.

In the $N = \infty$ case we have been able to map out[12] the whole set of unphysical singularities (all of them being branch points of second order) and in particular to locate the nearest ones at $\pm\tilde{\beta}_{\pm} \simeq \pm 0.321\,62(1 \pm i)$. This is perfectly consistent with our results for finite N. Since we also expect that the quartet structure of the nearest singularities of the $N = \infty$ case persists down to finite $N \ge 3$, the class of PA's we have considered, having at least four poles, appears to be the most reliable one.

C. The scaling behavior

In this section we shall test the consistency of our HT series with the low-temperature asymptotic behavior[9]

TABLE VI. The structure in N of HT expansion coefficients $b_r(N)$ in (2.4) for the second correlation moment in $d=4$ dimensions. For the use of the table see the caption of Table I.

	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$
$k_1,k_2,\ldots$	0	0	1	1	1,1	2,1
$p_0(r)$	8	128	2832	27 136	959 680	16 151 040
$p_1(r)N$			1408N	13 312N	702 192N	19 720 832N
$p_2(r)N^2$					115 584N^2	7 724 928N^2
$p_3(r)N^3$						951 296N^3

	$r=7$	$r=8$	$r=9$	$r=10$
$k_1,k_2,\ldots$	3,1,1	3,1,1	3,1,1,1	4,2,1,1
$p_0(r)$	1 575 485 952	12 493 123 584	777 512 669 184	47 674 193 215 488
$p_1(r)N$	2 955 271 552N	23 272 901 632N	1 534 940 407 296N	129 188 999 938 048N
$p_2(r)N^2$	2 140 822 464N^2	16 748 932 096N^2	1 207 213 671 296N^2	149 021 446 676 480N^2
$p_3(r)N^3$	742 289 920N^3	5 773 515 520N^3	480 413 164 736N^3	95 292 480 070 656N^3
$p_4(r)N^4$	121 964 512N^4	944 190 208N^4	101 269 408 288N^4	36 835 340 799 488N^4
$p_5(r)N^5$	7 541 760N^5	58 179 584N^5	10 674 780 688N^5	8 788 221 455 616N^5
$p_6(r)N^6$			439 466 880N^6	1 260 405 480 576N^6
$p_7(r)N^7$				99 111 836 544N^7
$p_8(r)N^8$				3 265 020 928N^8

	$r=11$	$r=12$	$r=13$	$r=14$
$k_1,k_2,\ldots$	5,3,1,1,1	5,3,1,1,1	5,3,1,1,1,1	6,3,2,1,1,1
$p_0(r)$	28 879 815 062 323 200	216 530 004 580 761 600	19 319 406 570 850 222 080	1 711 622 510 853 583 011 840
$p_1(r)N$	102 378 553 513 410 560N	764 285 889 698 856 960N	69 499 395 312 584 491 008 N	7 271 127 085 008 241 557 504N
$p_2(r)N^2$	161 239 676 408 152 064N^2	1 198 682 248 083 537 920N^2	111 676 686 703 381 053 440N^2	14 037 406 139 695 761 457 152N^2
$p_3(r)N^3$	148 666 001 675 055 104N^3	1 100 805 870 174 306 304N^3	105 798 518 171 713 929 216N^3	16 303 538 995 747 631 398 912N^3
$p_4(r)N^4$	89 006 105 574 980 608N^4	656 580 058 020 503 552N^4	65 691 477 730 512 060 416N^4	12 710 610 237 272 546 082 816N^4
$p_5(r)N^5$	36 266 849 530 573 824N^5	266 604 473 757 765 632N^5	28 110 327 506 185 798 656N^5	7 027 845 454 667 695 079 424N^5
$p_6(r)N^6$	10 244 203 588 587 008N^6	75 069 176 785 358 848N^6	8 484 424 698 101 913 600N^6	2 838 449 718 304 007 905 280N^6
$p_7(r)N^7$	2 002 390 906 712 832N^7	14 632 000 577 171 456N^7	1 816 568 754 087 937 792N^7	849 704 044 878 926 000 128N^7
$p_8(r)N^8$	264 955 564 933 312N^8	1 931 291 291 716 096N^8	273 333 893 436 789 504N^8	189 235 937 563 564 521 472N^8
$p_9(r)N^9$	22 562 936 229 056N^9	164 110 301 646 080N^9	28 140 364 968 530 048N^9	31 167 765 763 170 514 944N^9
$p_{10}(r)N^{10}$	1 111 049 564 160N^{10}	8 066 351 629 056N^{10}	1 878 876 000 983 936N^{10}	3 733 325 830 186 285 056N^{10}
$p_{11}(r)N^{11}$	23 932 495 872N^{11}	173 485 039 616N^{11}	72 969 437 442 240N^{11}	315 069 272 149 701 632N^{11}
$p_{12}(r)N^{12}$			1 245 846 703 104N^{12}	17 694 907 693 559 040N^{12}
$p_{13}(r)N^{13}$				591 540 304 853 120N^{13}
$p_{14}(r)N^{14}$				8 875 762 483 200N^{14}

predicted by the perturbative renormalization group at the three-loop level for the susceptibility $\chi(N,\tilde{\beta})$,

$$\chi_{\rm as}(N,\tilde{\beta}) = c(N)[b(N)\tilde{\beta}]^{-(N+1)/(N-2)} \times \exp[2b(N)\tilde{\beta}]\left[1+\frac{h}{\tilde{\beta}}+\cdots\right] \tag{3.5}$$

and for the correlation length $\xi(N,\tilde{\beta})$,

$$\xi_{\rm as}(N,\tilde{\beta}) = c'(N)[b(N)\tilde{\beta}]^{-1/(N-2)} \times \exp[b(N)\tilde{\beta}]\left[1+\frac{h'}{\tilde{\beta}}+\cdots\right], \tag{3.6}$$

where

$$b(N) = 2\pi N/(N-2) \tag{3.7}$$

and h,h' are nonuniversal $O(1/N)$ quantities that have been computed for the square lattice.[13] In both formulas, however, the $1/\tilde{\beta}$ corrections are quite small.

The presence of unphysical N-dependent singularities in the vicinity of the real $\tilde{\beta}$ axis might explain why the approach to asymptotic scaling has appeared so elusive in both HT (Refs. 14 and 15) and Monte Carlo[16,17] analyses of $O(N)$ models, for small N. Actually in the first such studies the behavior of the thermodynamic functions had usually been investigated over ranges of β not extending

TABLE VII. The structure in N of HT expansion coefficients $d_r(N)$ in (2.5) for the second field derivative of the susceptibility in $d=2$ dimensions. For the use of the table see the caption of Table I.

	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$
$k_1,k_2,\ldots$		1	1	2,1	2,1	3,1,1
$p_0(r)$	−32	−512	−3072	−122 624	−548 352	−27 150 336
$p_1(r)N$		−280N	−1792N	−166 848N	−746 624N	−54 673 920N
$p_2(r)N^2$				−71 616N^2	−322 112N^2	−42 256 128N^2
$p_3(r)N^3$				−9 352N^3	−41 984N^3	−15 468 352N^3
$p_4(r)N^4$						−2 639 744N^4
$p_5(r)N^5$						−167 328N^5

	$r=7$	$r=8$	$r=9$	$r=10$
$k_1,k_2,\ldots$	3,1,1	4,2,1,1	4,2,1,1	5,3,1,1,1
$p_0(r)$	−105 922 560	−25 281 232 896	−91 090 452 480	−25 509 872 271 360
$p_1(r)N$	−209 727 488N	−71 046 963 200N	−250 749 534 208N	−90 236 056 764 416N
$p_2(r)N^2$	−159 507 968N^2	−84 638 801 920N^2	−292 433 149 952N^2	−141 279 803 998 208N^2
$p_3(r)N^3$	−57 489 664N^3	−55 620 065 792N^3	−188 051 248 128N^3	−129 018 898 219 008N^3
$p_4(r)N^4$	−9 657 344N^4	−21 962 561 536N^4	−72 666 583 040N^4	−76 227 082 387 456N^4
$p_5(r)N^5$	−604 160N^5	−5 316 976 768N^5	−17 226 163 712N^5	−30 544 423 879 680N^5
$p_6(r)N^6$		−769 097 472N^6	−2 443 206 656N^6	−8 458 645 201 408N^6
$p_7(r)N^7$		−60 729 200N^7	−189 579 584N^7	−1 617 274 658 560N^7
$p_8(r)N^8$		−2 003 496N^8	−6 161 920N^8	−209 086 720 768N^8
$p_9(r)N^9$				−17 399 326 912N^9
$p_{10}(r)N^{10}$				−838 286 656N^{10}
$p_{11}(r)N^{11}$				−17 702 048N^{11}

	$r=11$	$r=12$	$r=13$	$r=14$
$k_1,k_2,\ldots$	5,3,1,1,1	6,3,2,1,1,1	6,3,2,1,1,1	7,4,3,1,1,1,1
$p_0(r)$	−87 235 189 800 960	−42 102 500 811 079 680	−416 219 862 788 997 120	−138 739 954 262 999 040
$p_1(r)N$	−302 120 698 904 576N	−174 022 970 025 443 328N	−561 523 419 632 369 664N	−1 496 819 676 609 830 191 104N
$p_2(r)N^2$	−462 634 718 855 168N^2	−325 422 570 506 551 296N^2	−1 026 945 647 521 038 336N^2	−3 417 055 430 559 324 438 528N^2
$p_3(r)N^3$	−412 802 695 299 072N^3	−364 504 234 447 077 376N^3	−1 123 604 171 268 816 896N^3	−4 782 129 114 510 564 261 888N^3
$p_4(r)N^4$	−238 127 192 064 000N^4	−272 929 226 814 914 560N^4	−820 895 488 379 191 296N^4	−4 595 089 247 620 632 150 016N^4
$p_5(r)N^5$	−93 129 870 897 152N^5	−144 393 618 748 334 080N^5	−423 367 602 534 203 392N^5	−3 218 011 327 485 241 196 544N^5
$p_6(r)N^6$	−25 175 575 072 768N^6	−55 629 526 732 750 848N^6	−158 909 666 164 817 920N^6	−1 701 792 134 400 127 533 056N^6
$p_7(r)N^7$	−4 702 254 317 568N^7	−15 850 247 225 260 032N^7	−44 105 169 752 195 072N^7	−694 844 122 797 083 099 136N^7
$p_8(r)N^8$	−594 699 031 552N^8	−3 356 268 047 056 896N^8	−9 100 893 088 256 000N^8	−222 086 594 520 095 752 192N^8
$p_9(r)N^9$	−48 508 057 344N^9	−525 713 098 575 360N^9	−1 390 497 680 946 176N^9	−55 999 938 891 368 628 224N^9
$p_{10}(r)N^{10}$	−2 296 123 904N^{10}	−59 980 137 570 816N^{10}	−154 984 159 232 000N^{10}	−11 171 567 854 489 735 168N^{10}
$p_{11}(r)N^{11}$	−47 753 216N^{11}	−4 834 680 108 672N^{11}	−12 228 237 683 712N^{11}	−1 759 969 529 538 605 056N^{11}
$p_{12}(r)N^{12}$		−260 263 368 064N^{12}	−645 801 055 744N^{12}	−217 447 513 487 444 992N^{12}
$p_{13}(r)N^{13}$		−8 373 860 992N^{13}	−20 432 227 328N^{13}	−20 793 034 272 038 912N^{13}
$p_{14}(r)N^{14}$		−121 445 920N^{14}	−292 061 184N^{14}	−1 505 364 135 603 968N^{14}
$p_{15}(r)N^{15}$				−79 619 382 179 456N^{15}
$p_{16}(r)N^{16}$				−2 896 359 558 848N^{16}
$p_{17}(r)N^{17}$				−64 639 771 520N^{17}
$p_{18}(r)N^{18}$				−665 697 152N^{18}

significantly beyond a neighborhood of the unphysical singularities. Since the distance of these singularities from the real β axis grows with N, their influence will be stronger for small N. More radical explanations of the aforementioned difficulties such as the one assuming the existence of a conventional phase transition[17] at finite $\tilde{\beta}$ find no support in our calculations.

The best way to cope with these singularities in a numerical study, as shown in Ref. 12 in the $N=\infty$ case, would be to introduce a convenient conformal transformation for $\tilde{\beta}$. This requires a detailed knowledge of the location and of the nature of the singularities that, for now, we only have in the $N=\infty$ case. Such an approach, however, has been attempted in the case of $O(3)$ and $O(4)$ (Ref. 18) with reasonable results. Here we prefer to try a different method.

Let us first gain some confidence in the actual possibility of computing the low-temperature asymptotic behavior from our HT series, by showing how to calculate the quantity $b(N)$ defined in (3.7) and appearing in (3.5) and

TABLE VIII. The structure in N of HT expansion coefficients $d_r(N)$ in (2.5) for the second field derivative of the susceptibility in $d=3$ dimensions. For the use of the table see the caption of Table I.

	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$	$r=7$
$k_1,k_2,\ldots$		1	1	2,1	2,1	3,1,1	3,1,1
$p_0(r)$	-48	-1248	$-12\,480$	$-851\,712$	$-6\,573\,312$	$-565\,908\,480$	$-3\,849\,744\,384$
$p_1(r)N$		$-660N$	$-6912N$	$-1\,128\,192N$	$-8\,786\,880N$	$-1\,137\,490\,944N$	$-7\,739\,114\,496N$
$p_2(r)N^2$				$-474\,000N^2$	$-3\,725\,856N^2$	$-877\,991\,616N^2$	$-5\,976\,661\,248N^2$
$p_3(r)N^3$				$-61\,236N^3$	$-483\,840N^3$	$-321\,566\,208N^3$	$-2\,190\,260\,352N^3$
$p_4(r)N^4$						$-55\,124\,016N^4$	$-375\,495\,552N^4$
$p_5(4)N^5$						$-3\,514\,968N^5$	$-23\,938\,560N^5$

	$r=8$	$r=9$	$r=10$	$r=11$
$k_1,k_2,\ldots$	4,2,1,1	4,2,1,1	5,3,1,1,1	5,3,1,1,1
$p_0(r)$	$-1\,607\,361\,822\,720$	$-10\,146\,634\,334\,208$	$-4\,986\,778\,413\,957\,120$	$-29\,957\,292\,231\,229\,440$
$p_1(r)N$	$-4\,630\,934\,495\,232N$	$-29\,145\,299\,066\,880N$	$-18\,502\,905\,604\,472\,832N$	$-110\,697\,821\,461\,807\,104N$
$p_2(r)N^2$	$-5\,658\,731\,108\,352N^2$	$-35\,515\,117\,682\,688N^2$	$-30\,434\,129\,019\,666\,432N^2$	$-181\,359\,443\,339\,575\,296N^2$
$p_3(r)N^3$	$-3\,815\,032\,300\,032N^3$	$-23\,883\,563\,143\,680N^3$	$-29\,229\,711\,376\,023\,552N^3$	$-173\,526\,582\,721\,855\,488N^3$
$p_4(r)N^4$	$-1\,545\,703\,906\,176N^4$	$-9\,654\,774\,400\,512N^4$	$-18\,173\,047\,179\,460\,608N^4$	$-107\,505\,154\,356\,572\,160N^4$
$p_5(r)N^5$	$-383\,951\,922\,240N^5$	$-2\,393\,329\,321\,728N^5$	$-7\,663\,189\,901\,412\,864N^5$	$-45\,183\,600\,633\,858\,048N^5$
$p_6(r)N^6$	$-56\,942\,341\,632N^6$	$-354\,292\,592\,640N^6$	$-2\,231\,748\,901\,824\,768N^6$	$-13\,119\,088\,167\,641\,088N^6$
$p_7(r)N^7$	$-4\,600\,824\,312N^7$	$-28\,580\,124\,768N^7$	$-448\,039\,233\,434\,880N^7$	$-2\,626\,537\,796\,155\,392N^7$
$p_8(r)N^8$	$-154\,867\,284N^8$	$-960\,719\,616N^8$	$-60\,661\,040\,264\,064N^8$	$-354\,741\,723\,247\,872N^8$
$p_9(r)N^9$			$-5\,267\,106\,682\,272N^9$	$-30\,735\,239\,578\,752N^9$
$p_{10}(r)N^{10}$			$-263\,609\,235\,360N^{10}$	$-1\,535\,369\,868\,288N^{10}$
$p_{11}(r)N^{11}$			$-5\,754\,914\,568N^{11}$	$-33\,465\,664\,512N^{11}$

	$r=12$	$r=13$	$r=14$
$k_1,k_2,\ldots$	6,3,2,1,1,1	6,3,2,1,1,1	7,4,3,1,1,1,1
$p_0(r)$	$-25\,424\,963\,775\,485\,706\,240$	$-147\,436\,255\,220\,890\,337\,280$	$-566\,463\,587\,862\,368\,653\,148\,160$
$p_1(r)N$	$-112\,568\,160\,566\,969\,892\,864N$	$-649\,808\,498\,848\,293\,715\,968N$	$-3\,043\,875\,242\,905\,724\,409\,348\,096N$
$p_2(r)N^2$	$-226\,122\,593\,284\,806\,672\,384N^2$	$-1\,299\,512\,129\,528\,339\,103\,744N^2$	$-7\,576\,777\,562\,322\,913\,437\,155\,328N^2$
$p_3(r)N^3$	$-272\,739\,288\,108\,751\,650\,816N^3$	$-1\,560\,658\,843\,095\,801\,004\,032N^3$	$-11\,598\,723\,404\,301\,679\,271\,608\,320N^3$
$p_4(r)N^4$	$-220\,335\,024\,310\,830\,366\,720N^4$	$-1\,255\,571\,803\,469\,692\,796\,928N^4$	$-12\,226\,130\,280\,460\,910\,006\,894\,592N^4$
$p_5(r)N^5$	$-125\,926\,002\,861\,175\,824\,384N^5$	$-714\,760\,376\,636\,548\,620\,288N^5$	$-9\,415\,725\,334\,597\,698\,602\,139\,648N^5$
$p_6(r)N^6$	$-52\,429\,589\,825\,842\,464\,768N^6$	$-296\,490\,805\,364\,682\,670\,080N^6$	$-5\,486\,072\,371\,880\,331\,913\,592\,832N^6$
$p_7(r)N^7$	$-16\,133\,360\,608\,103\,531\,520N^7$	$-90\,920\,656\,735\,938\,183\,168N^7$	$2\,470\,783\,134\,534\,602\,565\,992\,448N^7$
$p_8(r)N^8$	$-3\,682\,631\,735\,427\,354\,624N^8$	$-20\,688\,048\,472\,922\,093\,568N^8$	$-871\,335\,330\,057\,590\,064\,906\,240N^8$
$p_9(r)N^9$	$-619\,886\,922\,488\,017\,920N^9$	$-3\,472\,336\,842\,523\,894\,272N^9$	$-242\,243\,678\,099\,334\,349\,271\,040N^9$
$p_{10}(r)N^{10}$	$-75\,677\,317\,494\,258\,048N^{10}$	$-422\,813\,491\,673\,485\,824N^{10}$	$-53\,184\,931\,154\,543\,324\,258\,304N^{10}$
$p_{11}(r)N^{11}$	$-6\,492\,816\,409\,650\,048N^{11}$	$-36\,192\,084\,771\,724\,800N^{11}$	$-9\,193\,953\,185\,463\,740\,998\,656N^{11}$
$p_{12}(r)N^{12}$	$-369\,850\,295\,426\,784N^{12}$	$-2\,057\,419\,542\,670\,080N^{12}$	$-1\,241\,394\,935\,192\,535\,991\,296N^{12}$
$p_{13}(r)N^{13}$	$-12\,514\,445\,149\,200N^{13}$	$-69\,492\,377\,025\,984N^{13}$	$-129\,075\,746\,454\,058\,556\,160N^{13}$
$p_{14}(r)N^{14}$	$-189\,708\,636\,600N^{14}$	$-1051\,829\,121\,024N^{14}$	$-10\,102\,244\,514\,482\,628\,864N^{14}$
$p_{15}(r)N^{15}$			$-573\,981\,185\,124\,034\,560N^{15}$
$p_{16}(r)N^{16}$			$-22\,283\,021\,804\,865\,984N^{16}$
$p_{17}(r)N^{17}$			$-527\,221\,923\,353\,952N^{17}$
$p_{18}(r)N^{18}$			$-5\,719\,330\,613\,520N^{18}$

(3.6). Consider first $\chi(N,\tilde{\beta})$. If (3.5) is valid, for sufficiently large $\tilde{\beta}$ we have

$$B(N,\tilde{\beta})\equiv\frac{1}{2}\left[D\ln[\chi(N,\tilde{\beta})]+\frac{N+1}{(N-2)\tilde{\beta}}\right]\simeq b(N)\ . \quad (3.8)$$

However, even for rather small values of $\tilde{\beta}$, $b(N)$ may be estimated as follows. Let us approximate $D\ln[\chi(N,\tilde{\beta})]$ in (3.8) by a diagonal PA and fix $\tilde{\beta}$ at a value $\tilde{\beta}_s=\tilde{\beta}_s(N)$ which makes $B(N,\tilde{\beta})$ stationary. Then, if such a value can be found and is not too small or unreliably large, it is reasonable to take the quantity $B(N,\tilde{\beta}_s)$ as an optimal estimate of the constant $b(N)$. We have checked that in the $N=\infty$ case, in which arbitrarily long HT expansions are available, this procedure rapidly converges to the correct result.

If we form the [6/6] PA for $D\ln[\chi(N,\tilde{\beta})]$ and plot the quantity $B(N,\tilde{\beta})$ versus $\tilde{\beta}$ for various values of N, we obtain Fig. 3.

Our estimate of $b(N)$ by $B(N,\tilde{\beta}_s)$ has been plotted versus $1/(N-2)$ in Fig. 4. It should be pointed out that similarly good results are already obtained using the

TABLE IX. The structure in N of HT expansion coefficients $d_r(N)$ in (2.5) for the second field derivative of the susceptibility in $d=4$ dimensions. For the use of the table see the caption of Table I.

	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$	$r=7$
$k_1,k_2,\ldots$		1	1	2,1	2,1	3,1,1	3,1,1
$p_0(r)$	-64	-2304	-32256	$-3\,111\,424$	$-34\,077\,696$	$-4\,180\,119\,552$	$-40\,604\,565\,504$
$p_1(r)N$		$-1200N$	$-17408N$	$-4\,062\,336N$	$-44\,876\,032N$	$-8\,324\,496\,384N$	$-81\,048\,264\,704N$
$p_2(r)N^2$				$-1\,686\,528N^2$	$-18\,780\,800N^2$	$-6\,371\,176\,448N^2$	$-62\,174\,067\,712N^2$
$p_3(r)N^3$				$-216\,272N^3$	$-2\,420\,736N^3$	$-2\,317\,069\,184N^3$	$-22\,659\,064\,320N^3$
$p_4(r)N^4$						$-395\,282\,816N^4$	$-3\,871\,555\,584N^4$
$p_5(r)N^5$						$-25\,128\,192N^5$	$-246\,374\,400N^5$

	$r=8$	$r=9$	$r=10$	$r=11$
$k_1,k_2,\ldots$	4,2,1,1	4,2,1,1	5,3,1,1,1	5,3,1,1,1
$p_0(r)$	$-24\,255\,906\,840\,576$	$-219\,340\,863\,700\,992$	$-154\,597\,395\,781\,386\,240$	$-1\,332\,928\,768\,880\,148\,480$
$p_1(r)N$	$-69\,690\,457\,571\,328N$	$-630\,154\,460\,823\,552N$	$-575\,249\,505\,878\,278\,144N$	$-4\,954\,940\,633\,417\,515\,008N$
$p_2(r)N^2$	$-84\,935\,606\,534\,144N^2$	$-768\,003\,387\,981\,824N^2$	$-948\,788\,693\,853\,274\,112N^2$	$-8\,165\,049\,637\,374\,263\,296N^2$
$p_3(r)N^3$	$-57\,129\,480\,969\,216N^3$	$-516\,599\,037\,659\,136N^3$	$-913\,661\,759\,641\,092\,096N^3$	$-7\,856\,226\,314\,252\,320\,768N^3$
$p_4(r)N^4$	$-23\,103\,679\,267\,840N^4$	$-208\,930\,233\,880\,576N^4$	$-569\,523\,842\,320\,334\,848N^4$	$-4\,893\,418\,701\,997\,932\,544N^4$
$p_5(r)N^5$	$-5\,731\,696\,640\,256N^5$	$-51\,834\,928\,012\,288N^5$	$-240\,769\,879\,216\,277\,504N^5$	$-2\,067\,313\,537\,839\,226\,880N^5$
$p_6(r)N^6$	$-849\,465\,039\,872N^6$	$-7\,682\,271\,283\,200N^6$	$-70\,297\,676\,563\,985\,408N^6$	$-603\,224\,263\,886\,856\,192N^6$
$p_7(r)N^7$	$-68\,618\,721\,376N^7$	$-620\,552\,941\,184N^7$	$-14\,148\,384\,831\,046\,144N^7$	$-121\,340\,633\,334\,513\,664N^7$
$p_8(r)N^8$	$-2\,309\,856\,912N^8$	$-20\,888\,414\,208N^8$	$-1\,920\,311\,908\,716\,032N^8$	$-16\,461\,110\,770\,298\,880N^8$
$p_9(r)N^9$			$-167\,130\,489\,237\,888N^9$	$-1\,432\,044\,220\,229\,120N^9$
$p_{10}(r)N^{10}$			$-8\,382\,825\,183\,872N^{10}$	$-71\,800\,765\,981\,696N^{10}$
$p_{11}(r)N^{11}$			$-183\,366\,515\,072N^{11}$	$-1\,570\,073\,444\,352N^{11}$

	$r=12$	$r=13$	$r=14$
$k_1,k_2,\ldots$	6,3,2,1,1,1	6,3,2,1,1,1	7,4,3,1,1,1,1
$p_0(r)$	$-1\,624\,806\,042\,760\,202\,158\,080$	$-13\,539\,897\,576\,306\,140\,774\,400$	$-74\,794\,808\,420\,177\,295\,452\,405\,760$
$p_1(r)N$	$-7\,250\,504\,560\,045\,077\,823\,488N$	$-60\,336\,925\,144\,900\,561\,797\,120N$	$-406\,671\,970\,071\,656\,995\,252\,862\,976N$
$p_2(r)N^2$	$-14\,676\,830\,206\,665\,034\,825\,728N^2$	$-121\,976\,190\,862\,251\,679\,285\,248N^2$	$-1\,024\,082\,616\,739\,858\,104\,251\,842\,560N^2$
$p_3(r)N^3$	$-17\,835\,069\,642\,341\,027\,414\,016N^3$	$-148\,038\,970\,168\,862\,591\,418\,368N^3$	$-1\,585\,562\,940\,639\,389\,981\,891\,100\,672N^3$
$p_4(r)N^4$	$-14\,512\,293\,879\,109\,498\,437\,632N^4$	$-120\,317\,691\,762\,579\,335\,020\,544N^4$	$-1\,689\,863\,400\,509\,973\,307\,204\,829\,184N^4$
$p_5(r)N^5$	$-8\,351\,550\,692\,670\,013\,390\,848N^5$	$-69\,165\,378\,407\,145\,114\,927\,104N^5$	$-1\,315\,379\,322\,016\,849\,045\,127\,954\,432N^5$
$p_6(r)N^6$	$-3\,500\,180\,490\,056\,298\,971\,136N^6$	$-28\,958\,608\,067\,410\,813\,222\,912N^6$	$-774\,319\,294\,707\,161\,370\,498\,433\,024N^6$
$p_7(r)N^7$	$-1\,083\,812\,024\,563\,230\,830\,592N^7$	$-8\,958\,693\,612\,858\,095\,632\,384N^7$	$-352\,180\,187\,369\,188\,333\,891\,944\,448N^7$
$p_8(r)N^8$	$-248\,853\,772\,149\,371\,043\,840N^8$	$-2\,055\,303\,562\,471\,214\,096\,384N^8$	$-125\,367\,195\,749\,740\,355\,429\,400\,576N^8$
$p_9(r)N^9$	$-42\,119\,634\,210\,625\,754\,112N^9$	$-347\,611\,375\,618\,754\,758\,656N^9$	$-35\,164\,485\,418\,836\,553\,102\,950\,400N^9$
$p_{10}(r)N^{10}$	$5\,168\,223\,372\,754\,683\,904N^{10}$	$-42\,624\,913\,277\,048\,983\,552N^{10}$	$-7\,785\,187\,929\,458\,720\,078\,422\,016N^{10}$
$p_{11}(r)N^{11}$	$-445\,471\,112\,725\,139\,200N^{11}$	$-3\,671\,876\,230\,455\,658\,496N^{11}$	$-1\,356\,374\,454\,938\,285\,078\,888\,448N^{11}$
$p_{12}(r)N^{12}$	$-25\,481\,512\,871\,003\,648N^{12}$	$-209\,928\,435\,438\,289\,920N^{12}$	$-184\,479\,093\,797\,537\,375\,545\,344N^{12}$
$p_{13}(r)N^{13}$	$-865\,417\,069\,809\,536N^{13}$	$-7\,126\,529\,292\,423\,680N^{13}$	$-19\,310\,883\,210\,596\,332\,345\,344N^{13}$
$p_{14}(r)N^{14}$	$-13\,162\,025\,182\,720N^{14}$	$-108\,344\,872\,599\,552N^{14}$	$-1\,520\,743\,979\,256\,862\,806\,528N^{14}$
$p_{15}(r)N^{15}$			$-86\,892\,001\,158\,211\,352\,320N^{15}$
$p_{16}(r)N^{16}$			$-3\,390\,539\,345\,394\,453\,120N^{16}$
$p_{17}(r)N^{17}$			$-80\,589\,312\,225\,452\,800N^{17}$
$p_{18}(r)N^{18}$			$-877\,820\,376\,834\,048N^{18}$

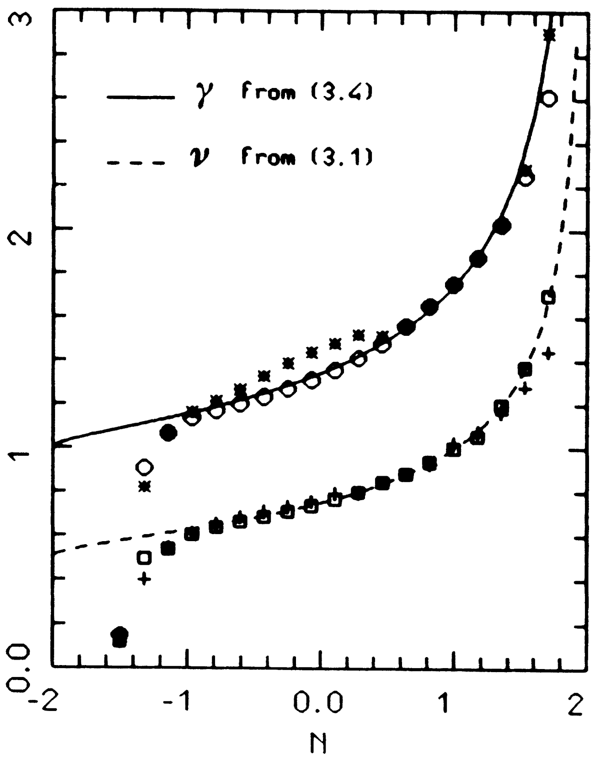

FIG. 1. The critical exponent γ of the susceptibility and ν of the correlation length computed by the method described in Sec. III A. The solid line is the Cardy-Hamber-Nienhuis prediction (3.4) for γ, the dashed line the prediction (3.1) for ν. In the case of γ we have presented the results from the [5/5] PA by stars and the ones from the [6/6] PA by open circles. In the case of ν we have represented the results from the [5/5] PA by crosses and the ones from the [6/6] PA by open squares.

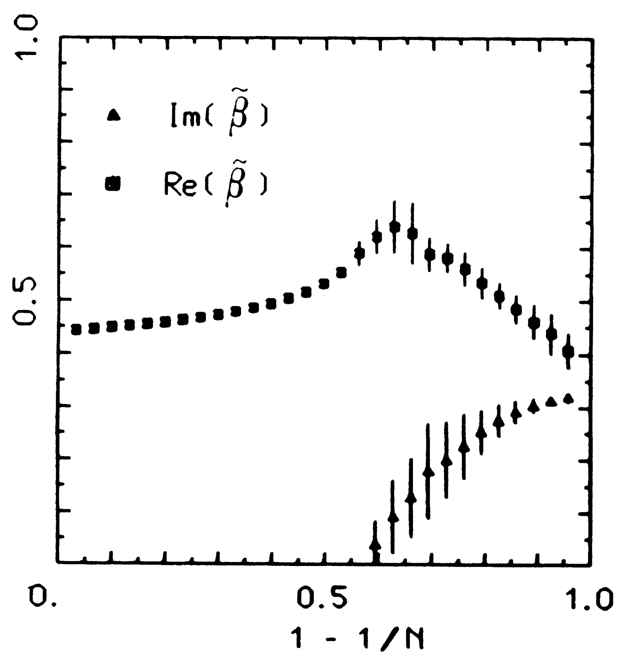

FIG. 2. The "average" real part (solid squares) and imaginary part (solid triangles) of the nearest pole (in the first quadrant of the complex $\tilde{\beta}$ plane) of the $[m/n]$ PA's to $D\ln[\chi(N,\tilde{\beta})]$ plotted vs $z=1-1/N$. (Only PA's with $m,n>3$ and $m+n>9$ are taken into account.)

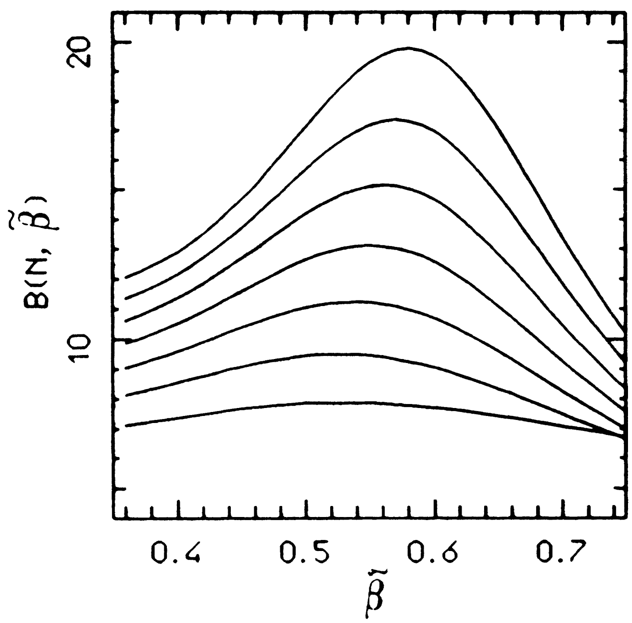

FIG. 3. The quantity $B(N,\tilde{\beta})$ defined in (3.8) plotted vs $\tilde{\beta}$ for various values of N. $D\ln[\chi(N,\tilde{\beta})]$ which enters in the definition of $B(N,\tilde{\beta})$ has been approximated by the [6/6] PA.

[5/5] PA since even relatively low-order approximants have a stationary point, which always occurs for $\tilde{\beta}\simeq 0.5$. Notice that this value of $\tilde{\beta}$ is somewhat larger than the radius of convergence of the HT series defined by the nearest singularities heretofore discussed.

An analogous procedure can be followed in the case of the correlation length, in which it will be convenient to compute the quantity

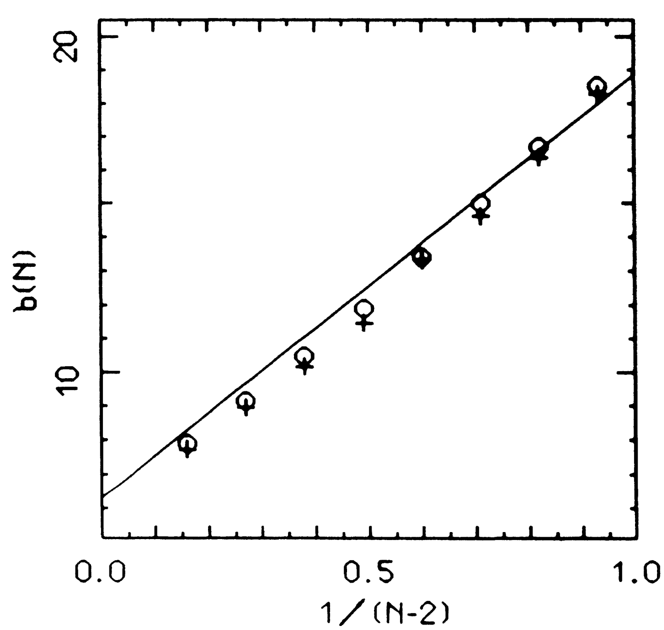

FIG. 4. The quantities $B(N,\tilde{\beta}_s)$ (circles) and $B'(N,\tilde{\beta}_s)$ (crosses), taken as approximations of $b(N)$, vs $1/(N-2)$. The solid line represents $b(N)$.

$$B'(N,\tilde{\beta})\equiv\frac{1}{2}\left[D\ln\left[\frac{\xi^2(N,\tilde{\beta})}{\tilde{\beta}}\right]+\frac{N}{(N-2)\tilde{\beta}}\right]\xrightarrow[\beta\to\infty]{}b(N)\ . \tag{3.9}$$

It should be observed that, in this case, for $N<3.5$, the [6/6] PA to $D\ln[\xi^2(N,\tilde{\beta})/\tilde{\beta}]$ has a real pole for $\tilde{\beta}\simeq 0.5$. This, however, appears to be an accidental feature of this approximation level and, in this range of N, we may consider the [5/5] PA instead. In Fig. 4 we have also reported, for various values of N, the quantity $B'(N,\tilde{\beta}_s)$. In conclusion it appears that both $B(N,\tilde{\beta}_s)$ and $B'(N,\tilde{\beta}_s)$ reproduce $b(N)$ rather accurately.

Once we have checked that our HT series are entirely consistent with the low-temperature asymptotic structures (3.5) and (3.6), we feel safer in trying to also compute the quantities $c(N)$ and $c'(N)$ in (3.5) and (3.6), for which we have no prediction from the perturbation theory. By applying the same procedure we estimate $c(N)$ as follows: we shall form the HT series for the quantity

$$C(N,\tilde{\beta})\equiv b(n)\tilde{\beta}[\chi(N,\tilde{\beta})]^{(N-2)/(N+1)}\exp\left[-4\pi\frac{N}{N+1}\tilde{\beta}\right]\xrightarrow[\beta\to\infty]{}[c(N)]^{(N-2)/(N+1)}\ . \tag{3.10}$$

As before, we shall estimate $c(N)$ by forming diagonal PA's to $C(N,\tilde{\beta})$ and fixing $\tilde{\beta}$ at a point $\tilde{\beta}_s$ which makes $C(N,\tilde{\beta})$ stationary. (In some less favorable case, in which a stationary point does not exist, we fix $\tilde{\beta}$ at the point of slowest variation.) Again we have checked that this procedure converges rapidly to the expected result in the case $N=\infty$.

A similar procedure may be followed for $\xi^2(N,\tilde{\beta})/\tilde{\beta}$ forming, however, the HT series for

$$C'(N,\tilde{\beta})\equiv b(N)\tilde{\beta}\left[\frac{\xi^2(N,\tilde{\beta})}{b(N)\tilde{\beta}}\right]^{(N-2)/N}\exp(-4\pi\tilde{\beta})\xrightarrow[\beta\to\infty]{}[c'(N)]^{2(N-2)/N}\ . \tag{3.11}$$

Also in these cases, the stationary points consistently occur for $\tilde{\beta}\simeq 0.5$. The results of these computations have been reported in Fig. 5, where we have plotted versus $1/(N-2)$ both the quantities

$$A(N,\tilde{\beta}_s)\equiv-\ln[C(N,\tilde{\beta}_s)^{(N+1)/(N-2)}/c(\infty)]$$
$$\simeq-\ln[c(N)/c(\infty)]$$

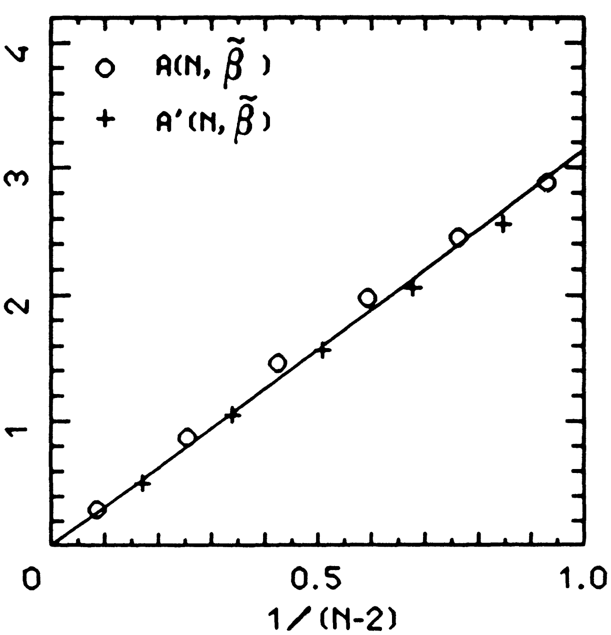

FIG. 5. The quantities $A(N,\tilde{\beta}_s)$ (circles) and $A'(N,\tilde{\beta}_s)$ (crosses), defined by (3.12) and (3.13) vs $1/(N-2)$. The solid line represents a fit to these quantities by the expressions $t/(N-2)$ with $t\simeq 3.1$.

and

$$A'(N,\tilde{\beta}_s)\equiv-\ln[C'(N,\tilde{\beta}_s)^{N/2(N-2)}/c'(\infty)]$$
$$\simeq-\ln[c'(N)/c'(\infty)]$$

where $C(N,\tilde{\beta}_s)$ and $C'(N,\tilde{\beta}_s)$ have been computed by the [7/7] PA. We recall that $c(\infty)=\pi/16$ and $c'(\infty)=\sqrt{1/32}$.

It has to be noticed that both $A(N,\beta_s)$ and $A'(N,\beta_s)$, for sufficiently large N, seem to be well represented by the same linear function of $1/(N-2)$. Our results appear then to be successfully fitted by the expression

$$c(N)/c(\infty)\simeq c'(N)/c'(\infty)\simeq\exp\left[-\frac{t}{(N-2)}\right]$$

with $t\simeq 3.1$. Of course, if we properly take into account the uncertainty of our estimates, which is larger for small N, observing that the [6/6] PA and the [7/7] PA results still differ up to 10%, the evidence for the above formula is not compelling. However, it is interesting to recall that, some time ago, arguments have been given for the validity of a similar exact formula (with $t=1+\pi/2$) in the case of $c'(N)$.[19]

Estimates of $c(N)$ and of $c'(N)$ have been previously obtained from various approaches including HT series,[14,15,18] Monte Carlo simulations,[16,17,20–25] finite volume approximation,[26–29] $1/N$ expansions,[30] HT series,[31] and Monte Carlo simulations[25,32] for Symanzik's "improved" Hamiltonians, designed to produce a faster approach to the scaling behavior.

Tables collecting the estimates for $c'(N)$, at various values of N, obtained by these computations can be found in Refs. 19, 20, and 28. We only recall that, although the various aforementioned methods produce results of the same order of magnitude, they still disagree by up to a

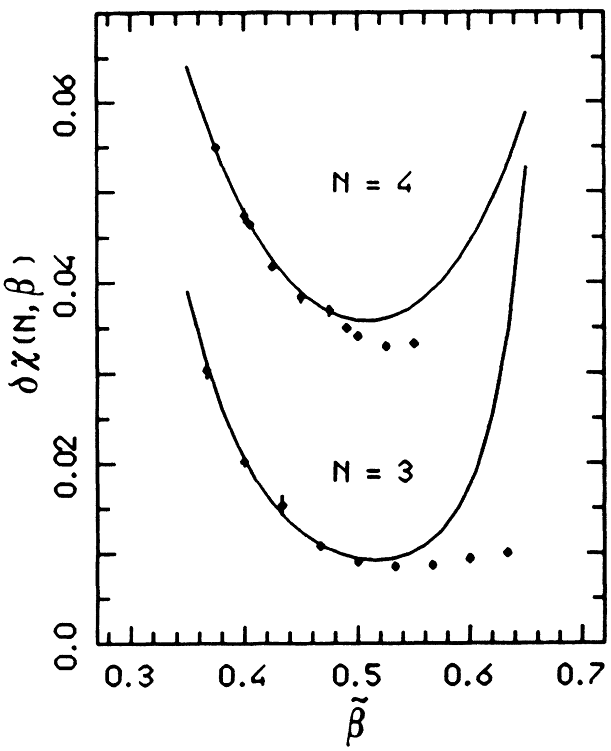

FIG. 6. The scaling defect of the susceptibility $\delta\chi(N,\tilde{\beta})$ computed as a function of $\tilde{\beta}$ by the [7/7] PA of the HT series for (3.10) compared to the same quantity as obtained from the Monte Carlo data of Refs. 24 and 25 and of Refs. 17 and 22 in the cases of $N=3$ and $N=4$, respectively.

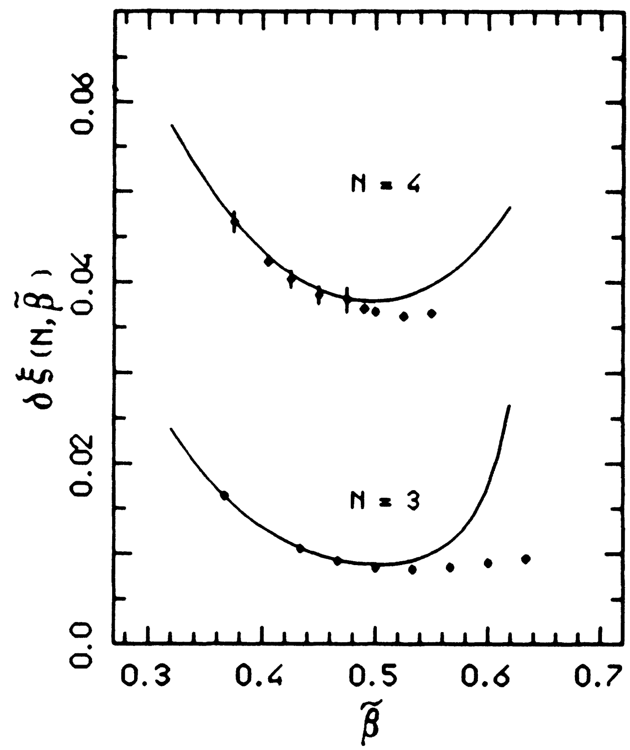

FIG. 7. The scaling defect of the correlation length $\delta\xi(N,\tilde{\beta})$ computed as a function of $\tilde{\beta}$ by the [7/7] PA of the HT series for (3.11) compared to the same quantity as obtained from the Monte Carlo data of Refs. 24 and 25 and of Refs. 17 and 22 in the cases of $N=3$ and $N=4$, respectively.

factor 3 for small N, while for $N>5$ the spread of the estimates is narrower and all of them essentially agree within the errors.

Keeping in mind that the probable errors are, in general, not smaller than 10%, let us now compare briefly some previous results to our best estimates for $c(N)$ and $c'(N)$ obtained by using the [7/7] PA with our HT series. As far as the susceptibility is concerned, we find $c(3)=0.009\,41$. This result may be compared with the value 0.011 suggested by the Monte Carlo simulation of Ref. 21 and with the value 0.0063 obtained in Ref. 18 by conformally transformed HT series. For $N=4$ we get $c(4)=0.0357$, to be compared with the estimate 0.0334 from the Monte Carlo simulation of Ref. 22 or the value of 0.032 obtained from the HT calculation of Ref. 18. For $N=5$ our value is $c(5)=0.0616$ while the only available Monte Carlo estimate[21] is 0.056. In Fig. 6 we have compared the so-called susceptibility scaling defect, namely the quantity $\delta\chi(N,\tilde{\beta})\equiv C(N,\tilde{\beta})^{(N+1)/(N-1)}$ computed as a function of $\tilde{\beta}$ by the [7/7] PA of the HT series for 3.10, with the same quantity extracted from the Monte Carlo data of Refs. 24 and 25 for $N=3$, and of Refs. 17 and 22 for $N=4$.

Similarly, studying the correlation length, we find $c'(3)=0.008\,76$ to be compared to 0.008 85 indicated by the Monte Carlo renormalization group method of Ref. 23, while the Monte Carlo simulation of Ref. 24 rather suggests some value between 0.012 and 0.014. For $N=4$ we find $c'(4)=0.0378$ to be compared to the value 0.0365 indicated by the Monte Carlo simulation of Ref. 22 For $N=5$ our value is $c'(5)=0.063$ and may be compared to the Monte Carlo estimate 0.072 in Ref. 21, or to the finite volume estimate 0.0615 in Ref. 27 or to the value 0.05 obtained in Ref. 28. Finally, in Fig. 7 we have reported the scaling defect of the correlation length, namely the quantity $\delta\xi(N,\tilde{\beta})\equiv C'(N,\tilde{\beta})^{N/2(N-2)}$, computed as a function of $\tilde{\beta}$ by the [7/7] PA of the HT series for 3.11, and we have compared it with the same quantity extracted from the Monte Carlo data of Refs. 24 and 25 for $N=3$ and of Refs. 17 and 22 for $N=4$.

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