Quantum levels and Zeeman splitting for two-dimensional hydrogenic donor states in a magnetic field

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We have presented exact solutions for two-dimensional (2D) hydrogenic donor states in a magnetic field and a study of quantum levels and Zeeman bifurcation of 2D donor states in different strength of magnetic field. The quadratic effect of the magnetic field partly lifts the degeneracy of 2D donor states in the pure Coulomb potential and results in an increase of all quantum levels. Then, the first-order term of a magnetic field, i.e., the Zeeman term, completely lifts the degeneracy. At strong magnetic field, the degeneracy of Landau levels is lifted by the Coulomb potential. As the magnetic field decreases, all of the values of the energies decrease linearly until the hybridization between Landau and Coulomb levels occurs. In addition, based on the comparison between energies of 2D and 3D states, we conclude that there are smaller diamagnetic shifts for 2D donor states than those for the corresponding 3D donor states.

I. INTRODUCTION

The spectrum of a hydrogenic impurity in bulk semiconductors in a constant magnetic field of arbitrary strength has been studied both theoretically[1–4] and experimentally[5–7] for many years. Using radiation generally in the millimeter and submillimeter region of the spectrum, magneto-optical studies of a variety of elemental and III-V bulk semiconductors have revealed a multiplicity of spectral lines, which can be assigned to transitions of electrons between the excited states of shallow donors.[5–7] The Zeeman splitting of inter-excited-state transitions has been identified by a number of calculations[1–4] of its dependence on magnetic field. In semiconductors, however, typical values of effective mass μ and dielectric constant ϵ make the effective Rydberg $\mathcal{R}^*$ about 10^3-10^4 times smaller and the dimensionless magnetic field $\gamma=\hbar\omega_c/2\mathcal{R}^*$ ($\omega_c=eB/\mu c$ is the cyclotron frequency of a free carrier in a field B) in a fixed field B about 10^4-10^5 times larger than those of the free hydrogenic atom, respectively. With the range of intermediate magnetic fields, the γ of semiconductors can be larger than 1 so that low-field ($\gamma \ll 1$) perturbation methods become inapplicable and other methods should be used to determine quantum levels and identify excited-state transitions and their Zeeman splitting. The adiabatic method was the first to be employed in the high-field regime $\gamma \gg 1$.[1,3] Calculations based on the method have usually been restricted to the ground state and first few excited states. Apart from purely numerical techniques, other workers have favored the use of the variational method. More recently Makado and McGill[4] have based their approach on that of Aldich and Greene,[2] and used a different set of basis functions to get quite good results with high accuracy.

The unique nature of electronic states associated with semiconductor quantum wells and superlattices has been the subject of a great deal of interest, both from theoretical and experimental viewpoints. In view of the potential applications of these structures, the understanding of impurity and exciton states found in these systems, with and without external fields, is an issue of technical as well as scientific importance.

Among quantum wells and superlattices, so far, the GaAs-$Ga_{1-x}Al_xAs$ system is the simplest and the most extensively studied. Then exciton and impurity states in the system have been calculated by a number of authors. For an infinite barrier height,[8] the binding energy approaches $4\mathcal{R}^*$ ($\mathcal{R}^*$ is effective Rydberg, i.e., the binding energy of the ground state in the bulk semiconductor) as the well size reduces. Then excitons and impurities become two-dimensional (2D). For a finite barrier height, however, several calculations[9–12] have shown that the binding energy goes through a maximum as the well size reduces instead of continuously increasing as is found in the infinite barrier calculation. The maximum of the binding energy is dependent on the barrier height[12] and is less than $4\mathcal{R}^*$. Then, the excitons and impurities become quasi-two-dimensional (Q2D) as the well width is smaller and the quantum confinement is stronger. Furthermore, Mailhiot *et al.*[11] have presented the energy spectra of the ground state and the low-lying excited states for a hydrogenic donor in a quantum well. All of the above calculated results have shown that the donor states strongly confined by GaAs-$Ga_{1-x}Al_xAs$ quantum wells can be correctly described by use of 2D or Q2D hydrogenlike atoms with properly variational parameters. Therefore, we can study 2D or Q2D hydrogenic donors in a magnetic field to understand the behavior of a donor strongly confined by a quantum well in different magnetic fields. Although the problem of 2D hydrogenic donors in magnetic fields can be separable, we are still prevented from obtaining an analytically exact solution of the eigenvalue problem. However, the weak-field regime can still be treated by considering the magnetic field as a perturbation, while in the strong-field regime the hydrogenic potential is treated as a perturbation. Using a two-point Padé approximation, MacDonald and Ritchie[13] have presented an interpolation between these two limiting situations and they have obtained analytic expressions for the magnetic-field dependence of both ground- and excited-state energies. In addition to the method mentioned above, using the method of series expansion, exact series forms in different regions of the equation can be obtained. Based on such series forms, the quantum levels and wave functions of 2D hydrogenic donors in a magnetic field can be exactly obtained by a numerical method in all magnetic field ranges. It should be interesting to compare the calculated results with MacDonald and Ritchie's and to see whether two independent methods yield equivalent results.

In this paper, we report for the first time exact solutions of a 2D hydrogenic donor in a magnetic field and a study of quantum levels and Zeeman bifurcation of 2D hydrogenic donor states in different strengths of magnetic field. In Sec. II, we present the exact series forms and the calculation techniques. The main results are presented and compared with MacDonald and Ritchie's,[13] and Makado and McGill's[4] results in Sec. III. A summary of the results and a conclusion are presented in Sec. IV.

II. FORMULA OF EXACT SOLUTIONS

Within the framework of an effective-mass approximation, the Hamiltonian of a 2D hydrogenic donor in the presence of a magnetic field that is perpendicular to the 2D plane can be written as

$$H=-\frac{\hbar^2\nabla_2^2}{2\mu}+\frac{e^2A^2}{2\mu c^2}+i\frac{\hbar e}{\mu c}\mathbf{A}\cdot\nabla_2-\frac{we^2}{\epsilon\rho}\,, \tag{1}$$

where A, μ, and ϵ are, respectively, the vector potential, electron effective mass, and static dielectric constant, and ∇_2 represents $(\hat{\mathbf{i}}\partial/\partial x+\hat{\mathbf{j}}\partial/\partial y)$, w is equal to 1.

If we choose the cylindrical gauge such that

$$\mathbf{A}=\tfrac{1}{2}\mathbf{B}\times\mathbf{r} \tag{2}$$

then the form of the Hamiltonian is (in cylindrical polar coordinates)

$$H=-\frac{\hbar^2}{2\mu}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial}{\partial\rho}\right]+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}\right]+\frac{e^2B^2}{8\mu c^2}\rho^2-i\hbar\frac{eB}{2\mu c}\frac{\partial}{\partial\varphi}-\frac{we^2}{\epsilon\rho}\,. \tag{3}$$

In order to solve the Schrödinger-like equation

$$H\Psi(\rho,\varphi)=E\Psi(\rho,\varphi) \tag{4}$$

the wave functions of an electron with well-defined magnetic quantum number m in the cylindrically symmetric potential, which is the magnetic field and 2D Coulomb potential, are written in the form

$$\Psi(\rho,\varphi)=e^{im\varphi}\psi(\rho)\,, \tag{5}$$

where we should take the m as both positive and negative integers, i.e., $m=0,\pm1,\pm2,\ldots$, to identify each of the donor states in the magnetic field. It is important for the study of the Zeeman splitting and the analysis of the quantum-level degeneracy in different conditions. Substituting Eq. (5) into Eq. (4), we find an equation for the function $\psi(\rho)$:

$$\frac{d^2\psi}{d\rho^2}+\frac{1}{\rho}\frac{d\psi}{d\rho}+\left[\frac{2w}{\rho}-\frac{m^2}{\rho^2}-m\gamma-\tfrac{1}{4}\gamma^2\rho^2+E(m)\right]\psi=0\,, \tag{6}$$

$$E(m)=E/\mathcal{R}^*\,, \tag{7}$$

and

$$\gamma=\hbar\omega_c/2\mathcal{R}^*\,, \tag{8}$$

where $\mathcal{R}^*=\mu e^4/2\hbar^2\epsilon^2$ is the effective Rydberg, ω_c ($=eB/\mu c$) is the cyclotron frequency, and ρ is in units of the effective Bohr radii ($a^*=\epsilon\hbar^2/e^2\mu$). In the absence of magnetic field ($\gamma=0$), the energy eigenvalues and eigenfunctions of Eq. (6) can be obtained exactly and are

$$E(m)=-(N-\tfrac{1}{2})^{-2},\quad N=1,2,3,\ldots \tag{9}$$

and

$$\psi_{Nm}(\rho)=\frac{2}{(N-\frac{1}{2})}\left[\frac{(N-|m|-1)!}{(N+|m|-1)!(2N-1)}\right]^{1/2}\left[\frac{2}{N-\frac{1}{2}}\right]^{|m|}\exp\left[\frac{-\rho}{N-\frac{1}{2}}\right]L_{N-|m|-1}^{2|m|}\left[\frac{2\rho}{N-\frac{1}{2}}\right]\,, \tag{10}$$

respectively, where $L_p^q(\rho)$ is the associated Laguerre polynomial. It is easily seen that the quantum levels are only dependent on the principal quantum number N and degenerate with respect to m. The degree of degeneracy is equal to $2N-1$ (excluding spin degeneracy), i.e., $m=0,\pm1,\ldots,\pm(N-1)$.

We are prevented from obtaining analytically exact solutions of the eigenvalue problem with both Coulomb potential and magnetic field, as was mentioned in the introduction, because Eq. (6), with suitable boundary conditions is beyond the analytic problem of confluent hypergeometric equations. However, using the method of series expansion, we can obtain exact series form in different regions of Eq. (6). It should be noted that zero and infinity are a regular and an irregular singular point of Eq. (6), respectively. In the region $0<\rho$, we have a series solution, which has a finite value at $\rho=0$, as follows:

$$\psi(\rho)=\rho^{|m|}\sum_{n=0}^{\infty} a_n\rho^n , \tag{11}$$

where a_0 is a constant. Noting that a_n equal to zero and n is equal to a negative integer, the other a_n can be determined by the following recurrence relation:

$$a_n=-\{2wa_{n-1}+[E(m)-m\gamma]a_{n-2}-\gamma^4 a_{n-4}/4\}/n(2|m|+n), \quad n=1,2,3,\ldots . \tag{12}$$

In the region $\rho<\infty$, we can obtain a normal solution.[14] It approaches zero at $\rho=\infty$ and is found in the form

$$\psi(\rho)=e^{-\gamma\rho^2/4}\rho^s\sum_{n=0}^{\infty} b_n\rho^{-n} , \tag{13}$$

where

$$s=E(m)/\gamma-m-1 \tag{14}$$

and

$$b_1=-wb_0 ,$$

$$b_n=-\{2wb_{n-1}+[(s-n+2)^2-m^2]b_{n-2}\}/n\gamma, \quad n=2,3,\ldots \tag{15}$$

and b_0 is a constant. The series appear suitable for numerical calculations for large ρ.[14] However, they are not suitable for small ρ. In order to get an exact value at small ρ, we can also find a solution of uniformly convergent Taylor series. It is as follows:

$$\psi(\rho)=\sum_{n=0}^{\infty} c_n(\rho-R_p)^n+\sum_{n=1}^{\infty} d_n(\rho-R_p)^n , \tag{16}$$

where R_p is a proper point for using Eq. (13) and c_0 and d_1 are constants. The other values of c_n and d_n can be determined by the recurrence relations.

Using Eqs. (11), (13), and (16) and the matching conditions at $\rho=R_p$ and R_0 for which Eqs. (13) and (16) are suitable for numerical calculations, we obtain the equation of the eigenenergies $E(m)$, which can be solved numerically. Once the nth eigenenergy $E_n(m)$ is known, the a_0, b_0, c_0, and d_1 [hence, $\psi_n(\rho)$] are known by the use of the normalized condition $\psi_n(\rho)$. Thus, $\psi_n(\rho)$ depends on the value of m, the magnetic field, the Coulomb potential, and the energy $E_n(m)$.

If there is no Coulomb potential in the Hamiltonian of Eq. (1), the energy eigenvalues and eigenfunctions of Eq. (6) can be obtained exactly and are:

$$E_L(m,w=0)=(2L+1)\gamma, \quad L=n+m;\ n=0,1,2,\ldots;\ m=-n,-n+1,\ldots,n,\ldots \tag{17}$$

and

$$\psi_{Lm}(\rho,w=0)=\left[\frac{(L-|m|)!\gamma}{L!}\right]^{1/2}\left[\frac{\gamma\rho^2}{2}\right]^{1/2}\exp\left[\frac{-\gamma\rho^2}{4}\right]L_{L-|m|}^{|m|}\left[\frac{\gamma\rho^2}{2}\right] , \tag{18}$$

respectively. It is easily seen that the quantum levels are only dependent on the Landau quantum number L and degenerate with respect to the m. The degree of degeneracy is equal to $L+1$ for positive m (i.e., $m=0,1,2,\ldots,L$). However, we should point out that the total degree of degeneracy approaches infinite because the Zeeman term of negative integer m can make the energy decrease from a higher level to any lower level. It is quite different from that in the purely Coulomb potential, mentioned above. In fact, using the equation of the eigenenergies $E(m)$, the same exact values of quantum levels can be obtained as $w=0$. Once quantum levels with and without a Coulomb potential [$E_N(m,w=1)$ and $E_N(m,w=0)$] are obtained, the binding energies [$E_{NB}(m)$] of the corresponding donor states in the magnetic field are given by

$$E_{NB}(m)=E_N(m,w=0)-E_N(m,w=1) . \tag{19}$$

III. QUANTUM LEVELS AND ZEEMAN SPLITTING

In order to check the calculation method, the energies of 2D donor ground and excited states in a purely Coulomb potential and those in a purely magnetic field have been calculated. The calculated results have shown that the eigenvalues can approach exactly those shown in Eqs. (9) and (17). It is interesting to point out that for the

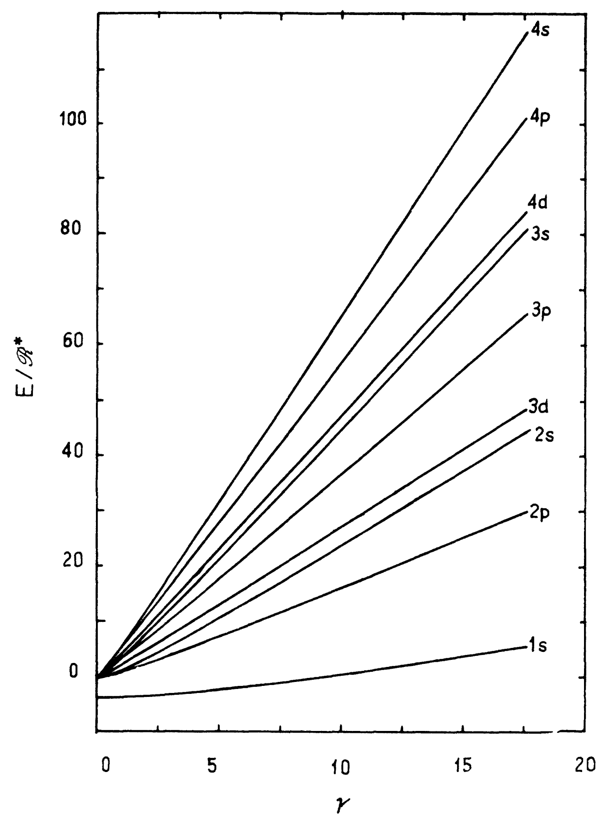

FIG. 1. The 2D donor energy in effective Rydberg units, excluding the Zeeman term $m\gamma$, is shown as a function of the normalized magnetic field γ for 1*s*, 2*p*, 2*s*, 3*d*, 3*p*, 3*s*, 4*d*, 4*p*, and 4*s* states.

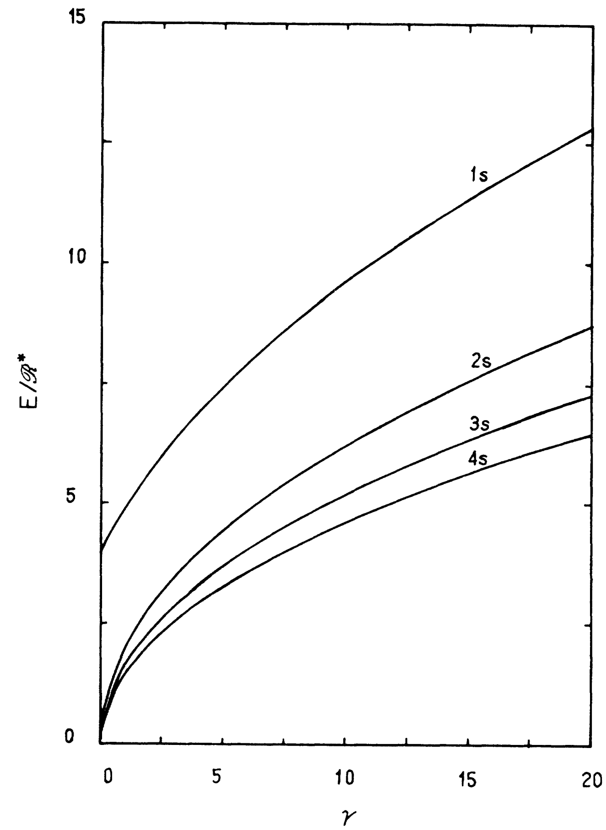

FIG. 2. The 2D donor binding energy in effective Rydberg units is shown as a function of γ for 1*s*, 2*s*, 3*s*, and 4*s* states.

case of a purely Coulomb potential, the other form should be used instead of using Eq. (13). It is as follows:

$$\psi(\rho)=e^{-\xi\rho}\rho^{z}\sum_{k=0}^{\infty}c_k\rho^{-k} \tag{20}$$

and

$$\xi=[-E(m)]^{1/2},\quad z=w/\xi-\tfrac{1}{2}\ , \tag{21}$$

where c_0 is a constant, and the other c_k can be determined by the recurrence relations.

Before calculated results are shown and discussed, it is worthwhile to note that there are several ways to indicate the quantum levels. In general, we can use m (magnetic quantum number) and n, which is equal to the number of the root of the equation of the eigenenergies in order of increasing magnitude, to label the quantum levels. In the limit of zero magnetic field, the states are labeled by the principal and magnetic quantum numbers N and m. The N is equal to $n+|m|$. As shown in Eq. (9), the quantum levels are only dependent on the N and degenerate with respect to m. The degree of degeneracy is $2N-1$. In the limit of zero Coulomb interaction, the states are labeled by the Landau and magnetic quantum numbers L and m. The L is equal to $n+m$. As shown in Eq. (17), the quantum levels are only dependent on the L and degenerate with respect to the m. However, the degree of degeneracy approaches infinite and is quite different from that in a purely Coulomb potential, mentioned above. In principle, we can use the N and m or the L and m to label 2D hydrogenic donor states in a magnetic field. Here, the quantum levels $E_N(m)$ are indicated by the principal and magnetic quantum numbers N and m. Therefore, we have levels (states) 1*s*, $2p_{\pm}$, $3d_{\pm}$, 2*s*, and so on if the notation $s, p_{\pm}, d_{\pm}, \ldots$ is used for the magnetic quantum number $m=0,\pm1,\pm2,\ldots$.

In order to study the quantum levels and their Zeeman splitting of 2D hydrogenic donor states in a magnetic field, we have performed a numerical calculation for the 2D donor spectra with and without the Zeeman term $m\gamma$ in the magnetic field. As a matter of fact, the spectra excluding the Zeeman term are of the 2D donor in a 2D parabolic quantum well, which is formed by the magnetic field. In Fig. 1, we have plotted quantum levels of 2D donor states excluding the Zeeman term $m\gamma$ as a function of the dimensionless magnetic field $\gamma=(\hbar\omega_c/2\mathcal{R}^*)$, where ω_c and R^* are, respectively, the cyclotron frequency and effective Rydberg, as defined in the introduction. In the figure, it is readily seen that as the γ approaches zero, the quantum levels approach those given by Eq. (9). As the γ increases, the degeneracy is lifted partly by the parabolic quantum well. The order of the quantum levels is 1*s*, 2*p*, 2*s*, 3*d*, 3*p*, 3*s*, and so on. It is easily understood that all of energies increase with γ because the width of the quantum well becomes narrower. It is interesting to note that the energies of the excited states increase much

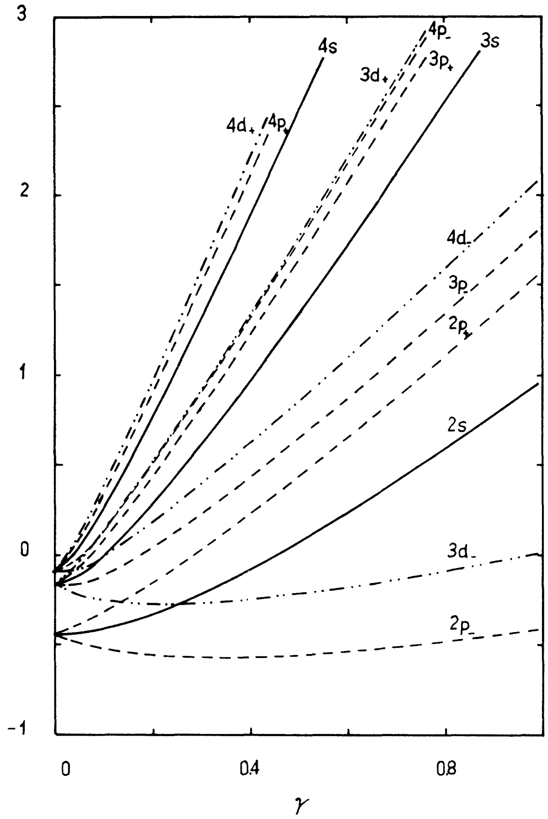

FIG. 3. The 2D donor energy in effective Rydberg units, including the Zeeman term $m\gamma$, is shown as a function of γ ranging from 0 to 1 for $2p_-$, $3d_-$, $2s$, $2p_+$, $3p_-$, $4d_-$, $3s$, $3p_+$, $4p_-$, $3d_+$, $4s$, $4p_+$, and $4d_+$ states.

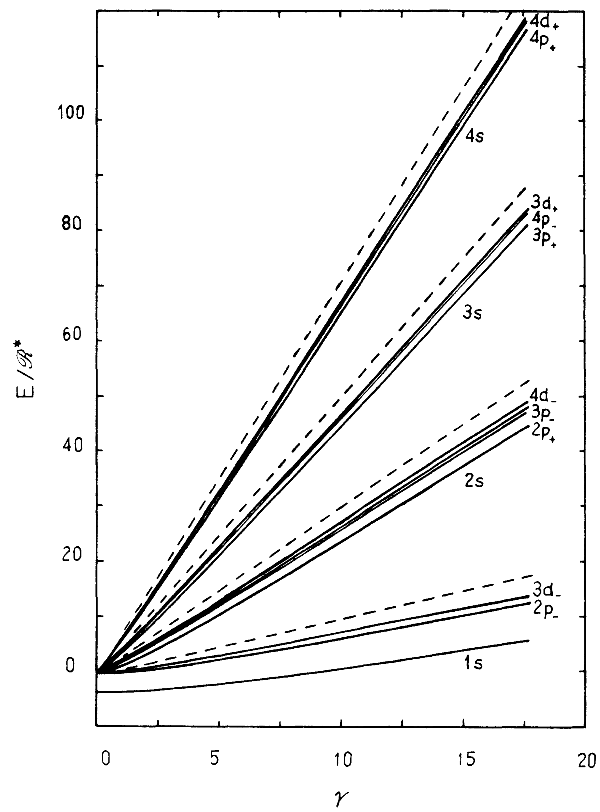

FIG. 4. Same as Fig. 3 but adding $1s$ state for γ ranging from 0 to 20. The corresponding Landau energy levels are shown by broken curves for comparison.

more rapidly with magnetic field than that of the ground state, and the energy differences between these states increase with applied field. In fact, the energies of the $s(m=0)$ states approach asymptotically the corresponding Landau levels. The Coulomb binding energy of a particular Landau level can be obtained using Eq. (19). The results are shown in Fig. 2 for the first four s states, i.e., $1s$, $2s$, $3s$, and $4s$ states. It can be seen that the binding energy increases with the γ for each of these states and that for a fixed field, the binding energy increases as we go to lower-lying donor states due to a stronger confinement of electrons in lower-lying states with a stronger field.

In Figs. 3 and 4, we have plotted quantum levels of 2D donor states, including the Zeeman term $m\gamma$, as a function of the γ. In Fig. 3, it is shown that as the γ increases slightly, the $2p_\pm$, $3p_\pm$, $3d_\pm$, $4p_\pm$, and $4d_\pm$ levels split, respectively, into two parts because of the Zeeman term $m\gamma$. Clearly the positive and negative m levels are not symmetrical about the corresponding $s(m=0)$ levels because the degeneracy of different $|m|$ states has been lifted by the parabolic quantum well, mentioned above. As the γ increases continuously, the energy values of the positive and negative m states close correspondently to those of lower and higher $s(m=0)$ states, and the curves of some negative m states, for example, $3d_-$, $4d_-$, and $4p_-$ states cross those of positive m states. Then, the hybrid levels between Coulomb field levels and Landau (magnetic field) levels are obtained. Clearly there are minima for the the $2p_-$ and $3d_-$ states. It is because of competition between two interactions of the Zeeman term and the parabolic quantum well in a range of small γ. In Fig. 4, it is shown that the degeneracy of Landau levels is lifted by the Coulomb field at high magnetic fields, and that as $m=0$ there is one-to-one correspondence between energy levels of s states at zero magnetic field and Landau levels at high magnetic fields. The energies for other states approach those of the corresponding Landau levels in a finite γ and lie in values between the energies of the s states and the corresponding Landau levels. It is readily seen that the order of the quantum levels for the $N=0$ and 1 Landau levels is, respectively, $1s$, $2p_-$, $3d_-$, and so on and $2s$, p_+, $3p_-$, $4d_-$, and so on. The split level order for the other Landau levels is similar, as shown in the figure. However, the split differences of higher Landau levels are much smaller than those of lower Landau levels. As the γ decreases, all of the values decrease linearly until the hybrid of the levels, mentioned above, happens in a small γ range.

To compare our results with MacDonald and Ritchie's results, we have plotted quantum levels for $1s$, $3d_-$, and $4s$ 2D donor states as a function of the transformed magnetic field $\gamma' = \gamma/(1+\gamma)$ in Figs. 5(a), 5(b), and 5(c). It is

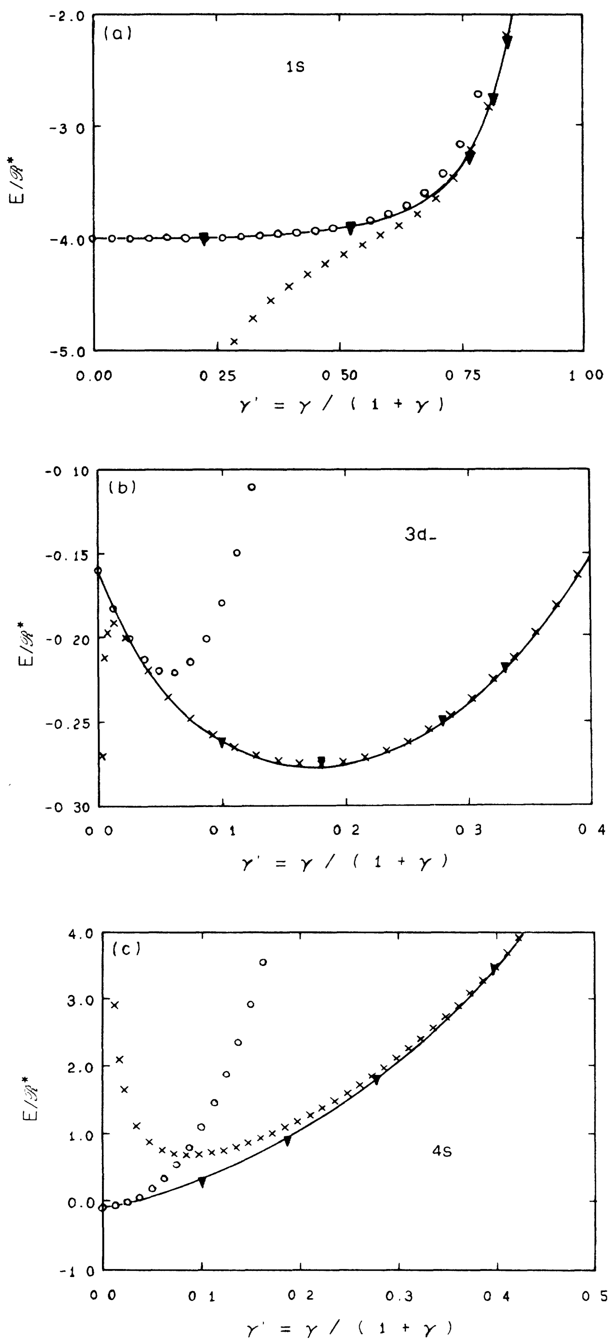

FIG. 5. The 2D donor energy in effective Rydberg units, including the Zeeman term $m\gamma$, is shown as a function of the transformed magnetic field $\gamma'=\gamma/(1+\gamma)$ for the (a) 1s, (b) 3d_-, and (c) 4s states. The circles, crosses, and solid curves represent, respectively, the prediction of the low-field expansion, the high-field expansion and the two-point approximation in Ref. 13. The solid triangles represent the prediction of the text.

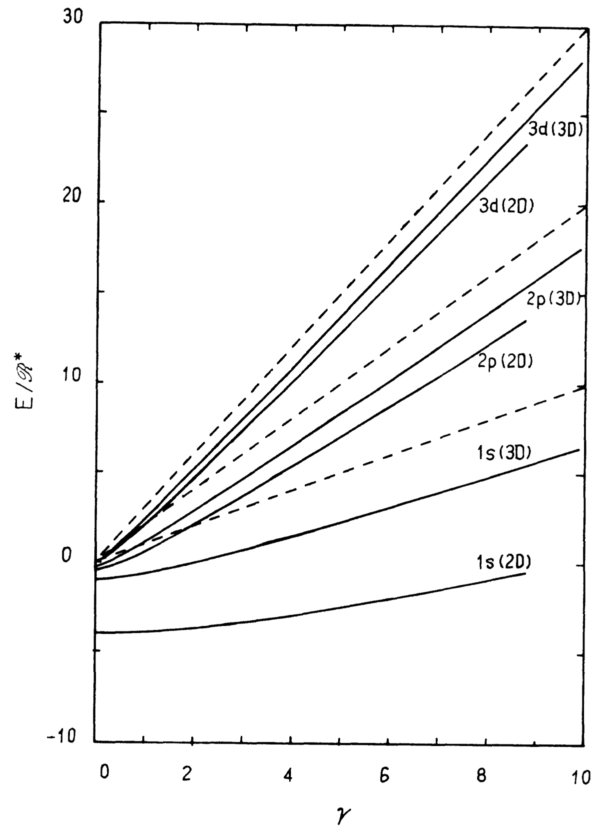

FIG. 6. The 2D and 3D donor energies in effective Rydberg units, excluding the Zeeman term $m\gamma$, are shown as a function of γ for 1s, 2p, and 3d states (see the text). The corresponding energy levels of a free 2D electron confined only by the parabolic quantum well are shown by broken curves for comparison.

readily seen that the difference between ours and MacDonald and Ritchie's[13] is very small in any case. The $m=1$ and -1 levels obtained by MacDonald and Ritchie are also very close to ours. Therefore, it can be concluded that the sequence of the two-point Padé approximation, chosen by them, is reliable and converging, and that the two independent methods yield quantum levels with high accuracy for 2D hydrogenic donor states in arbitrary magnetic fields.

To compare quantum levels of 2D donor states with those of 3D donor states in magnetic field, we have used some data from Makado and McGill.[4] They have chosen basis functions of the form

$$\psi_i=\rho^{|m|}e^{im\varphi}z^q e^{-\alpha_i\rho^2}e^{-\beta_i z^2} \tag{22}$$

and calculated eigenvalues for all states up to $n=4$. For the purpose of making a direct comparison of the results of 2D and 3D donor states, we have restricted ourselves to the case of $q=0$, which includes the ground state and some strongly localized states in the z direction, i.e., the direction of magnetic field. In Fig. 6, excluding the Zeeman term $m\gamma$, we have plotted quantum levels of 1s, 2p, and 3d states of 2D and 3D donors as a function of the γ. It is true that the energy levels of a free 2D electron confined only by the parabolic quantum well are higher than those of the corresponding states of 2D and 3D donors in the well because of the 2D and 3D Coulomb potentials. Clearly the energy of the 2D ground state is much smaller than that of the 3D ground state, and the energy shift of the 2D ground state is also much smaller than that of the 3D ground state. This is because of a stronger confinement in the 2D condition. The shifts of the $2p$ and $3d$ states of a 3D donor are similar to and larger than those of a 2D donor. However, the differences between 2D and 3D conditions are much smaller in the $2p$ and $3d$ states than in the ground states. It should be noted that the other 3D states, which are more extended in the direction of the magnetic field, will be quite different from 2D states.

IV. SUMMARY AND CONCLUSION

Using different series forms in different regions of Eq. (6), we have obtained exact solutions for 2D donor states in a magnetic field by a numerical method and studied the effect of the magnetic field on the quantum levels, binding energies, and the Zeeman splitting. Effects of second order of magnetic field become very appreciable in semiconductors even though it is in a range of intermediate magnetic fields. Calculated results have shown that the quadratic effect makes the degeneracy of 2D donor states in the purely Coulomb potential partly lift and all of the energies increase. Then, the first-order term of magnetic field, i.e., Zeeman term $m\gamma$, makes the degeneracy lift completely. It has been shown that the positive and negative m levels are not symmetrical about the corresponding $s(m=0)$ levels and the hybrid levels are obtained. There are energy minima for some negative m states, shown in Fig. 3. It has also been shown how the degeneracy of Landau levels is lifted by the Coulomb field at high magnetic fields and the kind of order of quantum levels that are obtained. The calculated results are found to be in good agreement with MacDonald and Ritchie's. In addition, based on the comparison between energies for 2D and 3D donor states, we conclude that there is much smaller diamagnetic shift of the ground state of a 2D donor than that of a 3D donor. The shift of some 3D states, which are strongly localized in the direction of magnetic field, are similar to and larger than those of the corresponding 2D donor states.

To close this paper, it is worthwhile to point out that our calculation method is much better than the other methods, such as the Wentzel-Kramers-Brillouin (WKB) approximation, the variation method, and the adiabatic method, and suitable for not only lower magnetic fields but also higher magnetic fields. Finally, it is also worthwhile to point out that using the obtained wave functions of a 2D donor in a magnetic field as trial functions with variation of parameter w shown in Eq. (1), the quantum levels and wave functions of shallow donors and heavy-hole excitons in quantum wells in the presence of magnetic field can be correctly calculated and compared with other calculations[15–17] and experiments.[18–21]

[1]R. J. Elliott and R. Loudon, J. Phys. Chem. Solids **15**, 196 (1960).
[2]C. Aldich and R. L. Greene, Phys. Status Solidi B **93**, 343 (1979).
[3]G. Wunner and H. Ruder, Phys. Lett. **85A**, 430 (1981).
[4]P. C. Makado and N. C. McGill, J. Phys. C **19**, 873 (1986).
[5]J. M. Chamberlain, P. E. Simmonds, R. A. Stradling, and C. C. Bradley, Solid State Commun. **11**, 463 (1972).
[6]A. M. Davison, P. Knowles, P. C. Makado, R. A. Stradling, Z. Wasilewski, and S. Porowski, *Solid State Sciences* (Springer, Berlin, 1981), Vol. 24, p. 84.
[7]G. J. Armistead, P. C. Makado, S. P. Najda, and R. A. Stradling, J. Phys. C **19**, 6023 (1986).
[8]G. Bastard, Phys. Rev. B **24**, 4714 (1981).
[9]S. Chaudhuri, Phys. Rev. B **28**, 4480 (1983).
[10]Wen-Ming Liu and J. J. Quinn, Phys. Rev. B **31**, 2348 (1985).
[11]C. Mailhiot, Y. C. Chang, and T. C. McGill, Phys. Rev. B **26**, 4449 (1982).
[12]Jia-Lin Zhu, J. Phys. Condens. Matter **1**, 1539 (1989).
[13]A. H. MacDonald and D. S. Ritchie, Phys. Rev. B **33**, 8336 (1986), and references quoted therein for the one- and two-point Padé approximation.
[14]A. Erdelyi, *Asymptotic Expansions* (Dover, New York, 1956), Chap. III.
[15]G. Duggan, Phys. Rev. B **37**, 2759 (1988).
[16]S.-R. E. Yang and L. J. Sham, Phys. Rev. Lett. **58**, 2598 (1987).
[17]F. Ancilotto, A. Fasolino, and J. C. Mann Phys. Rev. B **38**, 1788 (1988).
[18]J. C. Mann, G. Belle, A. Fasolino, M. Altarelli, and K. Ploog, Phys. Rev. B **30**, 2253 (1984).
[19]D. C. Rogers, J. Singleton, R. J. Nicholas, C. T. Foxon, and K. Woodbridge, Phys. Rev. B **34**, 4002 (1986).
[20]A. Petrou, G. Waytena, X. Liu, J. Ralston, and G. Wicks, Phys. Rev. B **34**, 7436 (1986).
[21]D. J. Mowbray, J. Singleton, M. S. Skolnick, N. J. Pulsford, S. J. Bass, L. L. Taylor, R. J. Nicholas, and W. Hayes, Superlatt. Microstruct. **3**, 471 (1987).