Asymptotic limits for the penetration depth of strong-coupling superconductors

F. Marsiglio
Physics Department, University of California at San Diego, La Jolla, California 92093

J. P. Carbotte
Physics Department, McMaster University, Hamilton, Ontario, Canada L8S 4M1
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We calculate the asymptotic limit of some select electromagnetic properties when the electron-boson mass renormalization parameter of Eliashberg theory $\lambda \to \infty$. Our results apply for any reduced temperature $t>0$ and $\sqrt{\lambda}t \gg 1$. Local- and London-limit penetration depths are considered as well as the electromagnetic coherence length. Exact numerical results are given as well as approximate analytic expressions in a one-gap model for t near 1. For simplicity a δ function is employed for the electron-boson spectral density, and the Coulomb pseudopotential is neglected.

I. INTRODUCTION

In two previous papers, we have derived expressions for the thermodynamics[1] and for the second upper critical magnetic field[2] of a superconductor in the limit when the electron-boson mass renormalization parameter λ tends towards infinity. In this limit, these properties take on particularly simple forms. The calculations are valid for any reduced temperature $t>0$ with the restriction that $\sqrt{\lambda}\, t \gg 1$, which excludes zero temperature. Similar work has also been described by Bulaevskii *et al.*,[3] but is restricted to temperature $T=0$ and T very near T_c, so that the two works mentioned above are complementary.

In this paper, we extend our previous work to select electromagnetic properties. We will consider the local-limit penetration depth, which is also related to the dc critical current flowing in a superconductor-insulator-superconductor (*SIS*) Josephson[4] tunnel junction. Also, the London-limit penetration depth and the electromagnetic coherence length are investigated.

In Sec. II, we introduce the necessary formalism. This is followed with a general discussion (Sec. III) of the asymptotic form of the basic equations that reduce to universal dimensionless forms independent of material parameters. Section III also contains results for a simple one Matsubara gap solution to the universal equations that give qualitatively valid results for temperatures (T) near the critical temperature (T_c). In Sec. IV, we present our exact results in numerical form for all $t>0$ such that $\sqrt{\lambda}\, t \gg 1$ while Sec. V is a short conclusion.

II. FORMALISM

Sang Boo Nam[5] derived formulas for the electromagnetic properties of a superconductor within Eliashberg theory. Here we will be interested in the magnetic-field penetration depth $\lambda(T)$ in the local and London limit and in the electromagnetic coherence length $\xi(T)$ as a function of temperature T. The local limit is characterized by the condition $\xi(0) \gg l$ where l is the electron mean free path equal to $v_F \tau_N$ with v_F the electron Fermi velocity and τ_N the normal-state scattering time due to normal impurity scattering. The formula is[5,6]

$$\lambda_l(T) = \left[4\pi \frac{\sigma_N}{\hbar} T \mu_0 \sum_{n=1}^{\infty} \frac{\Delta^2(i\omega_n)}{\omega_n^2 + \Delta^2(i\omega_n)} \right]^{-1/2}, \quad (1)$$

where the subscript "l" denotes the local limit and σ_N is the normal-state conductivity equal to

$$\sigma_N = \tfrac{2}{3} N(0) e^2 v_F^2 \tau_N$$

with e the charge on the electron and $N(0)$ the single spin electronic density of states at the Fermi surface. In Eq. (1), $\Delta(i\omega_n)$ is the nth Matsubara gap and $i\omega_n = i\pi T(2n-1)$, $n=0,\pm1,\pm2,\ldots$ the nth Matsubara frequency. Except for a different numerical factor formula (1) is also related to the critical current $J_c(T)$ of an *SIS* Josephson function and we have[4]

$$\frac{J_c(T)}{J_c(0)} = \left[\frac{\lambda_l(0)}{\lambda_l(T)} \right]^2 \quad (2)$$

so that knowing the asymptotic limit for $\lambda_l(T)$ also implies that we know it for $J_c(T)$.

The London-limit penetration depth which applies when $\lambda_L(T) \gg \xi(0)$ is given by[5,7–9]

$$\lambda_L(T) = \left[\tfrac{4}{3}\pi N(0) e^2 v_F^2 T \mu_0 \sum_{n=1}^{\infty} \frac{\Delta^2(i\omega_n)}{Z(i\omega_n)[\omega_n^2 + \Delta^2(i\omega_n)]^{3/2}} \right]^{-1/2}, \quad (3)$$

which requires for its evaluation the nth Matsubara renormalization factor $Z(i\omega_n)$ which with $\Delta(i\omega_n)$ is given by solutions of the imaginary frequency axis Eliashberg gap equations. These are[10–12]

$$\Delta(i\omega_n)Z(i\omega_n)=\pi T\sum_m[\lambda(n-m)-\mu^*(\omega_c)]\frac{\Delta(i\omega_m)}{[\omega_m^2+\Delta^2(i\omega_m)]^{1/2}}+\pi t_+\frac{\Delta(i\omega_n)}{[\omega_n^2+\Delta^2(i\omega_n)]^{1/2}} \quad (4)$$

and

$$Z(i\omega_n)=1+\frac{\pi T}{\omega_n}\sum_m\lambda(n-m)\frac{\omega_m}{[\omega_m^2+\Delta^2(i\omega_m)]^{1/2}}+\pi t_+\frac{1}{[\omega_n^2+\Delta^2(i\omega_n)]^{1/2}}\ , \quad (5)$$

where

$$\lambda(n-m)=\int_0^\infty\frac{2\omega\alpha^2F(\omega)d\omega}{\omega^2+(\omega_n-\omega_m)^2}\ . \quad (6)$$

In Eq. (4), $\mu^*(\omega_c)$ is the Coulomb pseudopotential which comes with a cutoff at ω_c and $\pi t_+=\hbar/2\tau_N$ describes impurity scattering. In Eq. (6), $\alpha^2F(\omega)$ is the electron-boson spectral density. Throughout the work to be described here, we will use, for this quantity, a delta function of weight A at the Einstein frequency ω_E, i.e., $\alpha^2F(\omega)=A\delta(\omega-\omega_E)$ and so $\lambda(0)\equiv\lambda=2A/\omega_E$. For simplicity, we will also take $\mu^*(\omega_c)=0$. [Retaining a finite $\mu^\alpha(\omega_c)$ does not lead to any qualitative changes in the results although it does of course affect numerical values somewhat.]

A first important fact to notice about the gap equations is that on substituting (5) into (4) the πt_+ impurity term drops out of the single equation for $\Delta(i\omega_n)$, which takes on the form

$$\Delta(i\omega_n)=\pi T\sum_m\lambda(n-m)\frac{[\Delta(i\omega_m)-(\omega_m/\omega_n)\Delta(i\omega_n)]}{[\omega_m^2+\Delta^2(i\omega_m)]^{1/2}} \quad (7)$$

so that $\Delta(i\omega_n)$ is completely independent of πt_+ and hence, the local-limit penetration depth is independent of τ_N except for the factor of σ_N appearing in (1). This is consistent with the fact that the dirty limit $\pi t_+\gg 1$ has already been built into the prescription for $\lambda_l(T)$, which only depends on $\Delta(i\omega_n)$. Of course, πt_+ does not drop out of $Z(i\omega_n)$ and, in fact, for $\pi t_+\gg 1$, we can make the approximation

$$Z(i\omega_n)\cong\pi t_+\frac{1}{[\omega_n^2+\Delta^2(i\omega_n)]^{1/2}}\ . \quad (8)$$

On substitution of Eq. (8) into formula (3) for the London penetration depth, we recover immediately our formula (1) for the local limit. Thus the London-limit penetration depth is indeed dependent on impurity factor πt_+ and tends toward the local limit as $\pi t_+\gg 1$.

The final quantity of interest here is the electromagnetic coherence length $\xi(T)$ which is given by the formula[5,13]

$$\xi(T)=\frac{v_F\hbar}{2}\frac{\left[\sum_{n=1}^{\infty}\frac{\Delta^2(i\omega_n)}{Z(i\omega_n)[\omega_n^2+\Delta^2(i\omega_n)]^{3/2}}\right]}{\left[\sum_{n=1}^{\infty}\frac{\Delta^2(i\omega_n)}{\omega_n^2+\Delta^2(i\omega_n)}\right]} \quad (9)$$

and hence

$$\left[\frac{\lambda_l(T)}{\lambda_L(T)}\right]^2=\frac{\xi(T)}{l}\ , \quad (10)$$

where $l=v_F\tau_N$ is the mean free path. We have used Eqs. (1) and (3), and (9) to establish this last equation. Thus we see that local and London penetration depth and electromagnetic coherence distance are not independent quantities. Also in the dirty limit equation (8) for $Z(i\omega_n)$ can be used and the coherence length reduces to

$$\xi(T)=\frac{v_F\hbar}{2\pi t_+}=l\ .$$

This is a well-known result which states that for $l\ll\xi_0(T)$, where $\xi_0(T)$ is the intrinsic coherence length, the coherence distance becomes the mean free path. This makes sense since $\xi(T)$ measures the distance over which information can be transferred between two electrons.

III. ASYMPTOTIC LIMIT NEAR T_c IN THE ONE-GAP MODEL

We note first that the $m=n$ term in the sum on the right-hand side of Eq. (7) drops out since the combination

$$[\Delta(i\omega_m)-(\omega_m/\omega_n)\Delta(i\omega_n)]$$

is then identically zero. So the gap equation becomes

$$\Delta(i\omega_n)=\pi T\sum_{m\neq n}\lambda(m-n)\frac{\left[\Delta(i\omega_m)-\frac{\omega_m}{\omega_n}\Delta(i\omega_n)\right]}{[\omega_m^2+\Delta^2(i\omega_m)]^{1/2}}\ . \quad (11)$$

But for $n\neq m$ and a delta function spectral density $\lambda(n-m)$ reads

$$\lambda(m-n)=\frac{2A\omega_E}{\omega_E^2+(2\pi T)^2(m-n)^2}\ ,\quad n\neq m\ . \quad (12)$$

On the assumption that $\omega_E\ll 2\pi T$, which will turn out later to be equivalent to the condition $\sqrt{\lambda}\ t\gg 1$, we can neglect the ω_E^2 factor in the denominator of (12) and get

$$\lambda(m-n)=\frac{2}{(2\pi\bar{T})^2(m-n)^2}\ ,\quad m\neq n\ , \quad (13)$$

where the dimensionless temperature $\bar{T}\equiv T/\sqrt{A\omega_E}$ has been introduced. If we further write

$$\bar{\Delta}(n)\equiv\Delta(i\omega_n)/\sqrt{A\omega_E}$$

we get from Eq. (11) the dimensionless equation

$$\bar{\Delta}(n)=\pi\bar{T}\sum_{m\neq n}\frac{2}{(2\pi\bar{T})^2(m-n)^2}\frac{[\bar{\Delta}(m)-(\bar{\omega}_m/\bar{\omega}_n)\bar{\Delta}(n)]}{[\bar{\omega}_m^2+\bar{\Delta}^2(m)]^{1/2}}, \tag{14}$$

where $\bar{\omega}_n\equiv(2n-1)\pi\bar{T}$. This last equation is for a universal set of dimensionless functions $\bar{\Delta}(n)\equiv f_n(t)$ of the reduced temperature $t\equiv T/T_c$ which do not refer in any way to specific normal-state parameters, since $\bar{T}_c$ is given by a number.

For evaluation of the electromagnetic properties introduced in Sec. II, we need not only the gaps but also the renormalization $Z(i\omega_n)$, and it is therefore necessary to deal with the separate Eqs. (4) and (5) rather than with the combined Eq. (7) which nevertheless plays a critical role in our work. We can rewrite these in the dimensionless form

$$\bar{\Delta}(n)Z(n)=\frac{\pi\bar{T}\lambda\bar{\Delta}(n)}{[\bar{\omega}_n^2+\bar{\Delta}^2(n)]^{1/2}}+\pi\bar{T}\sum_{m\neq n}\frac{2}{(2\pi\bar{T})^2(m-n)^2}\frac{\bar{\Delta}(m)}{[\omega_m^2+\bar{\Delta}^2(m)]^{1/2}}+\pi\bar{t}_+\frac{\bar{\Delta}(n)}{[\bar{\omega}_n^2+\bar{\Delta}^2(n)]^{1/2}} \tag{15}$$

and

$$Z(n)=1+\frac{\pi\bar{T}\lambda}{[\bar{\omega}_n^2+\bar{\Delta}^2(n)]^{1/2}}+\frac{\pi\bar{T}}{\bar{\omega}_n}\sum_{m\neq n}\frac{2}{(2\pi\bar{T})^2(m-n)^2}\frac{\bar{\omega}(m)}{[\bar{\omega}_m^2+\bar{\Delta}^2(m)]^{1/2}}+\pi\bar{t}_+\frac{1}{[\bar{\omega}_n^2+\bar{\Delta}^2(n)]^{1/2}} \tag{16}$$

with $\bar{t}_+\equiv t_+/\sqrt{A\nu_E}$. Now we know from the form of Eq. (14) that $\bar{\Delta}(n)$ is *completely* independent of any material parameter. Thus, only the first term on the right-hand side of Eq. (15) and the second on the right-hand side of (16) refer to material parameters, except, of course, for the impurity factor $\pi\bar{t}_+$. In the asymptotic limit, we can assume $\pi\bar{t}_+=\pi t_+/\sqrt{A\nu_E}$ to be much smaller than $\pi\bar{T}\lambda$. This is equivalent to the condition

$$\sqrt{\lambda}t\lambda\gg\frac{t_+}{\omega_E},$$

which can always be satisfied for $\lambda\gg1$ and $\sqrt{\lambda}t\gg1$ assuming t_+ represents a finite impurity content. If we think of A as fixed and $\omega_E\to0$ so that $\lambda\to\infty$ we rewrite the condition as $\sqrt{\lambda}t\gg t_+/A$ which can always be arranged for any $t>0$, for fixed t_+ and sufficiently large λ. If we take ω_E as fixed and $A\to\infty,\lambda^{3/2}\,t\gg t_+/\omega_E$ can also always be arranged if λ is taken large enough and $t>0$. In this case, the $\pi\bar{t}_+$ term can be dropped in both (15) and (16) so that, in the asymptotic limit, impurities will not change the London limit penetration depth or the coherence distance. This theorem makes sense since for $\lambda\gg1$ the electron-boson scattering will dominate over impurity scattering except at $T=0$, a case we do not treat here ($\sqrt{\lambda}t\gg1$).

Noting the symmetry $\bar{\Delta}(-n)=\bar{\Delta}(n+1)$ and $\bar{\omega}(-n)=-\bar{\omega}(n+1)$, we can limit the sum in Eq. (14) to the interval $m=1$ to ∞ only and obtain

$$\bar{\Delta}(n)=\pi\bar{T}\sum_{m=1}^{\infty}\frac{2}{(2\pi\bar{T})^2}\left[\frac{1-\delta_{n,m}}{(m-n)^2}+\frac{1}{(m+n-1)^2}\right]\frac{\bar{\Delta}(m)}{[\bar{\omega}_m^2+\bar{\Delta}^2(m)]^{1/2}}-\pi\bar{T}\sum_{m=1}^{\infty}\frac{2}{(2\pi\bar{T})^2}\left[\frac{1-\delta_{n,m}}{(m-n)^2}-\frac{1}{(m+n-1)^2}\right]\frac{(\bar{\omega}_m/\bar{\omega}_n)\bar{\Delta}(n)}{[\bar{\omega}_m^2+\bar{\Delta}^2(m)]^{1/2}}. \tag{17}$$

To get exact results, we need to solve this equation numerically. For T near T_c, we can expand in powers of $\bar{\Delta}(m)$ to get

$$\bar{\Delta}(n)=\pi\bar{T}\sum_{m=1}^{\infty}\frac{2}{(2\pi\bar{T})^2}\left[\frac{1-\delta_{n,m}}{(m-n)^2}+\frac{1}{(m+n-1)^2}\right]\frac{\bar{\Delta}(m)}{|\bar{\omega}_m|}\left[1-\frac{1}{2}\frac{\bar{\Delta}^2(m)}{\bar{\omega}_m^2}\right]-\pi\bar{T}\sum_{m=1}^{\infty}\frac{2}{(2\pi\bar{T})^2}\left[\frac{1-\delta_{n,m}}{(m-n)^2}-\frac{1}{(m+n-1)^2}\right]\left[1-\frac{1}{2}\frac{\bar{\Delta}^2(m)}{\bar{\omega}_m^2}\right]\frac{\bar{\Delta}(n)}{\bar{\omega}_n}. \tag{18}$$

A rough approximation to (18) can be obtained if we assume that $\bar{\Delta}(1)=\bar{\Delta}(T)$ and that all other gaps are zero. No unit term appears in the last curly bracket because, in that case, the sum over m must be carried out to ∞ and it gives zero because terms cancel in pairs. This gives

$$1=\frac{1}{2(\pi\bar{T})^2}\left[1-\frac{\bar{\Delta}^2(T)}{(\pi\bar{T})^2}\right]. \tag{19}$$

To get the critical temperature, we set $\bar{\Delta}^2(T)=0$ in Eq. (19) and get $\bar{T}_c=1/\sqrt{2}\pi$ or

$$T_c=\frac{1}{\sqrt{2}\pi}\sqrt{A\omega_E}=\frac{1}{2\pi}\sqrt{\lambda}\omega_E.$$

For T near T_c the gap can be written as

$$\bar{\Delta}(T)=\sqrt{1-t}\quad\text{and}\quad\Delta(T)=\frac{\sqrt{\lambda}\omega_E}{\sqrt{2}}\sqrt{1-t} \tag{20}$$

with t the reduced temperature $t=T/T_c$. It is instructive to use this simple model to get a first approximation for the electromagnetic properties introduced in Sec. II. Equation (1) for $\lambda_l(t)$ reduces to

$$\lambda_l^{-2}(t)\cong 4\sigma_N\omega_E\sqrt{\lambda}\frac{\mu_0}{\hbar}(1-t)\ ,\quad t\to 1\ . \tag{21}$$

Equation (3) for the London limit is a little more difficult to dispose of since, in addition to a dependence on the gaps, it also contains an explicit factor of $Z(i\omega_n)$. But in the limit $\lambda\to\infty$ Eq. (16) gives us

$$Z(n)\cong\frac{\pi\bar{T}\lambda}{[\bar{\omega}_n^2+\bar{\Delta}^2(n)]^{1/2}} \tag{22}$$

and we get

$$\lambda_L^{-2}(T)=4\left[\frac{ne^2}{m}\mu_0\right]\frac{1}{\lambda}(1-t)=\frac{4\sigma_N\mu_0}{\tau_N\lambda}(1-t)\ ,\quad t\to 1\ , \tag{23}$$

where we have used the relation $n=\frac{2}{3}N(0)mv_F^2$ with n the number of free electrons per unit volume and m the electron mass. From Eqs. (21) and (23) and reference to the general relationship (10), we immediately obtain

$$\xi(t)=\frac{v_F\hbar}{\lambda^{3/2}\omega_E}\ ,\quad t\to 1\ . \tag{24}$$

All these relationships are approximate and hold only for T near T_c. In the next section, we will get exact numerical results for any $t>0$ and compare our results near T_c with the preceding approximate estimates.

IV. EXACT NUMERICAL RESULTS

We start with Eq. (17), which is dimensionless and universal. Linearization in $\bar{\Delta}(n)$ leads to an equation for $\bar{T}_c$ which, through numerical solution, gives

$$\bar{T}_c=0.2584 \quad\text{or}\quad T_c=0.183\sqrt{\lambda}\omega_E \tag{25}$$

a result first obtained by Allen and Dynes.[14] The condition $\omega_E\ll 2\pi T$, which is the fundamental approximation made throughout our work, can be transformed into a more useful form using (25). We see that $\omega_E\ll 2\pi T$ can be rewritten as $1\ll\sqrt{\lambda}t$ or $\sqrt{\lambda}\ t\gg 1$, which restricts the range of values of λ and reduced temperature $t\equiv T/T_c$ for which our work is valid.

In general, at reduced temperature $t<1$, the full nonlinear form (17) must be retained and, as stated before, we get a universal dimensionless function $\bar{\Delta}(n)\equiv f_n(t)$ from which we can evaluate the local limit penetration depth given in Eq. (1). It can be written as

$$\lambda_l^{-2}(t)=\frac{\sqrt{\lambda}\omega_E}{\sqrt{2}}\frac{\sigma_N\mu_0}{\hbar}g^{-2}(t) \tag{26}$$

with

$$g^{-2}(t)=4\pi\bar{T}_c t\sum_{n=1}^{\infty}\frac{f_n^2(t)}{(\pi\bar{T}_c t)^2(2n-1)^2+f_n^2(t)}\ , \tag{27}$$

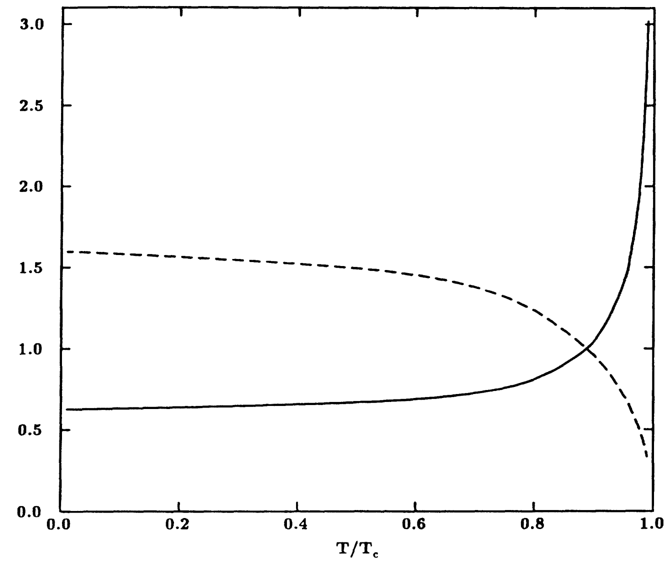

FIG. 1. The universal dimensionless function $g(t)$ (solid line) which determines the temperature dependence of the local penetration depth in the asymptotic limit. The inverse function $g^{-1}(t)$ (dashed curve) is also shown. While our calculations are valid only for $\sqrt{\lambda}\ t\gg 1$ and so, $t>0$ for any large but finite value of λ, the curves for g and g^{-1} are flat over a large temperature range around $t\to 0$ and so the value of $g(0)$ can probably safely be obtained by extrapolation of our lowest-temperature results ($t=0.01$) on the assumption that at yet lower temperatures the behavior does not change unexpectedly.

which is a dimensionless universal function that plays a crucial role in asymptotic expressions for electromagnetic properties at any $t>0$ with the condition $\sqrt{\lambda}\ t\gg 1$ understood.

On comparing (26) and (27) with our one-gap result (21) for $\lambda_l^{-2}(t)$ with $t\to 1$, we see that in the one-gap model

$$g^{-2}(t)\equiv 4\sqrt{2}(1-t)=5.66(1-t)\ . \tag{28}$$

In the exact case, the number $4\sqrt{2}$ is different, as we will see shortly. From (26), we have

$$\left[\sigma_N\frac{\sqrt{\lambda}\omega_E}{\sqrt{2}}\right]^{1/2}\lambda_l(t)=\left[\frac{\hbar}{\mu_0}\right]^{1/2}g(t)$$
$$=7.33\times 10^{-4}g(t)\ , \tag{29}$$

where the units are $\text{meV}^{1/2}\,\text{m}/(\Omega\,\text{m})^{1/2}$.

The function $g(t)$ is plotted in Fig. 1 where its inverse is also shown for the convenience of the reader. As we know from our one-gap model results, this function goes to ∞ as $t\to 1$. In fact, we find that as $t\to 1$

$$g^{-2}(t)=9.64(1-t)\ . \tag{30}$$

The difference in numerical coefficient from the one-gap model result [Eq. (28)] is significant although the temperature law is, of course, the same. As the temperature is lowered, the one-gap model becomes less and less valid and our full numerical calculations cannot be compared with any analytic qualitative result. We see from Fig. 1 that as $t \to 0$ the curve for $g(t)$ becomes rather flat. While, as established above, our numerical work is only valid for $\sqrt{\lambda}\, t \gg 1$ and so we must have $t > 0$, it nevertheless, seems reasonable to extrapolate to zero temperature from our lowest temperature which was $t = 0.01$. If we do this, we get 0.67 for $g(0)$ and the curve remains nearly flat for a considerable low-temperature range. This feature has often been interpreted in actual experiments, as a signature of s-state pairing with a finite gap everywhere on the Fermi surface. Of course, in experiments, we are not necessarily in the local limit. Sometimes the London limit is more appropriate.

The London penetration depth given by the full Eq. (3) can also be related to the same function $g(t)$ that we have just introduced. On reference to (22), we obtain in the asymptotic limit

$$\lambda_L^{-2}(t) = \frac{e^2 \mu_0 n}{m \lambda t 2\pi \bar{T}_c} g^{-2}(t) . \tag{31}$$

While, as previously stated, $\lambda_L(t)$ is given by the same function $g(t)$ as λ_l, the reader should note the extra factor of t in the denominator of (31). Thus

$$\frac{\sqrt{n}\, \lambda_L(t)}{\sqrt{\lambda}} = 1.274 \bar{\Lambda} \sqrt{t}\, g(t) , \tag{32}$$

where

$$\bar{\Lambda} = 0.5317 \times 10^{-7}\ \mathrm{m}(10^{22}/\mathrm{cm}^3)^{1/2}$$

is the classical penetration depth first introduced by London.

The function $\sqrt{t}\, g(t)$ is given in Fig. 2 as its inverse. The very low-temperature dependence of this function is very interesting. While our work is restricted to the asymptotic region with $\sqrt{\lambda}\, t \gg 1$ and so to $t > 0$ for any large, but fixed value of λ, we see again that $\sqrt{t}\, g(t)$ appears to be well behaved as $t \to 0$ and seems to go towards zero. On the other hand, the inverse function and therefore, $\lambda_L^{-1}(t)$ goes to ∞ like $1/\sqrt{t}$ as $t \to 0$. This is a completely different behavior from Bardeen-Cooper-Schreiffer (BCS) and is characteristic of the asymptotic limit. It is valid for any finite impurity content and does not map directly into our previous results for the local limit. A word of explanation is perhaps in order. If we assume πt_+ to dominate in the $Z(i\omega_n)$ channel then, as we have demonstrated, the London limit goes into the local limit. But in the asymptotic limit, we first assume $\lambda \to \infty$ and so the πt_+ term in the Eq. (16) for $Z(i\omega_n)$ can be dropped against the second term for $\bar{T}\lambda \gg \pi \bar{t}_+$ which we assumed is valid for finite t_+. It is a question of which limit is taken first. The physical limit is $\bar{T}\lambda \gg \pi \bar{t}_+$, of course, indicating that electron-boson scattering dominates, in this case, over any impurity scattering.

Finally, we look at the asymptotic limit of the coherence length given by Eq. (9). Noting that as $\lambda \to \infty$, $Z(i\omega_n)$ can be approximated by Eq. (22), we obtain immediately

$$\xi(t) = \frac{v_F \hbar}{2\pi T \lambda} . \tag{33}$$

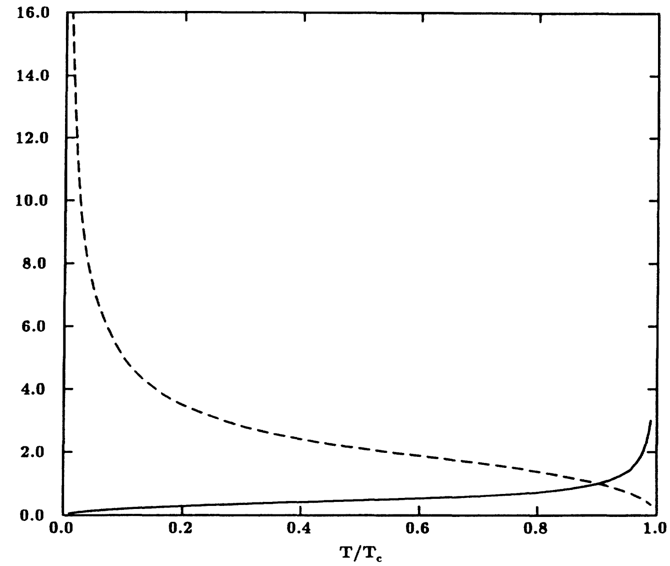

FIG. 2. The universal dimensionless function $\sqrt{t}\, g(t)$ (solid line) and its inverse (dashed line) which determine respectively the reduced temperature dependence of the London penetration depth and of its inverse. The calculations are valid only for $\sqrt{\lambda}\, t \gg 1$ and so for any large but finite value of λ we need $t > 0$. Nevertheless, the curve for $\sqrt{t}\, g(t)$ indicates that as $t \to 0$ this function appears to go smoothly to zero as $0.67\sqrt{t}$ at least up to $t = 0.01$.

But $T_c = 0.183\sqrt{\lambda}\omega_E$ and so

$$\xi(t) = \frac{0.87 v_F \hbar}{\omega_E \lambda^{3/2} t} , \tag{34}$$

which is to be contrasted with the approximate result (24) valid only for $t = 1$ in which case the numerical factor is 1 rather than 0.87. Note that (34) was derived as for all other formulas in this paper under the approximation $\sqrt{\lambda}\, t \gg 1$ so that the $1/t$ factor is not a problem.

V. CONCLUSIONS

We have calculated the temperature variation of the local- and London-limit penetration depth in the asymptotic limit $\lambda \to \infty$ where λ is the electron-boson mass renormalization parameter. We find that $\sqrt{\lambda}$ times the local limit penetration depth $[\lambda_l(t)]$ is proportional to a universal dimensionless function $g(t)$ which does not depend on material parameters and which determines the temperature variation of $\lambda_l(t)$. The numerical work is limited to $\sqrt{\lambda}\, t \gg 1$ which implies that for any large, but finite value of λ, we need $t > 0$. Nevertheless, the behavior of $\lambda_l(t)$ down to $t = 0.01$, where we have stopped the numerical work, is very flat and would indicate that a straight mathematical extrapolation to $t = 0$ may be valid provided it is assumed that no unexpected special behavior sets in at yet lower temperatures.

We have found that the temperature variation of the London limit is also determined by the same universal function that we have computed numerically but now it is multiplied by $\sqrt{t}$. This result is valid for any finite impurity content and indicates a very different behavior around $t=0$ for $\lambda_L(t)$ than found in BCS theory.

Finally, the electromagnetic coherence length takes on a particularly simple form, namely,

$$\xi(t)=0.87\ v_F\hbar/\lambda^{3/2}\omega_E t\ ,$$

which holds for $\sqrt{\lambda}\ t \gg 1$. A simple one-gap approximation gives the same result near $t=1$ except that the numerical factor 0.87 is to be replaced by 1.0. This same model gives similar rough estimates for the penetration depth near $t=1$. At lower temperatures, it is necessary to do complete numerical calculations as we have presented here.

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