High-temperature series for random-anisotropy magnets

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High-temperature series expansions for thermodynamic functions of random-anisotropy-axis models in the limit of infinite anisotropy are presented, for several choices of the number of spin components, m. In three spatial dimensions there is a divergence of the magnetic susceptibility χ_M for $m=2$. We find $T_c/J=1.78\pm0.01$ on the simple cubic lattice, and on the face-centered cubic lattice, we find $T_c/J=4.29\pm0.01$. There is no divergence of χ_M at finite temperature for $m\geq 3$ on either lattice. We also give results for simple hypercubic lattices.

I. INTRODUCTION

The random-anisotropy-axis model was introduced by Harris, Plischke, and Zuckermann[1] (HPZ) in 1973 to describe the behavior of amorphous magnetic alloys with randomly oriented single-site anisotropy. The Hamiltonian is

$$H=-J\sum_{\langle ij\rangle}\mathbf{S}_i\cdot\mathbf{S}_j-D\sum_i[(\hat{\mathbf{n}}_i\cdot\mathbf{S}_i)^2-1]\ , \tag{1}$$

where $\langle ij\rangle$ indicates a sum over neighbors on some lattice, and $\mathbf{S}_i$ is an m-component spin of unit length. The $\hat{\mathbf{n}}_i$ are uncorrelated random m-component unit vectors which are fixed in space. HPZ considered Eq. (1) for $m=3$. The $m=2$ case is also of experimental relevance for modeling materials which possess a combination of random anisotropy and nonrandom in-plane anisotropy.

This Hamiltonian may give rise to spin-glass behavior under certain conditions, as was made clear by later work.[2,3] The calculations of Pelcovits, Pytte, and Rudnick[4,5] showed that the isotropic ferromagnetic phase is destabilized by even a small amount of random anisotropy when the number of spatial dimensions, d, is less than or equal to 4. It was widely believed that this meant that Eq. (1) was relatively uninteresting for $d=3$, which is, of course, where we can compare calculations with experiments. It has recently become clear, however, that there is still much to be learned about the behavior of this model.

If we go to the strong anisotropy limit, $D/J\rightarrow\infty$, each spin is constrained to be parallel to its local anisotropy axis. Equation (1) then reduces to

$$H_\infty=-J\sum_{\langle ij\rangle}(\hat{\mathbf{n}}_i\cdot\hat{\mathbf{n}}_j)S_iS_j \tag{2}$$

in the absence of an external magnetic field. Each S_i is now an Ising variable, which takes on only the values ±1. Equation (2) was solved in the infinite-range case by Derrida and Vannimenus,[6] and it is convenient for both computer modeling[7,8] and high-temperature series expansions.[9-11]

In this work we will give a more detailed description of our calculation[11] of the magnetic susceptibility, χ_M, for Eq. (2). We will also present high-temperature series for the free energy and for some other functions of the spin correlations. Although we have not answered all of the interesting questions, we believe that a reasonably coherent picture of the behavior of random-anisotropy magnets is beginning to emerge.

II. EXPERIMENTAL SYSTEMS

Equation (1) was originally inspired by the experiments of Rhyne, Pickart, and Alperin[12] on amorphous $TbFe_2$. This material is ferrimagnetic and it does not show a divergence of the transverse magnetic correlation length,[13] due to the presence of a coherent uniaxial aniso-tropy[14] in addition to the random anisotropy. The composition $Tb_{75}Fe_{25}$, which has a coherent in-plane anisotropy,[14] does appear to display a divergence of the magnetic correlation length[15] at the transition temperature T_c. The HPZ model is particularly appropriate for the conceptually simpler amorphous DyCu system,[16] which contains no transition-metal ions, and negligible coherent anisotropy. Careful measurements[17,18] on DyCu show no divergence of χ_M, but exhibit typical spin-glass behavior at low temperatures.

It should be no surprise that the HPZ model has been applied to a large number of amorphous metallic alloy systems[19-23] which are similar to those we have just mentioned. A less obvious use of the model is for AuFe alloys[24-26] in the range 2–20 at. % Fe. In this system, most of the Fe is found in platelets called Guinier-Preston zones, which are distributed in a Au-rich matrix. Here, each "spin" in Eq. (1) corresponds to an Fe-rich platelet, rather than to a single ion.

For the $m=2$ case, it is convenient to transform the spin $\mathbf{S}_i$ into a phase angle θ_i. Equation (1) can then be simply generalized to

$$H(\boldsymbol{\kappa}) = -J \sum_{\langle ij \rangle} \cos[\theta_i - \theta_j - \boldsymbol{\kappa}\cdot(\mathbf{r}_i - \mathbf{r}_j)] - D \sum_i [\cos^2(\theta_i - \phi_i) - 1] , \quad (3)$$

where $\mathbf{r}_i$ is the spatial location of site i, and ϕ_i is the angular coordinate of $\hat{\mathbf{n}}_i$. Equation (3) is the Hamiltonian of a circularly polarized spin-density wave (SDW), of wave vector $\boldsymbol{\kappa}$, in the presence of random pinning potentials. If we were to relax the fixed-length-spin constraint, we could write down the Hamiltonian of a linearly polarized SDW.[27] Ioffe and Feigel'man[28] have used the linearly polarized SDW with a random pinning potential to describe the behavior of alloys like YGd,[29,30] YTb,[31] and CuMn (Ref. 32) with more than about 1% of the magnetic element.

III. HIGH-TEMPERATURE SERIES

We have used the infinite anisotropy Hamiltonian, Eq. (2), as the starting point for our series expansions. From the appearance of H_∞, the reader might think that these calculations would be very similar to doing high-temperature series for the Ising spin glass.[33–35] Our configuration averages, however, are over random angles rather than random bonds. This results in the calculations being more similar to series for isotopic classical m-component ferromagnets.[36] The key fact is that we are forced to use J/T rather than $\tanh(J/T)$ as the fundamental expansion variable, in order to do the angular averages.

In the past, it has sometimes been assumed[2,4] that the average over random angles could be replaced by an average over an effective distribution of random bonds. If that assumption were valid, then the random anisotropy model would be relatively uninteresting, since it would reduce to a simpler problem without giving any new behavior. We will see, however, that this is not correct, except in the limit $m \to \infty$.[4,11]

A. Free energy and specific heat

The high-temperature series for the free energy of H_∞ can be calculated by a fairly standard star graph expansion. This was shown by Shender,[9] who calculated the specific heat for $m=3$ on several three-dimensional lattices. We write the free energy in the form

$$F = -T\left\{ \frac{z}{2}\left[\ln\left[\cosh\left[\frac{J}{T} \right] \right] \right]_c + \sum_{n=3}^{\infty} a_n u^n \right\} , \quad (4)$$

where z is the number of neighbor spins, $u = J/(mT)$, and $[\ ]_c$ indicates the configuration average over the random-anisotropy axes. It is convenient to absorb the factor of $1/m$ into u, since this causes the leading-order contributions to appear independent of m. The sum in Eq. (4) is often referred to as the "singular part" of F. Using the star graph expansion, we have calculated the coefficients a_n up to $n=14$ for simple-cubic (sc) and hypercubic (shc) lattices, and $n=10$ for the face-centered cubic (fcc) lattice, for $m=2$–6. It is not very difficult to extend the calculations to larger values of m, but the series takes longer to "settle down" as m increases. On the simple-cubic-type lattices, $a_n = 0$ for odd n. We checked our computer programs for these series, and all the others which will be presented here, by also obtaining the results for the $m=1$ case, which is precisely the standard Ising model, for which the results can either be found in the literature or computed by simpler methods.

If the radius of convergence of the sum in Eq. (4) is determined by the critical point, $u_c = J/(mT_c)$, then for a ferromagnetic transition we expect that for large n all of the nonzero a_n should be negative. For a spin-glass transition the a_n are positive[37] for large n. Thus, a ferromagnet has a peak its the specific heat at T_c, but a spin glass has an inflection point. A nonuniform sign pattern indicates that either the transition is first order or else the large-order behavior is not controlled by the physical critical point.

In Table I we display the nonzero a_n coefficients for

TABLE I. Free-energy series for various lattices with $m=2$ and 3. The series are defined by Eq. (4).

sc lattice $m=2$	$-12u^4 - 40u^6 - 166u^8 - 509.\overline{3}u^{10} + 2\,814.2\overline{51\,8}u^{12} + 74\,549.025\,185u^{14} + \cdots$
sc lattice $m=3$	$-54u^4 - 7.2u^6 + 1\,073.180\,\overline{571\,428}u^8 + 14\,500.758\,\overline{857\,142}u^{10} + 146\,570.251\,169\,7u^{12} - 157\,412.377\,73u^{14} + \cdots$
fcc lattice $m=2$	$-16u^3 - 66u^4 - 288u^5 - 1\,420u^6 - 7\,456u^7 - 40\,781u^8 - 229\,370.\overline{074}u^9 - 1\,311\,196u^{10} + \cdots$
fcc lattice $m=3$	$-24u^3 - 99u^4 - 374.4u^5 - 1\,471.44u^6 - 4\,970.0\overline{57\,142\,8}u^7 - 6\,512.3\overline{74\,285\,71}u^8 + 109\,839.971\,265\,3u^9 + 1\,629\,812.278\,139u^{10} + \cdots$
4d shc lattice $m=2$	$-48u^4 - 416u^6 - 6712u^8 - 127\,477.\overline{3}u^{10} - 2\,789\,297.6\overline{59\,2}u^{12} - 67\,555\,910.921\,48u^{14} + \cdots$
4d shc lattice $m=3$	$-108u^4 - 590.4u^6 - 7\,660.038\,\overline{857\,142}u^8 - 97\,686.555\,\overline{428\,571}u^{10} - 1\,240\,487.619\,626u^{12} - 16\,843\,460.691\,83u^{14} + \cdots$

the sc, fcc, and 4d shc lattices for $m=2$ and 3. The specific heat, c_H, is obtained, as usual, by differentiating F twice with respect to T. In agreement with Shender,[9] we do not find any singular behavior in c_H for the 3d lattices for any $m \geq 2$. Our series for the 4d shc lattice with $m=2$ indicates the presence of a cusp in c_H at T_c. We estimate that the critical exponent α is -0.2 ± 0.2 in this case. The c_H series for $m \geq 3$ in 4d are too irregular to be analyzed. For the 5d shc lattice the series do not yield any useful information since the specific heat has a jump discontinuity at T_c for $m<5$.

B. Magnetic susceptibility

At high temperatures, the magnetic susceptibility is given by

$$T\chi_M = \frac{1}{m}\left[1+\frac{1}{N}\sum_{i\neq j}^{N}\langle \mathbf{S}_i\cdot\mathbf{S}_j\rangle\right], \tag{5}$$

where the $\langle\ \rangle$ denote a thermal expectation value, and N is the total number of sites, which we let go to infinity. The factor of $1/m$ in Eq. (5) results from the fact that the external field is uniform, while each spin points along its own anisotropy axis. This reduces the effectiveness of the external field in aligning the spins, and means that the uniform magnetic field is not really the field conjugate to the order parameter, even when the system is in a ferromagnetic state. Thus, the susceptibility exponent γ_M may not be equal to the exponent γ which appears in the various critical exponent scaling relations.

The χ_M series is defined by

$$T\chi_M = \frac{1}{m}\left[1+\sum_{n=1}^{\infty} c_n u^n\right] \tag{6}$$

and can be calculated by various methods. For the fcc lattice, we obtained coefficients out to $n=9$ by a fairly standard weak graph expansion, using the lattice embedding constants of Baker *et al.*[38] For the sc and shc lattices, we calculated the coefficients to $n=15$ using the no-free-end method,[39] which formulates the closed graph theorem[40] as a cumulant expansion. Since this program was rather slow, we used a method due to Horwitz and Callen[41] to obtain the cumulants for the graphs with 15 bonds. The Horwitz-Callen algorithm calculates the nth-order term of the cumulant for an n-bond graph in a very efficient fashion. (All of the lower-order terms of the cumulant are zero). The series coefficients are shown in Table II.

The series were analyzed using standard ratio methods.[42] Since it is clear by inspection that the calculated terms of these series are rather smoothly varying, the ratio techniques work rather well. We have no *a priori* reason to expect that we must find a singularity of simple power-law type, so the use of Padé methods is rather problematical.

By looking at the ratios of successive coefficients, one readily concludes that for the sc and fcc lattices there is no divergence of χ_M on the positive real u axis when $m \geq 3$. On the square lattice we find similar results for all $m \geq 2$, in agreement with the conclusions of Bray and Moore[43] for $d=2$.

The situation for the 3d lattices when $m=2$ is much more interesting. There does indeed seem to be a divergence of χ_M at some positive value of u, but the fit to a simple power-law form, $(T-T_c)^{-\gamma_M}$, is not satisfactory. We already know that the magnetization M is zero at $T=0$ for this case (see *Note added in proof*),[8] and we did not find any indication of a singularity in the specific heat. Consequently, it is not surprising that a power-law form does not work. It is natural to try to fit these series to a fractional exponential, $\exp[A/(T-T_c)^{\zeta}]$. This form is appropriate at the lower critical dimension, and it works quite well here. For the sc lattice we find $T_c/J=1.78\pm0.01$ and $\zeta=0.69\pm0.05$. As is often the case when performing a ratio analysis for a two-colorable lattice, we found it helpful to make a transformation to allow for the small odd-even alternation of the c_n. On the fcc lattice we find $T_c/J=4.29\pm0.01$ and $\zeta=0.45\pm0.03$. At this point we cannot be sure that the difference in our values for ζ on these two lattices is a real effect. Our quoted error estimates do not allow for systematic errors, i.e., they assume that the phase transition is continuous and that we are using the correct functional form.

If the difference in the value of ζ for $D/J=\infty$ on these two lattices is real, the implications are quite profound. It would imply that for a given three-dimensional lattice, the value of ζ is a function of D/J. We could then drive ζ down to some minimum value (which might or might not be zero) by reducing D/J. This minimum value of ζ should be universal, the same for all 3d lattices. We could also increase ζ by diluting the lattice, i.e., replacing some of the spins by noninteracting sites. Presumably, there is also a universal maximum value of ζ. It would be interesting to know if this maximum is reached at the percolation threshold, or whether there is another type of behavior which takes over even before the infinite cluster of spins is totally destroyed. It should be possible to investigate this question by doing a calculation similar to the work of Klein *et al.*[44] for the diluted Ising spin glass.

Before we take the speculations in the last paragraph too seriously, we must also consider the possibility that the system has another phase transition at some higher temperature. The values of T_c which we have found for these $m=2$ transitions are, however, somewhat higher than even the mean-field estimates of the spin-glass freezing temperatures for these lattices. Also, the existence of long-range spin correlations[8] in the ground state on the sc lattice when $m=2$ argues for the reality of the divergence of χ_M.

We believe that what is going on for $m=2$ in 3d is similar to the situation in the closely related 3d random-field Ising model, where the critical behavior is controlled by a zero-temperature fixed point.[45] In the $m=2$ random anisotropy case, however, we seem to be dealing with a zero-temperature fixed line. The exponent η_1 which describes the power-law decay of spin correlations in the ground state apparently changes as D/J is varied. Similar long-range spin correlations are found in 3d $m=2$ models with threefold or fourfold random anisotropy.[46] These results are somewhat similar to the ideas of Aharo-

TABLE II. Magnetic susceptibility series for $3d$ lattices, with $m=2$ and 3, for the $4d$ shc lattice with $m=2$, 3, and 4, and for the $5d$ shc lattice with $m=2$, 3, and 4. The series are defined by Eq. (6).

	$3d$ lattices
sc lattice $m=2$	$1+6u+30u^2+144u^3+666u^4+3\,020u^5+13\,436u^6$ $+58\,918.\overline{6}u^7+255\,460.\overline{6}u^8+1\,095\,867.2u^9+4\,662\,697.\overline{3}u^{10}$ $+19\,674\,854.18\overline{6}u^{11}+82\,500\,121.\overline{3}u^{12}+343\,685\,731.923\,808u^{13}$ $+1\,424\,431\,147.907\,72u^{14}+5\,872\,789\,753.311\,03u^{15}+\cdots$
sc lattice $m=3$	$1+6u+30u^2+139.2u^3+618u^4+2\,622.1\overline{71\,428\,5}u^5+10\,751.79\overline{4\,285\,71}u^6$ $+42\,217.536u^7+160\,460.60\overline{5\,714\,28}u^8+583\,308.554\,805\,194u^9$ $+2\,027\,333.898\,745\,8\,2u^{10}+6\,637\,797.310\,305\,46u^{11}$ $+20\,264\,446.693\,317\,0u^{12}+56\,161\,109.833\,998\,2u^{13}$ $+130\,827\,918.366\,620u^{14}+206\,252\,296.859\,672u^{15}+\cdots$
fcc lattice $m=2$	$1+12u+132u^2+1\,392u^3+14\,292u^4+143\,992u^5+1\,430\,256u^6$ $+14\,048\,493.\overline{3}u^7+136\,736\,137.\overline{3}u^8+1\,320\,751\,369.0\overline{6}u^9+\cdots$
fcc lattice $m=3$	$1+12u+132u^2+1\,382.4u^3+13\,965.6u^4+137\,048.50\overline{2\,857\,14}u^5$ $+1\,312\,032u^6+12\,286\,661.869\,7143u^7$ $+112\,746\,484.355\,265u^8+1\,014\,963\,605.083\,62u^9+\cdots$
	$4d$ shc lattice
$m=2$	$1+8u+56u^2+384u^3+2\,584u^4+17\,274.\overline{6}u^5+114\,613.\overline{3}u^6$ $+757\,768.\overline{8}u^7+4\,989\,673.\overline{7}u^8+32\,783\,035.3\overline{7}u^9$ $+214\,851\,732.6\overline{2}u^{10}+1\,405\,984\,012.397\,04u^{11}$ $+9\,185\,249\,515.30\overline{6}u^{12}+59\,942\,779\,289.8\overline{2}u^{13}$ $+390\,714\,537\,058.417u^{14}+2\,544\,687\,649\,225.37u^{15}+\cdots$
$m=3$	$1+8u+56u^2+377.6u^3+2\,494.4u^4+16\,245.0\overline{28\,571\,4}u^5$ $+104\,768u^6+670\,029.494\,\overline{857\,142}u^7$ $+4\,260\,084.745\,142\,85u^8+26\,933\,817.114\,597\,4u^9$ $+169\,584\,501.785\,457u^{10}+1\,063\,439\,719.414\,12u^{11}$ $+6\,648\,582\,831.526\,47u^{12}+41\,441\,798\,194.844\,1u^{13}$ $+257\,712\,485\,935.393u^{14}+1\,598\,946\,446\,345.96u^{15}+\cdots$
$m=4$	$1+8u+56u^2+370.\overline{6}u^3+2\,397.\overline{3}u^4+15\,149.\overline{3}u^5$ $+94\,363.\overline{5}u^6+578\,623.8\overline{2}u^7+3\,513\,150.4u^8$ $+21\,087\,548.2\overline{6}u^9+125\,474\,485.9\overline{85\,1}u^{10}+739\,539\,461.778\,625u^{11}$ $+4\,327\,363\,316.120\,44u^{12}+25\,131\,276\,843.494\,0u^{13}$ $+144\,997\,577\,118.357u^{14}+831\,353\,961\,886.386u^{15}+\cdots$
	$5d$ shc lattice
$m=2$	$1+10u+90u^2+800u^3+7\,030u^4+61\,593.\overline{3}u^5$ $+537\,580u^6+4\,685\,984.\overline{4}u^7+40\,771\,643.\overline{3}u^8$ $+354\,501\,116.\overline{4}u^9+3\,079\,061\,760.\overline{4}u^{10}+26\,732\,346\,726.459\,3u^{11}$ $+231\,934\,439\,686.193u^{12}+2\,011\,744\,682\,033.48u^{13}$ $+17\,441\,474\,389\,861.1u^{14}+151\,185\,117\,213\,440.u^{15}+\cdots$
$m=3$	$1+10u+90u^2+792u^3+6\,886u^4+59\,490.\overline{285\,714}u^5$ $+511\,680.3\overline{42\,857\,1}u^6+4\,388\,266.83\overline{4\,285\,71}u^7$ $+37\,551\,133.817\,142\,8u^8+320\,813\,601.633\,246u^9$ $+2\,737\,190\,089.395\,83u^{10}+23\,330\,234\,446.264\,4u^{11}$ $+198\,681\,806\,564.362u^{12}+1\,690\,842\,766\,050.73u^{13}$ $+14\,380\,972\,772\,381.2u^{14}+122\,254\,717\,152\,247.u^{15}+\cdots$

TABLE II. (*Continued*).

	5d shc lattice
$m=4$	$1+10u+90u^2+783.\bar{3}u^3+6\,730u^4+57\,236.\bar{6}u^5$ $+484\,036.\bar{6}u^6+4\,073\,090.\bar{8}u^7+34\,172\,559.\bar{3}u^8$ $+285\,886\,684.\overline{962}u^9+2\,387\,084\,126.\overline{148}u^{10}+19\,894\,744\,810.976\,4u^{11}$ $+165\,593\,522\,586.389u^{12}+1\,376\,582\,958\,700.72u^{13}$ $+11\,432\,803\,682\,793.1u^{14}+94\,866\,649\,711\,913.0u^{15}+\cdots$

ny and Pytte[47] and Villain and Fernandez,[48] although those authors did not predict that the behavior would only occur for $m=2$ in $3d$.

Since we found a power-law singularity in the specific heat on the $4d$ shc lattice for $m=2$, it is natural to expect a simple power-law form to work for χ_M also. This turns out to be true, and we find that $T_c/J=3.215\pm0.005$, with $\gamma_M=1.192\pm0.008$ gives a very good fit to the calculated c_n. If we use the power-law form for $m=3$ on this lattice, we find $T_c/J=2.005\pm0.005$, with $\gamma_M=1.46\pm0.04$. The fit is not that good, and since we did not find a specific-heat singularity in this case, we have reasonable grounds to doubt that the power-law form is correct here. An attempt to use the fractional exponential form does not work this time, however, as the value of ζ turns out to be indistinguishable from zero. The evidence for some kind of a divergence of χ_M is very strong, but we are not sure what the precise form is. We will return to this point in the next section. For $m\geq 4$ we see no indication of any divergence in χ_M on this lattice.

It is interesting to compare our results for $d=4$ with the weak-anisotropy calculation of Pelcovits, Pytte, and Rudnick.[4,5] For strong anisotropy M is not really the order parameter, even in a ground state with a large magnetic moment. The true order parameter is parallel to the ground state at each site. Therefore, the order-parameter exponent, γ, may be greater than γ_M if the overlap between M and the true order parameter decays to zero in the infinite-volume limit. The isotropic ferromagnetic phase may be thought of as a "floating" phase, since it has well-defined massless spin waves at long wavelengths. For strong anisotropy the ferromagnetic phase is "pinned," and the spin-wave mass is proportional to D. So these phases are distinct, and at low temperatures it is not possible to go from one to the other by varying D/J without passing through a phase boundary. Since there is no symmetry difference between these phases, the boundary may have a critical endpoint. The "proof" of Pelcovits[5] that M must be zero for $m=2$ in $4d$ just *assumed,* based on replica symmetry, that a magnetization could not exist without also having massless spin waves.

For shc lattices with $d\geq 5$ and $m=2$, 3, and 4, simple power-law fits to the χ_M series give $\gamma_M=1.06$, 1.10 and 1.17, respectively. We must consider two possibilities. Either corrections to scaling are surprisingly large in $5d$, or else γ_M does not reach its mean-field theory value of 1 until $d=6$. Pytte[49] has claimed that, although the upper critical dimension is 6, $\gamma_M=1$ for $d=5$. Fischer and Zippelius,[50] however, have shown that Pytte's approximations are not reliable for the case of strong anisotropy. More analytical work on this point would certainly be a useful contribution to our understanding of the problem.

A simple power-law fit to χ_M for the Ising model gives $\gamma_M=1.03$ for $d=5$, although the correct answer is exactly 1. We believe, therefore, that allowing for corrections to simple power-law behavior would give a slightly lower value of γ_M for $m=2$, perhaps 1.04. Our values for $m=3$ and 4, however, are probably accurate to better than ±0.02. For $m\geq 5$, spin-glass effects become important on this lattice. We estimate $\gamma_M\approx 1.3$ for $m=5$, but the simple power-law fits are not convincing. The values of mT_c/J are 8.635, 8.445, 8.200, and 7.89 for $m=2$–5, respectively, with uncertainties of ±0.005, except for $m=5$, where we have less understanding of the nature of the transition. Our results for T_c are summarized in Table III.

C. Q susceptibility

In order to obtain more information, we want to calculate other functions. For instance, one could calculate

TABLE III. Ferromagnetic critical temperatures of random anisotropy axis models in the infinite anisotropy limit, Eq. (2), for d-dimensional simple hypercubic lattices. The numbers displayed are mT_c/J, where m is the number of spin components. The values given for $m=1$ are for the standard Ising model.

d / m	3	4	5	6
1	4.5120[a]	6.6817[b]	8.774	10.830
2	3.56	6.430	8.635	10.735
3		6.015[c]	8.445	10.615
4			8.200	10.480
5			7.89[c]	10.31

[a]Reference 54.
[b]Reference 55.
[c]Ferromagnet–spin-glass multicritical point.

higher derivatives of the free energy with respect to the uniform external field. Since we were particularly interested in exploring the possibility of spin-glass order, we chose to calculate the Q susceptibility,[33–35] χ_Q. At high temperatures, the longitudinal part, χ_{Q_L}, is defined as

$$T\chi_{Q_L}=\frac{1}{m^2}\left[1+\frac{1}{N}\sum_{i\neq j}^{N}\langle \mathbf{S}_i\cdot\mathbf{S}_j\rangle^2\right] . \tag{7}$$

χ_{Q_L} is often referred to as the Edwards-Anderson[51] susceptibility. If we include the terms which are off diagonal in the spin components, we have

$$T\chi_Q=\frac{1}{m^2}\left[m+\frac{1}{N}\sum_{i\neq j}^{N}\sum_{\alpha=1}^{m}\sum_{\beta=1}^{m}\langle S_i^\alpha S_j^\beta\rangle^2\right] . \tag{8}$$

Although χ_Q is usually associated with the spin-glass phase, it also diverges at a ferromagnetic transition. We will discuss the behavior of χ_Q for the Ising ferromagnet in the Appendix.

For $D/J=\infty$ there is not much difference in the behavior of the transverse and longitudinal parts of χ_Q. In particular, they both diverge with the same exponent, γ_Q. For that reason, and in order to save space, we will only give results for χ_Q. The series coefficients are defined by

$$T\chi_Q=\frac{1}{m}\left[1+\sum_{n=1}^{\infty}f_n u^n\right] \tag{9}$$

and they are given in Table IV. For the simple-cubic-type lattices, we were able to compute the f_n through $n=14$. Because all of the odd-n terms are zero on these lattices, we do not have as many terms to work with in doing the asymptotic analysis as we did for χ_M. For the fcc lattice we again computed terms through $n=9$.

For $m=2$ on the sc lattice, our estimate of T_c from the χ_Q series is 9% higher than the estimate from the χ_M series. On the fcc lattice, our estimate of T_c is 7% higher in this case. Given the short length and irregular behavior of these series, we cannot say that these differences are meaningful. For $m\geq 3$ the series for the 3d lattices are too irregular for us even to decide whether or not χ_Q diverges.

The situation improves for $d=4$. The estimates of T_c for $m=2$ and 3 agree quite well with the values from the χ_M series, although they are not as precise. We used the method of n shifts[42] to obtain values of γ_Q, assuming that T_c was exactly the same. We obtain $\gamma_Q=0.28\pm0.04$ for $m=2$ and $\gamma_Q=1.15\pm0.10$ for $m=3$. For $m\geq 4$, χ_Q appears to diverge, but we do not believe that we can make meaningful quantitative estimates of the behavior in these cases.

If M behaves like the order parameter for the phase transition, then the standard scaling analysis for ferromagnets gives the relation

$$\gamma_Q=2\gamma_M-2+\alpha . \tag{10}$$

The derivation of Eq. (10) uses the hyperscaling relation, i.e., the terms "$-2+\alpha$" come from a "$-d\nu$." Even though M is not, strictly speaking, the order parameter when $m\neq 1$, Eq. (10) will still be satisfied as long as $\gamma=\gamma_M$. We will show in the Appendix that our χ_Q series for the Ising ferromagnet gives values of γ_Q which satisfy Eq. (10).

If we insert our 4d $m=2$ values of γ_M and γ_Q into Eq. (10), we obtain $\alpha=-0.10\pm0.05$. The agreement with the value of α which we found directly from the specific-heat series, $\alpha=-0.2\pm0.2$, is very good, although we must admit that the error estimates are large. We conclude that this case is displaying a typical ferromagnetic critical behavior, contrary to the prediction of Pelcovits *et al.*[4,5]

On the other hand, if we do the same thing for $m=3$ we find $\alpha=0.23\pm0.12$. A positive value of α is rather implausible for this model. It is also inconsistent with our analysis of the specific-heat series, which shows no evidence of singular behavior at T_c. The most likely explanation of this apparent breakdown of Eq. (10) is that the system is rather close to the ferromagnet–spin-glass multicritical point. The value $\gamma_Q=1.15$ is somewhat larger than one might anticipate at a simple ferromagnetic critical point, but it is quite reasonable for spin-glass freezing. This could also explain why the fit to a simple power law does not work as well for χ_M as we had expected. This explanation implies that we will find a smaller value of γ_Q if we calculate χ_Q on the 4d face-centered hypercubic lattice, for which $z=24$.

An alternative explanation of the results for $m=3$ is that hyperscaling is not obeyed for $m\geq 3$. Random-anisotropy models are closely related to random-field models, for which hyperscaling is known to be violated.[45,52] One could certainly argue that the apparent success of Eq. (10) for $m=2$ is more surprising than its failure for larger values of m.

For the 5d shc lattice, the singularity in the χ_Q series appears to be at a temperature which is lower than T_c when $m=2$, 3, and 4. We expect that there is a jump discontinuity in χ_Q at T_c in these cases. For $m=5$, both χ_M and χ_Q seem to diverge at the same T_c, and $\gamma_Q\approx 1.1$.

The high-temperature series analysis for H_∞ on Cayley trees[10] predicts that the spin-glass transition occurs at a higher temperature than the ferromagnetic transition when the number of neighbors on the lattice is less than the number of spin components, i.e., $z<m$. One of the questions that we wish to consider is how well this result works for lattices which can be embedded in a finite-dimensional space. To do this properly, we should do calculations for different lattices with the same value of d. For $d=3$, we found that with a given value of m both the sc and fcc lattices have similar behavior.

The Cayley tree criterion clearly overestimates the stability of the ferromagnetic phase relative to the spin-glass phase. Our shc lattices have $z=2d$, but for $d=4$ we find only a spin-glass transition for $m\geq 4$, and for $d=5$ χ_Q appears to diverge at a higher temperature than χ_M when

$m=6$. It is worthy of note that we find $\gamma_Q \geq 1$ whenever χ_Q diverges at a higher temperature than χ_M does. This is as it should be for a spin-glass transition. The values which we find for γ_Q depend on m, and they are not the same as the γ_Q of the Ising spin glass, which corresponds to $m=\infty$. It is possible that this dependence of γ_Q on m is real, since the calculation of Chen and Lubensky[2] assumes that the configuration average over random axes can be replaced by an average over a distribution of effective bonds. This is known to be incorrect.[3,50,53]

We must emphasize that our χ_Q series are very short compared to what is necessary to obtain reliable results for the Ising spin glass.[35] We cannot be sure that these series have settled down to their asymptotic forms by order $(J/T)^{14}$. The best argument that we have which indicates that our results are meaningful is the excellent answers which we obtained with series of the same length for the standard Ising ferromagnet. For the Ising spin glass, on the other hand, it is possible to calculate twice as many terms on shc lattices as we can obtain for H_∞, and the results from a series of the length we have here are sometimes misleading in that case. We believe that our quantitative results for $m=2$ are accurate, but for higher m our analysis of the χ_Q series should be considered primarily qualitative. We note, in particular, that for $d=4$ the apparent value of the temperature at which χ_Q diverges on a given lattice seems to be approximately proportional to $1/m$. The Cayley tree analysis predicts

TABLE IV. Q susceptibility series for $3d$ lattices with $m=2$ and 3, for the $4d$ shc lattice with $m=2,3$, and 4, and for the $5d$ shc lattice with $m=2,3$, and 4. The series are defined by Eq. (9).

	$3d$ lattice
sc lattice $m=2$	$1+12u^2+240u^4+4\,845.\overline{3}u^6+90\,584u^8+1\,647\,000.74\overline{6}u^{10}$ $+29\,100\,522.500\,740\,8u^{12}+503\,550\,321.132\,919u^{14}+\cdots$
sc lattice $m=3$	$1+18u^2+421.2u^4+9\,153.2\overline{57\,142\,8}u^6+173\,606.790\,\overline{857\,142}u^8$ $+3\,031\,841.091\,740\,26u^{10}+46\,369\,286.424\,027\,3u^{12}$ $+644\,180\,609.401\,716u^{14}+\cdots$
fcc lattice $m=2$	$1+24u^2+192u^3+2\,064u^4+19\,968u^5+194\,154.\overline{6}u^6$ $+1\,867\,392u^7+17\,869\,040u^8+170\,090\,360.\overline{8}u^9+\cdots$
fcc lattice $m=3$	$1+36u^2+288u^3+3\,434.4u^4+34\,329.6u^5+349\,129.23\overline{4\,285\,71}u^6$ $+3\,444\,526.08u^{\&}+33\,528\,882.610\,285\,7u^8+320\,894\,637.014\,204u^9+\cdots$
	$4d$ shc lattice
$m=2$	$1+16u^2+480u^4+16\,604.\overline{4}u^6+605\,102.\overline{2}u^8+22\,449\,330.77\overline{3}u^{10}$ $+844\,264\,873.\overline{481}u^{12}+32\,104\,725\,202.703\,7u^{14}+\cdots$
$m=3$	$1+24u^2+849.6u^4+31\,836.3\overline{42\,857\,1}u^6+1\,202\,982.336u^8$ $+45\,241\,885.499\,844\,2u^{10}+1\,686\,798\,100.550\,33u^{12}+62\,562\,015\,672.231\,0u^{14}+\cdots$
$m=4$	$1+32u^2+1\,301.\overline{3}u^4++53\,169.\overline{7}u^6+2\,109\,073.0\overline{6}u^8$ $+81\,782\,088.24\overline{8}u^{10}+3\,061\,309\,598.205\,29u^{12}+112\,179\,174\,971.382u^{14}+\cdots$
	$5d$ shc lattice
$m=2$	$1+20u^2+800u^4+39\,195.\overline{5}u^6+2\,138\,742.\overline{2}u^8+124\,151\,979.0\overline{2}u^{10}$ $+7\,488\,338\,983.980\,25u^{12}+464\,192\,194\,031.359u^{14}+\cdots$
$m=3$	$1+30u^2+1\,422u^4+75\,615.\overline{428\,571}u^6+4\,283\,727.18\overline{8\,571\,42}u^8$ $+252\,425\,720.234\,805u^{10}+15\,240\,357\,916.624\,9u^{12}+937\,039\,669\,235.428u^{14}+\cdots$
$m=4$	$1+40u^2+2\,186.\overline{6}u^4+126\,968.\overline{8}u^6+7\,576\,732.\overline{4}u^8+460\,892\,690.0\overline{14\,8}u^{10}$ $+28\,260\,869\,626.536\,8u^{12}+1\,745\,694\,346\,469.55u^{14}+\cdots$

TABLE V. Q susceptibility series for the Ising model, with $w=\tanh(J/T)$.

Square lattice	$1+4w^2+36w^4+236w^6+1556w^8+9956w^{10}+62\,796w^{12}+391\,724w^{14}+\cdots$
Simple-cubic lattice	$1+6w^2+102w^4+1998w^6+38\,118w^8+740\,454w^{10}+14\,587\,614w^{12}+289\,922\,718w^{14}+\cdots$
$4d$ shc lattice	$1+8w^2+200w^4+6584w^6+238\,376w^8+8\,952\,904w^{10}+345\,828\,280w^{12}+13\,652\,328\,568w^{14}+\cdots$
$5d$ shc lattice	$1+10w^2+330w^4+15\,290w^6+826\,250w^8+48\,326\,730w^{10}+2\,961\,595\,610w^{12}+187\,519\,852\,250w^{14}+\cdots$

that the spin-glass freezing temperature should go as $1/\sqrt{m}$, and we believe that this would be observed if our series were longer.

IV. SUMMARY

In this work we have presented high-temperature series calculations for the random-anisotropy-axis model in the infinite D/J limit, Eq. (2), for several values m. We have given results for the free energy (from which the specific heat can easily be derived), the magnetic susceptibility, and the Q susceptibility. We believe that our most significant results are for $3d$ lattices with $m=2$. We find that χ_M diverges at $T_c>0$. The asymptotic form near T_c appears to be $\exp[A/(T-T_c)^{\zeta}]$. ζ is between 0 and 1 and seems to depend on the lattice type, and therefore, implicitly, on D/J. These results can be tested both by real experiments and by Monte Carlo simulations. For $m\geq 3$ we find no divergence of χ_M on $3d$ lattices, in agreement with prior work. We conclude that the lower critical dimension for this problem is 3.

For the $4d$ shc lattice with $m=2$, we find a surprisingly normal ferromagnetic phase transition, with power-law singularities in c_H, χ_M, and χ_Q. For $m=3$, χ_M and χ_Q both diverge, and the behavior seems to be that of a ferromagnet–spin-glass multicritical point. For $m\geq 4$, only χ_Q diverges on this lattice.

For the $5d$ shc lattice we find ferromagnetic transitions for $m=2$, 3, and 4. For $m\geq 6$ we have a spin-glass transition, and $m=5$ is very close to the multicritical point. The exponents we find for these transitions do not appear to be mean-field-like, so the upper critical dimension is probably 6.

Note added in proof: See Ref. 46 for more recent and detailed results. There may indeed be a finite M at $T=0$ for this case.

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APPENDIX: Q SUSCEPTIBILITY FOR THE ISING MODEL

The Q susceptibility for the Ising model can be calculated by standard methods. In Table V we give the χ_Q series coefficients for the square, simple cubic, and $4d$ and $5d$ simple hypercubic lattices, using the expansion variable $w=\tanh(J/T)$. A ratio analysis of the coefficients, using the method of n shifts and values of T_c from the literature,[54,55] gives $\gamma_Q=1.48$, 0.58, and -0.01 in $d=2$, 3, and 4, respectively. These are slightly lower than the correct values[56] of 1.50, 0.59, and 0.00, but, considering the length of our series, the agreement can only be described as excellent. If one were to calculate the χ_Q series out to as high an order as possible for the bcc and fcc lattices, it might produce a more precise set of values for the critical exponents than the currently accepted one. For $d=5$ we expect a jump discontinuity at T_c, just as for the specific heat.

For completeness, we give the Ising model χ_M series for the $5d$ and $6d$ shc lattices to 15 terms in Table VI. These extend the 11-term series of Fisher and Gaunt.[57] Analysis of these series gives $T_c/J=8.774\pm0.003$ for $d=5$ and $T_c/J=10.830\pm0.002$ for $d=6$. In the notation of Fisher and Gaunt, these give $\omega(5)=8.812$ and $\omega(6)=10.861$, where $\omega(d)=1/w_c(d)$.

TABLE VI. Magnetic susceptibility series for the Ising model, with $w=\tanh(J/T)$.

$5d$ shc lattice	$1+10w+90w^2+810w^3+7210w^4+64\,170w^5+568\,970w^6$ $+5\,044\,810w^7+44\,649\,930w^8+395\,180\,650w^9+3\,494\,051\,130w^{10}$ $+30\,893\,156\,970w^{11}+272\,971\,707\,930w^{12}+2\,411\,975\,074\,570w^{13}$ $+21\,302\,972\,395\,370w^{14}+188\,151\,452\,434\,090w^{15}+\cdots$
$6d$ shc lattice	$1+12w+132w^2+1452^3+15\,852w^4+173\,052w^5+1\,884\,972w^6$ $+20\,532\,252w^7+223\,437\,852w^8+2\,431\,526\,492w^9+26\,447\,593\,812w^{10}$ $+287\,669\,976\,492w^{11}+3\,128\,064\,123\,732w^{12}+34\,013\,987\,172\,972w^{13}$ $+369\,792\,173\,040\,492w^{14}+4\,020\,299\,656\,610\,636w^{15}+\cdots$

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