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# Annealed versus quenched diffusion coefficient in random media

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We examine two *a priori* different definitions of a diffusion coefficient for tracer particles in static random media. The "quenched" diffusion coefficient $D^Q$ characterizes the spread of a packet in a single environment. The "annealed" coefficient $D^A$ characterizes the spread of the configuration-averaged packet. For one-dimensional diffusion with a drift $D^A > D^Q$ whereas for $d>1$ $D^A = D^Q$. For some models displaying anomalous diffusion we also find $D^A > D^Q$.

In this paper, we examine whether diffusion of mutually independent particles in a random environment with static disorder can be characterized by a *unique* diffusion tensor. This question is important in view of the considerable current experimental interest in diffusion in disordered porous materials, such as studies of tracer dispersion in natural rocks or bead packs[1,2] diffusion of polymers in porous glasses[3,4] and gels. Because of spatial inhomogeneities it is not clear *a priori* whether different experiments designed to measure a diffusion coefficient will actually measure the same quantity, even if performed on the same sample. For instance, it is not obvious that out-of-equilibrium experiments measuring the spread in time of a packet of test particles initially concentrated in a small region[1] should give the same result for the diffusion coefficient (DC) as a light-scattering experiment[3] realized on the system in a stationary state.

In this Rapid Communication we study nearest-neighbor random-hopping models described by the master equation:

$$\frac{\partial P_x(t)}{\partial t} = \sum_{x'} (W_{xx'} P_{x'}(t) - W_{x'x} P_x(t)) , \tag{1}$$

where $P_x(t)$ is the occupation probability on site $x$ at time $t$ and the transition rates $W_{xx'}$ are quenched random variables. Such models and their discrete-time or continuous-space versions have been extensively studied in order to describe tracer diffusion and hydrodynamic dispersion in random media.[5,6] In $d=1$, an exact expression for a diffusion coefficient has been obtained by Derrida[7] for (1) with general bond disorder from a study of the stationary state ($t=\infty$) in a periodic medium of period $L\to\infty$. The general site-disordered problem with a bias has been studied by Lehr, Machta, and Nelkin[8] using a different method, involving the opposite order of limits $L=\infty$, $t\to\infty$. Perturbative methods have been developed[9] for $d>1$. However, to our knowledge, it has not been asked whether these methods computed the same quantity.

We now introduce a general definition for the diffusion tensor as a function of the initial distribution $U_y$ of the tracer. Let $P_{xy}(t)$ be the solution of (1) which is concentrated at $y$ at $t=0$: $P_{xy}(t=0)=\delta_{x,y}$. The Laplace transform $P_{xy}(z)$ is the Green's function (GF). We denote the "thermal" average with respect to $P_{xy}$ by $\langle \cdots \rangle_y$, e.g., $\langle x(t)\rangle_y = \sum_x x P_{xy}(t)$. For $d=1$ (with obvious generalization to $d>1$) we define the diffusion coefficient as

$$D(U) = \lim_{t\to\infty} \lim_{\Omega\to\infty} \frac{1}{2t} \left[ \sum_y U_y \langle (x(t)-y)^2\rangle_y - \left[ \sum_y U_y \langle x(t)-y\rangle_y \right]^2 \right], \tag{2}$$

where $\Omega$ is the volume of the system. We now consider two limiting cases of this definition. The first corresponds to the spread of a packet which is initially located at the origin, $U_y=\delta_{y0}$. With this choice (2) yields the "quenched" diffusion coefficient $D^Q$ given by

$$D^Q = \lim_{t\to\infty} \lim_{\Omega\to\infty} (\langle x(t)^2\rangle - \langle x(t)\rangle^2)/2t \tag{3}$$

(here and below $\langle \cdots \rangle \equiv \langle \cdots \rangle_0$). The second case corresponds to tracer particles that are initially spread uniformly through the system, $U_y = 1/\Omega$. The "annealed" coefficient $D^A$ is defined by

$$D^A \equiv \lim_{t\to\infty} \lim_{\Omega\to\infty} \frac{1}{2t} \left[ \frac{1}{\Omega} \sum_y \langle (x(t)-y)^2\rangle_y - \left[ \frac{1}{\Omega} \sum_y \langle x(t)-y\rangle_y \right]^2 \right]. \tag{4}$$

To carry out analytical calculations and gain a better understanding of $D^A$ and $D^Q$ it is convenient to relate these quantities to configurational averages. Let the average over realizations of $\{W_{xx'}\}$ be denoted by $[\cdots]$. If we suppose that $D^Q$ is a self-averaging quantity, i.e., independent of the configuration for almost all configurations, then we have

$$D^Q = \lim_{t\to\infty} \{[\langle x(t)^2\rangle] - [\langle x(t)\rangle^2]\}/2t . \tag{5}$$

This self-averaging property can be verified in some special cases.[7,9] The validity of self-averaging hinges upon the infinite time limit in (3) which insures that the tracer explores a large region of space. An average of many independent spatial regions occurs in $D^A$ because the initial distribution is extended and the infinite volume limit is taken in (4) before the infinite time limit.[10] If we suppose that the spatial averages at fixed time of the first two moments are equal to their configurational averages then we obtain[11]

$$D^A = \lim_{t\to\infty} \{[\langle x(t)^2\rangle] - [\langle x(t)\rangle]^2\}/2t\,. \tag{6}$$

Why might $D^Q$ and $D^A$ differ? Consider a packet of tracer particles initially localized near the origin. After some time the packet $P_{x0}(t)$ can be expected to take a Gaussian scaling form[12] centered at $\langle x(t)\rangle$. The quenched diffusion coefficient gives the width of the packet, $\sim 2D^Qt$. On the other hand, according to (6), the annealed diffusion coefficient gives the width of the configuration-averaged[12] packet $[P_{x0}(t)]$. If the mean displacement $\langle x(t)\rangle$ fluctuates from environment to environment and if these fluctuations grow linearly with time then the two diffusion coefficients will differ. Specifically, combining (5) and (6) we see that

$$D^A - D^Q = \lim_{t\to\infty} \{[\langle x(t)\rangle^2] - [\langle x(t)\rangle]^2\}/2t$$

so that $D^A > D^Q$.

Although the definitions (3) and (4) refer to special initial conditions and infinite time and space limits, the foregoing considerations lead to the conclusion that $D^A$ will be measured in experiments where the spatial extent of the initial distribution is much greater than the diffusion length $(2D^Qt)^{1/2}$, since then tracer particles start in widely separated regions and explore different environments.[10] On the other hand, $D^Q$ will be observed whenever the initial packet is localized in a region much smaller than the diffusion length so that all the tracer particles explore nearly the same environment.

More formally we conjecture[13] that $D(U) = D^A$ if $U$ is "extended" while $D(U) = D^Q$ if $U$ is "localized." Extended distributions are defined via a limiting procedure in which $U$ is normalized to unity for all $\Omega$ and $\Omega U_y$ goes to a finite limit for each $y$ as $\Omega \to \infty$. Localized distributions go to a finite limit normalized to unity, as $L \to \infty$. An example of an extended but nonuniform distribution is the stationary distribution with a current discussed in Ref. 8 and below.

To illustrate the difference between $D^A$ and $D^Q$ we consider a directed walk on a one-dimensional (1D) lattice such that the walkers jump only to the right. Each site $x$ is characterized by a thermal distribution of release time $\psi_x(t)$ with a finite mean $\tau_x$ and variance $\sigma_x$. Model (1) with $W_{x+1,x} = \tau_x^{-1}$, $W_{x-1,x} = 0$ is a directed walk, with $\psi_x(t) = \tau_x^{-1} e^{-t/\tau_x}$ depending on a single time scale $\tau_x = \sqrt{\sigma_x}$. A configuration of disorder is specified by the set of functions $\{\psi_x(t)\}_{x\in Z}$ chosen independently from site to site. For a single particle in a given environment the first passage time at $x$ is $T_x = t_0 + \cdots + t_{x-1}$, each $t_z$ being chosen according to $\psi_z(t)$ (one has $\tau_z = \langle t_z\rangle, \ldots$). $T_x$ is a sum of random variables with *different* distributions. The velocity $V = \lim_{x\to\infty} x/T_x = [\tau_x]^{-1}$ does not depend on the particle or the environment. Thus there is no ambiguity in the definition of a velocity. The diffusion coefficients are related to the fluctuations of the first passage time:

$$D^{A,Q} = V^3 \lim_{x\to\infty} \mathrm{var}_{A,Q}(T_x)/2x\,.$$

We believe that this identity, trivially valid for pure systems, holds quite generally for random media[14] (this is confirmed by some results below). In order to evaluate $D^Q$ one must use $\mathrm{var}_Q(T_x) = [\langle T_x^2\rangle - \langle T_x\rangle^2]$ and $\mathrm{var}_A(T_x) = [\langle T_x^2\rangle] - [\langle T_x\rangle]^2$ to evaluate $D^A$. Since for a given environment $\langle T_x\rangle = \tau_0 + \cdots + \tau_{x-1}$ and $\langle T_x^2\rangle - \langle T_x\rangle^2 = \sigma_0 + \cdots + \sigma_{x-1}$, one finds that

$$D^Q = \frac{[\sigma_z]}{2[\tau_z]^3}\,, \quad D^A = \frac{[\sigma_z] + [\tau_z^2] - [\tau_z]^2}{2[\tau_z]^3}\,. \tag{7}$$

Thus $0 < D^Q < D^A$ as a combined effect of disorder and bias ($D^Q = 0$ if no thermal noise is present).

In dimension $d > 1$ we expect that $D^A_{\mu\nu} = D^Q_{\mu\nu}$, even in the presence of a bias. Here and below we consider the $d$-dimensional version of the biased site-disordered model (BSDM) of Ref. 8; $W_{x'x} = \tau_x^{-1} e^{\epsilon(x_1' - x_1)} \equiv \tau_x^{-1} W^0_{x'x}$, with hopping between nearest neighbors on a hypercubic lattice and $x = (x_1, \ldots, x_d)$. The $\tau_x$ are identically distributed independent random variables and $\epsilon$ is the dimensionless bias. We are free to set $[\tau_x] = 1$ and define $\Delta = [\tau_x^2] - [\tau_x]^2$. Let us first give an intuitive argument, exact for $\epsilon \to \infty$, but which we believe to be generally valid for normal biased diffusion. A packet initially concentrated at $x = 0$ reaches the velocity $V$ in the longitudinal direction 1 but spreads diffusively in the $d-1$ transverse directions as $x_T \sim \sqrt{D_T t}$. Along the longitudinal direction the walk is similar to the directed 1D model, except that the thermal-averaged release time from the plane at $x_1$ now involves an average over $x_T^{d-1} \sim (x_1/V)^{(d-1)/2}$ sites with independent $\tau_x$. The distribution of this quantity as a function of the environment thus goes to a delta function; thus, for $d > 1$, $D^A_{11} = D^Q_{11}$. This argument also shows that there are some exceptions, such as (i) quasi-1D media with finite transverse length $L_T$ for which $D^A_{11} - D^Q_{11} \sim (L_T/\xi)^{1-d}$ (this could be observed in porous media which are heterogeneous on macroscopic scales $\xi$) and (ii) layered or long-range-correlated media: If all $\tau_x$ in a plane are equal, $D^A_{11} \neq D^Q_{11}$.

In Ref. 8 a perturbation expansion in the disorder of an averaged GF was introduced. It permits one to compute $D(U)$ for the 1D BSDM in the case of the stationary distribution $U_y \sim \tau_y/\Omega$. This method is readily extended to any $d$. Details will be given in a future publication. We have shown that only terms up to second order contribute to $D(U)$. The result does not depend on the details of the distribution $U$ (we also checked for $U_y = 1/\Omega$). Thus this method gives an exact determination of $D^A$ and the leading behaviors of $[\langle x_\mu(t)\rangle]$ and $[\langle x_\mu x_\nu(t)\rangle]$. $D^Q$ is extracted from the large-$t$ behavior of $[\langle x_\mu(t)\rangle\langle x_\nu(t)\rangle]$ obtained from a similar perturbative expansion of an average of a convolution of two GF's. We have also applied Derrida's method[7,9] to the BSDM and found that it yields $D^Q$. We believe this is because the initial distribution, though not specified in this method,[7] extends over a finite length $L$, the limit $L \to \infty$ being taken *after* $t \to \infty$.

Based on the methods of Refs. 8 and 9 we show that the final result for the BSDM is $D^{A,Q}_{\mu\nu} = D^0_{\mu\nu}$ for $(\mu\nu) \neq (1,1)$ and $D^{A,Q}_{11} = D^0_{11} + \Delta V^2 G^{A,Q}$ with

$$G^A = \lim_{z\to 0^+} \frac{1}{(2\pi)^d} \int_0^{2\pi} \cdots \int_0^{2\pi} dq_1 \cdots dq_d \frac{1}{z + A(q)}\,,$$
$$G^Q = \lim_{L\to\infty} \frac{1}{L} \sum_{q\neq 0} \frac{1}{A(q)}\,, \tag{8}$$

where $D^0_{\mu\nu} = \cosh(\epsilon)\delta_{\mu\nu}$ is the diffusion tensor of the pure problem $W \equiv W^0$. $V = 2\sinh\epsilon$ is the velocity and

$$A(q) = \sum_{\mu=2,d} 2(1-\cos q_\mu) + 2\cosh\epsilon - 2\cosh(iq_1+\epsilon)\,.$$

Both $G^A$ and $G^Q$ in (8) represent the diagonal element $P^0_{xx}(z=0)$ of the pure GF, but with distinct limiting procedures reflecting the different methods: In $G^A$ the GF is computed in the infinite medium, then the limit $t \to \infty$ $(z \to 0)$ is taken while in $G^Q$ the GF is computed in a periodic medium at $z=0$ (the summation is over $q_\mu = 2\pi n_\mu / L$, $\mu = 1, \dots, d$ with $n_\mu$ integer $0 \le n_\mu < L$) with $L \to \infty$ at the end. This gives the same result for $d > 1$ (leading to $D^A = D^Q$) but not in $d=1$. Both $G^A$ and $G^Q$ can be written[8] as

$$\frac{i}{2\pi} \int \frac{ds}{e^\epsilon (s-r_+)(s-r_-)}\,,$$

with $s = e^{iq_1}$ (the integral is over the positively oriented unit circle and $r_+ = 1$, $r_- = e^{-\epsilon}$). In $G^A$ the pole at $r_+$ is moved outside the unit circle and $G^A = 1/|V|$ from the residue theorem. However in $G^Q$, $r_+$ contributes as a principle part leading to $G^Q = 1/2|V|$. Thus for the 1D BSDM, $D^A = \cosh\epsilon + \Delta|\sinh\epsilon|$ and $D^Q = \cosh\epsilon + (\Delta/2) \times |\sinh\epsilon|$. One recovers (7) (with $[\sigma_z] = [\tau_z^2]$) in the directed limit $\epsilon \to \infty$ (up to trivial rescalings). In Ref. 14, $D^A$ and $D^Q$ are computed using the first-passage time method [used to derive (7)] for arbitrary 1D hopping models. The above results (8) are recovered for the BSDM. For bond disorder (as in Ref. 7, each pair $W_{x\pm 1,x} \equiv \{W_\rightarrow, W_\leftarrow\}$ is chosen independently from bond to bond) one finds $D^A = 2D^Q - \frac{1}{2}(1 + [W_\leftarrow / W_\rightarrow]) \times [1/W_\rightarrow]^{-1}$, the quenched coefficient $D^Q$ being identical to the expression (84) of Ref. 7. Note that $D^Q \le D^A \le 2D^Q$ holds for this model: both coefficients thus diverge with the same exponent at the onset of the regime of anomalous diffusion[7] $[(W_\leftarrow / W_\rightarrow)^2] = 1$.

For a given model, anomalous diffusion occurs below the critical dimension $d_c$ (such that for $d > d_c$ unbiased diffusion is governed by a homogeneous mean-field fixed point corresponding usually to ordinary diffusion $x \sim t^{1/2}$). $d_c = 2$ for (1) with weak uncorrelated bond disorder with asymmetric rates but $d_c$ is higher if disorder has long-range correlations.[6] The annealed and quenched second moments are

$$M^A(t) = [\langle x(t)^2 \rangle] - [\langle x(t) \rangle]^2 \sim D^A t^{2\nu_A}\,,$$

$$M^Q(t) = [\langle (x(t) - \langle x(t) \rangle)^2 \rangle] \sim D^Q t^{2\nu_Q}\,.$$

$D^A$ and $D^Q$ are the *generalized* DC's. In the presence of a bias the average motion of a packet center $[\langle x(t) \rangle]$ may also exhibit[15] anomalous behavior $\sim t^{\nu_1}$, although in general[6] for $d > 1$ it is proportional to $Vt$. These situations are *a priori* possible: (i) $\nu_A = \nu_Q$, $D^A = D^Q$; (ii) $\nu_A = \nu_Q$, $D^A > D^Q > 0$; (iii) $\lim_{t\to\infty} M^Q / M^A = 0$. For $d \le d_c$ the fixed point is disordered and one expects (ii) or (iii).[16] These behaviors can occur with or without a bias as illustrated by the following examples.

(1) *Layered hydrodynamic flow.* We consider the model introduced in Ref. 17 in a time and space discretized version. Brownian particles diffuse in a 2D velocity flow $v(x) = (v_1(x_2), 0)$ parallel to the first axis, the flow velocity being constant in each layer of constant $x_2$. Anomalous diffusion occurs[17] with $x_1 \sim t^{3/4}$. Here $v_1$ is chosen to be $\pm 1$, independently from layer to layer, and the Brownian motion along $x_1$ is neglected as irrelevant. One has $x_1(t) = \sum_k v(k) n(k,t)$, where $n(k,t)$ is the total number of passages at $x_2 = k$ between 0 and $t$ of the pure random walk that each particle performs along $x_2$. One easily shows that

$$M^A(t) = \sum_k \langle n(k,t)^2 \rangle \sim D^A t^{3/2}$$

and

$$M^Q(t) = \sum_k \langle n(k,t)^2 \rangle - \langle n(k,t) \rangle^2 \sim D^Q t^{3/2}\,,$$

with $D^A > D^Q > 0$. One is in situation (ii): The width of a packet (along $x_1$) is comparable with the fluctuations of its center $(\sim t^{3/4})$.

(2) *Broadly distributed waiting time models.* We consider (1) with uncorrelated site disorder $W_{x'x} = \tau_x^{-1}$ and a broad distribution of mean release time $\rho(t) \sim_{\tau\to\infty} \tau^{-(1+\alpha)}$. For $0 < \alpha < 1$ this unbiased model exhibits[18] anomalous diffusion $x \sim t^\nu$ with $\nu = \alpha/2$ $(d > 2)$ and $\nu^{-1} = 2 + 2/\alpha - d$ $(d < 2)$. According to the renormalization-group analysis of Ref. 18, for $d > d_c = 2$ this model behaves on a large scale as a continuous time random walk with a single thermal waiting time distribution $\psi(t)$ at each site. Thus on long time scales there is no difference between quenched and annealed averages. Anomalous diffusion arises from the fact that $\psi(t) \sim_{t\to\infty} t^{-(1+\alpha)}$ $(\langle t \rangle = \infty)$. For $d < 2$ the fixed point is disordered and each site $x$ is decreased by a mean waiting time $\tau_x$ itself distributed with a Levy stable distribution of index $\alpha$. The following argument suggests that this implies $D^A > D^Q$. After time $t$ the configuration-averaged packet is symmetric around the origin and occupies a region of size $t^\nu$. However, in a given environment, sites within this region are occupied in proportion to $\tau_x$. Since a sum of Levy stable variables (for $0 < \alpha < 1$) is dominated by the largest summand and since this largest $\tau_x$ may be anywhere in the region we see that the mean displacement in a given environment also scales as $t^\nu$ so that $D^A - D^Q > 0$. Thus we have case (i) for $d > d_c$ and case (ii) for $d < d_c$.

(3) *Sinai's model.* In $d = 1$, model (1) with uncorrelated bond disorder leads[19] to Sinai's diffusion $[\langle x^2(t) \rangle] \sim \ln^4 t$ in the unbiased case $F = [\ln(W_\rightarrow / W_\leftarrow)] = 0$. However, it seems to be less well known that for a related model Golosov and Sinai proved[19] that in a *given* environment the width of a packet *does not grow with time* and one has $M^A(t) \sim \ln^4 t$ but $M^Q(t) \sim O(1)$, an example of case (iii). We emphasize that this is a remarkable example of "classical" *localization.* Note that the average packet $[P_x(t)]$, whose scaling form is computed in Ref. 15, is very different from the typical one[19] which roughly concentrates on the deepest minimum available at time $t$. In the presence of a bias $F > 0$, this model renormalizes[6,14,15] towards a directed, broadly distributed, waiting time model {with $\alpha$ such that $[(W_\leftarrow / W_\rightarrow)^\alpha] = 1$} and should be in situation (ii).

(4) *Dispersion in long-range correlated flow.* We con-

sider convection and diffusion of a tracer in a stationary flow, $v(x)=v_0+\Delta v(x)$ with $\Delta v(x)$ a long-range correlated random variable; $[\Delta v(x)]=0$ and $[\Delta v(x)\Delta v(x')]\sim|x-x'|^{-g}$ with $0<g<1$. In the absence of thermal noise $M^Q$ vanishes; however, $M^A$ is nonvanishing since tracers in different environments experience different flows. For small fluctuations in the velocity, the deviation $\Delta r(t)$ from the configuration-averaged displacement is $\Delta r(t)=\int_0^t dt'\Delta v(v_0t')$ so that

$$M^A(t)=v_0^{-2}\int_0^{v_0t}\int_0^{v_0t}dx\,dx'[\Delta v(x)\Delta v(x')]\sim\int_0^{v_0t}dR\int_{-R/2}^{R/2}dr\,r^{-g}\sim t^{2-g}, \tag{9}$$

and $\nu_A=1-g/2$. When a small amount of thermal noise is added, a packet initially at the origin spreads, first due to the thermal noise and then due to velocity fluctuations on the length scale of the packet. Velocity fluctuations on longer length scales do not contribute to $M^Q$ since these merely convect the packet as a whole. We can implement this idea as a self-consistent calculation for $M^Q$. Suppose that $M^Q\sim t^{2\nu_Q}$ and then calculate $M^Q$ in the same way as $M^A$ except that the integral over $r$ in (9) is cut off at $r\sim t^{\nu_Q}$. This leads to the relation $2\nu_Q=-\nu_Qg+\nu_Q+1$ or $\nu_Q=1/(1+g)<\nu_A$ so that we have case (iii). Note that one obtains normal diffusion for $g=1$. More precise approaches are needed to test this argument.

As far as the majority of experiments are concerned our conclusions are reassuring: Independent particles in random media are generally described by a unique diffusion tensor. However, we have analyzed some exceptional situations of possible experimental interest, where fluctuations in the environment lead to correlated particle motions and there are differences between the quenched and annealed rms displacement of tracer particles. An alternative way of viewing this phenomena is to regard the environment as inducing an effective interaction between tracer particles which is accounted for in the quenched diffusion coefficient but not in the annealed one. It would be interesting to apply similar ideas to other problems such as diffusion on percolating clusters,[20] diffusion of macromolecules, etc.

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[10]$D(U)$ can be estimated through the useful relation

$$D(U)=D^Q+\lim_{t\to\infty}(4t)^{-1}\sum_{ij}U_iU_j(\langle x(t)-i\rangle_i-\langle x(t)-j\rangle_j)^2.$$

For large but fixed $t$, the spatial summation is evaluated as follows: Two packets originating from points $i$ and $j$ very far apart, $|i-j|\gg(D^Qt)^{1/2}$; see *different environments* and thus give a contribution $\sim 2U_iU_j(D^A-D^Q)$. On the other hand, if $|i-j|\sim(D^Qt)^{1/2}$ this contribution is vanishingly small.

[11]$D^A$ is an "annealed" coefficient since the configurational and thermal average are taken simultaneously whereas $D^Q$ is "quenched" since the configurational average is not taken until the end of the calculation. $D^A$ must not be confused with the diffusion coefficient of the pure model defined by $\tilde{W}_{xx'}^{-1}=[W_{xx'}^{-1}]$ often called the *annealed model.*

[12]By Gaussian scaling form we mean that the rescaled variable $X\equiv(x-\langle x(t)\rangle)/(2D^Qt)^{1/2}$ is such that, in a single environment

$$\lim_{t\to\infty}\text{Prob}\{X<c\}=\int_{-\infty}^{c}du\,(2\pi)^{-1/2}\exp(-u^2/2).$$

Note however, that most of the rigorous results in 1D obtained by probability-theory techniques including the existence of a scaling form (Ref. 15), concern the average packet $[P_{x0}(t)]$, the corresponding rescaled variable being then $X\equiv(x-Vt)/(2D^At)^{1/2}$.

[13]$U$ may have both a localized and extended component, e.g., $U_y=p\delta_{y0}+(1-p)/\Omega$ yields (Ref. 10) $D(U)=p^2D^Q+(1-p^2)D^A$.

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