ϵ expansion for the Nishimori multicritical point of spin glasses

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The renormalization-group recursion relations obtained by Chen and Lubensky for the multicritical point associated with simultaneous critical fluctuations in both the spin-glass and ferromagnetic order are reanalyzed. To first order in $\epsilon \equiv 6-d$ we find that the multicritical fixed point is located inside the Nishimori manifold and that the scaling axes agree with those obtained recently from general arguments. It is confirmed that the scaling along the Nishimori line and at the paramagnetic-spin-glass transition are related. We also point out some universal properties of the multicritical point of possible experimental interest.

This paper concerns the nature of the fluctuations in a random magnet near the multicritical point where the paramagnetic, ferromagnetic, and spin-glass phases coexist. In a recent Letter[1] it was shown that under rather general conditions this multicritical point lies on the so-called Nishimori line,[2] defined by the occurrence of a particular type of gauge symmetry. In addition, it was shown that constraints on the phase diagram led to an unambiguous identification of the scaling axes near this multicritical point. It was also noted that the scaling axes obtained by this general argument agreed with the results of the renormalization-group ϵ expansion of Chen and Lubensky.[3] Here we show this explicitly, since the scaling axes were not given in Ref. 3. In addition, we verify the prediction[4] that the scaling behavior on the Nishimori line (or within the Nishimori manifold if more coupling constants are allowed) can be related to the thermal scaling exponent at the spin-glass–paramagnetic critical point. Finally, in order to manifest the gauge symmetry in replica space which characterizes the Nishimori manifold, we display explicitly the dependence of the results on the number of replicas n, rather than immediately taking the limit $n \to 0$.

We consider an Ising magnet with quenched random bonds. For a given configuration of exchange integrals $J(x,x')$, the Hamiltonian is

$$\mathbf{H} = -\sum_{x,x'} J(x,x') S(x) S(x') , \tag{1}$$

where $S(x)$ is a spin variable at site x which can assume the values $+1$ and -1, and $J(x,x')$ for any given nearest-neighbor bond x—x' is a random variable such that the exchange integral has an average value J and a variance which we denote Δ^2. We use the so-called replica trick which involves calculating $Z^{(n)} \equiv [Z^n]_{\text{av}}$, where $[\ ]_{\text{av}}$ indicates an average over the random variables $J(x,x')$, and $Z = \text{Tr}\exp(-\beta \mathbf{H})$. The quantity $Z^{(n)}$ can be conveniently calculated as the partition function for the x-replicated Hamiltonian $\mathbf{H}^{(n)}$, where

$$\mathbf{H}^{(n)} = -\sum_{x,x'} \sum_{\alpha=1}^{n} J(x,x') S_\alpha(x) S_\alpha(x') . \tag{2}$$

In the limit $n \to 0$ the annealed free energy obtained from $[Z^n]_{\text{av}}$ is identical to the quenched free energy $-kT[\ln Z]_{\text{av}}$ of the model of Eq. (1). It is convenient to recast the partition function for $\mathbf{H}^{(n)}$ in terms of a field theory by performing a Hubbard-Stratonovich transformation in which fields $M_\alpha(\mathbf{r})$ conjugate to $S_\alpha(\mathbf{r})$ and fields $Q_{\alpha,\beta}(\mathbf{r})$ conjugate to $S_\alpha(\mathbf{r}) S_\beta(\mathbf{r})$ are introduced following closely the procedure used by Bray and Moore.[5] The result is then that

$$[Z^n]_{\text{av}} = \int DMDQ e^{-F(M,Q)} , \tag{3}$$

where DM indicates integration over all fields $M_\alpha(\mathbf{r})$ and DQ integration over all fields $Q_{\alpha\beta}(\mathbf{r})$, and the new free-energy functional is

$$F(M,Q) = \tfrac{1}{2}\int d\mathbf{r} \sum_{\alpha=1}^{n} [r_M M_\alpha(\mathbf{r})^2 + |\nabla M_\alpha(\mathbf{r})|^2] + \tfrac{1}{2}\int d\mathbf{r} \sum_{1 \le \alpha < \beta \le n} [r_Q Q_{\alpha,\beta}(\mathbf{r})^2 + |\nabla Q_{\alpha,\beta}(\mathbf{r})|^2]$$
$$-2w_1 \int d\mathbf{r} \sum_{1 \le \alpha < \beta \le n} M_\alpha(\mathbf{r}) M_\beta(\mathbf{r}) Q_{\alpha\beta}(\mathbf{r}) - 6w \int d\mathbf{r} \sum_{1 \le \alpha < \beta < \gamma \le n} Q_{\alpha\beta}(\mathbf{r}) Q_{\beta,\gamma}(\mathbf{r}) Q_{\alpha,\gamma}(\mathbf{r}) . \tag{4}$$

Apart from the sign convention for the cubic potentials, this is exactly the free energy analyzed in some detail by Chen and Lubensky[3] and we follow their treatment closely. (In any case one has $\langle Q_{\alpha\beta}(\mathbf{r}) \rangle = \gamma[\langle S(\mathbf{r}) \rangle^2]_{\text{av}}$. The sign of the cubic potentials we choose is such that the constant γ is positive. This choice of sign is essentially arbitrary.) Also in Eq. (4), $r_M = a(T - T_M^0)$, where α is a constant and T_M^0 is the mean-field transition temperature for ferromagnetic ordering, so that $r_M=a[(kT/zJ)-1]$. Similarly, $r_Q=a'(T-T_Q^0)$, where a' is a constant and T_Q^0 is the mean-field transition temperature for spin-glass ordering, so that $r_Q=a'[(kT/\Delta)^2/z-1]$. The multicritical point we wish to study corresponds in mean-field theory to $r_M=r_Q=0$. The Nishimori line[2] is characterized by the condition[1,4]

$$P(-J)=e^{-2\beta J}P(J) \tag{5}$$

or a generalization thereof when more complicated couplings are allowed.[4] When Eq. (5) (or its generalization) is satisfied, the free-energy functional is invariant under the local gauge transformation $Q_{1\alpha}(\mathbf{r})\rightleftarrows M_\alpha(\mathbf{r})$. Note that this symmetry can occur for integer n and not just for $n\to 0$. In fact, this gauge symmetry must be preserved under the rescaling of the renormalization group.[1,4] Our results for arbitrary n will be consistent with this symmetry. One can easily see that this condition implies that the Nishimori line occurs in the field theory when

$$r_M=r_Q \tag{6a}$$

and

$$w=w_1/3\ . \tag{6b}$$

This gauge symmetry also implies that

$$\eta_Q=\eta_M\ , \tag{6c}$$

where η_Q and η_M are the exponents governing the power-law decay of Q and M correlations, respectively, exactly at the multicritical point. From the general arguments in Ref. 1 it was concluded that under transformation of length scales (as in the renormalization group), one scaling direction lies in the Nishimori subspace and another lies along the temperature axis. The purpose of this paper is to verify that the ϵ-expansion results of Chen and Lubensky[3] agree with this and that Eqs. (6a)–(6c) hold at the multicritical point, where $\epsilon=6-d$, where d is the spatial dimension. Furthermore, it is shown in Ref. 6 that

$$[Z^n(\beta)]_{\text{av,Nishimori}}=\frac{[Z^{n+1}(\beta)]_{\text{av},J=0}}{[Z(\beta)]_{\text{av},J=0}}\ . \tag{7}$$

Here the left-hand side is evaluated at an arbitrary temperature and for a distribution of J's which satisfy the Nishimori condition of Eq. (5). The right-hand side of Eq. (7) is evaluated for $J=0$, which corresponds to the usual spin glass. We wish to check that the ϵ expansion reproduces this interesting result which relates the properties of the spin-glass–ferromagnet–paramagnetic multicritical point (on the Nishimori line[1,4]) to those of the spin-glass–paramagnetic fixed point.

We write the recursion relations of Chen and Lubensky for the Ising case ($m=1$ in their notation) as

$$r_M'=b^2\{r_M-4(n-1)w_1^2[A(0)-K_d(r_M+r_Q)]\ln b -\tfrac{4}{3}(n-1)w_1^2K_dr_M\ln b\}\ , \tag{8a}$$

$$r_Q'=b^2\{r_Q-36(n-2)w^2[A(0)-2K_dr_Q\ln b] -4w_1^2[A(0)-2K_dr_M\ln b] -\tfrac{1}{3}[36(n-2)w^2+4w_1^2]K_dr_Q\ln b\}\ , \tag{8b}$$

$$w'=\{1+(\epsilon/2)\ln b-\tfrac{1}{2}[36(n-2)w^2+4w_1^2]K_d\ln b\} \times\{w+[36(n-2)w^3+\tfrac{4}{3}w_1^3]K_d\ln b\}\ , \tag{8c}$$

$$w_1'=\{1+(\epsilon/2)\ln b-\tfrac{1}{6}[36(n-2)w^2+4w_1^2]K_d\ln b -\tfrac{4}{3}(n-1)w_1^2K_d\ln b\} \times\{w_1+4[w_1^3+3(n-2)ww_1^2]K_d\ln b\}\ , \tag{8d}$$

where $A(0)$ and K_d are constants whose values are not needed here. Also they give

$$\eta_M=\tfrac{4}{3}(n-1)w_1^2K_d\ , \tag{9a}$$

$$\eta_Q=\tfrac{1}{3}[36(n-2)w^2+4w_1^2]K_d\ . \tag{9b}$$

One should note that the recursion relation does preserve the gauge symmetry of the Nishimori line. That is, if we set $r_M=r_Q$ and $w=w_1/3$, then $r_M'=r_Q'$ and $w'=w_1'/3$. At the multicritical point the fixed point values (indicated by the subscript "c") are[7]

$$w_c=\frac{1}{6}\left[\frac{\epsilon}{K_d(1-n)}\right]^{1/2}, \tag{10a}$$

$$w_{1c}=\frac{1}{2}\left[\frac{\epsilon}{K_d(1-n)}\right]^{1/2}. \tag{10b}$$

Note that these values from the ϵ expansion satisfy the expected relation, Eq. (6b), for the Nishimori line. Also when Eq. (6b) is satisfied, $\eta_M=\eta_Q$, again as expected for the Nishimori line [see Eq. (6c)]. Both these statements hold before the limit $n\to 0$ is taken as expected.

The expression in Eq. (10) becomes imaginary for $n>1$. We believe that this reflects the fact that the transition is no longer continuous for $n>1$. In fact, within mean-field theory Sherrington[8] has found that the transition is discontinuous for $n>1$ for values of J and Δ corresponding to the Nishimori point. (In the notation of his Fig. 2, $J^{(2)}=J^{(1)}$.)

Linearizing Eqs. (9c) and (9d) about these fixed point values we get

$$\delta w'=\delta w\{1+\ln b[\tfrac{1}{2}\epsilon+54(n-2)w_c^2K_d-2w_{1c}^2K_d]\} +4K_d\delta w_1(w_{1c}^2-w_cw_{1c})\ln b\ , \tag{11a}$$

$$\delta w_1'=\delta w_1\{1+\ln b[\tfrac{1}{2}\epsilon+(14-4n)w_{1c}^2K_d +24(n-2)w_{1c}w_cK_d+6(2-n)w_c^2]\} -12(2-n)K_d\delta w(w_{1c}^2-w_cw_{1c})\ln b\ , \tag{11b}$$

where $\delta w=w-w_c$ and $\delta w_1=w_1-w_{1c}$. The fixed point of Eq. (10) is stable with respect to variations in the cubic coefficients w and w_1 and the corresponding correction to scaling exponents are

$$\lambda_1=-\epsilon \tag{12a}$$

and

$$\lambda_2=-\epsilon\left[\frac{5-n}{3(1-n)}\right] . \tag{12b}$$

Likewise we write $t_M=r_M-r_{M,c}$ and $t_Q=r_Q-r_{Q,c}$, where the subscript c indicates the value at the multicritical point, in which case Eq. (8) yields

$$t_M'=b^2[(1-(\tfrac{2}{3}\epsilon\ln b)t_M-(\epsilon\ln b)t_Q] , \tag{13a}$$

$$t_Q'=b^2\left[(2\epsilon\ln b)(1-n)^{-1}t_M + \left[1+\frac{(5n-11)}{3(1-n)}\epsilon\ln b\right]t_Q\right] . \tag{13b}$$

To display the scaling vectors g_1 and g_2 unambiguously, we write

$$g_1'\equiv t_M'-t_Q'=b^{\mu_1}(t_M-t_Q)\equiv g_1 , \tag{14a}$$

$$g_2'\equiv t_M'-\tfrac{1}{2}(1-n)t_Q'=b^{\mu_2}[t_M-\tfrac{1}{2}(1-n)t_Q]\equiv g_2 , \tag{14b}$$

with $\mu_1=2-\frac{8}{3}\epsilon(1-\frac{1}{4}n)/(1-n)$ and $\mu_2\equiv[2-(5\epsilon/3)]$. This gives the correlation length exponent $\nu=1/\mu_1=\frac{1}{2}+\frac{2}{3}\epsilon(1-\frac{1}{4}n)/(1-n)$ and a crossover exponent ϕ,

$$\phi\equiv\mu_2/\mu_1=1+\tfrac{1}{2}\epsilon(1+n)/(1-n) . \tag{15}$$

(In Ref. 3 the inequality $\mu_1<\mu_2$ is incorrectly stated.)

Now the initial values are $t_M=(kT/zJ)-1$ and $t_Q=(kT)^2/[z(\Delta^2)]$. The Nishimori direction is $JkT=\Delta^2$, or $t_M=t_Q$, i.e., $g_1=0$. As expected, this condition, $g_1=0$, does not involve n. Along the Nishimori direction scaling is governed by the larger exponent μ_2. If we start at the multicritical point (see Fig. 1) and vary the temperature by an amount dT we find $t_M=dT/T_N$ and $t_Q=2dT/T_N$, where T_N is the temperature at the multicritical point. Thus along the temperature direction we have $g_2=0$. Along this line scaling is governed by the exponent μ_1.

We now point out two simple properties of the multicritical fixed point which could be of experimental interest. The first of these is that spin-glass susceptibility χ_Q and the ferromagnetic susceptibility χ_M are asymptotically equal as the multicritical point is approached along any straight line not tangent to the phase boundary. This is a consequence of (i) the fact that since μ_2 is larger than μ_1, approaching the multicritical point along any straight line except that tangent to the paramagnetferromagnet phase boundary is equivalent to approaching the multicritical point along the Nishimori line, and (ii) the fact that χ_Q and χ_M are equal on this line.[1] The second property concerns the nonlinear susceptibility, χ_{NL}. For spin glasses one often measures the dependence of the uniform susceptibility χ_M on the uniform field H. Near the transition from the ferromagnetic to spin-glass phase, one has

$$\chi_M(H)=\chi_0+\chi_{NL}H^2\cdots , \tag{16}$$

where χ_0 is finite and χ_{NL} diverges with the same exponent γ_Q as the Q susceptibility. Near the multicritical point the situation is quite different. Here χ_0 diverges as

$$\chi_0\sim|T-T_c|^{-\gamma}\equiv t^{-\gamma} , \tag{17}$$

and furthermore we expect that the free energy depends on H through the scaling variable H/t^{Δ}, where $\Delta=\beta+\gamma$. Of course, in view of the symmetry of the Nishimori point we do not have to distinguish between exponents for M and those for Q. Thus at the Nishimori point we have

$$\chi_{NL}\sim t^{-3\gamma-2\beta} . \tag{18}$$

Thus the nonlinear susceptibility is predicted to be more strongly divergent at the multicritical point than at the usual spin-glass critical point.

Furthermore, from Eq. (7) we see that the partition function for the spin glass (for $n\to 0$ and $J=0$) is proportional to $[Z^n]_{av}$ for $n\to -1$ evaluated at the Nishimori point. The relation (7) already implies the remarkable relation that the $n\to -1$ replica Nishimori point occurs at the same temperature as the $n\to 0$ replica spin-glass transition.[4,9] However, since we find that μ_2 is independent of n, we assume that our result $\mu_2=2-5\epsilon/3$ holds even as $n\to 1$. This argument therefore predicts that μ_2 is equal to the thermal scaling exponent λ_t for the spin-glass thermal order-disorder fixed point, in agreement with the first order in ϵ result[10] $\lambda_t=2-5\epsilon/3$. More generally, $\lambda_t=\mu_2(n=-1)$.

Let us make the following remark which is important for discussing possible application of these results to real spin glasses. It is obvious from Eq. (5) that unlike the models with Gaussian or $\pm J$ distributions,[1] the vast majority of models do not possess a Nishimori line (or manifold). For example, consider the model, for which

$$P(J)=(1-p)\delta(J-1)+p\delta(J+a) , \tag{19}$$

where $0<a\neq 1$. For suitable values of p and T this model has a multicritical point of the type we have been considering but Eq. (5) is never satisfied: there is no Nishimori manifold for this model. The fact that the multicritical fixed point lies outside the parameter space of this model is a familiar phenomenon: at this multicritical point this

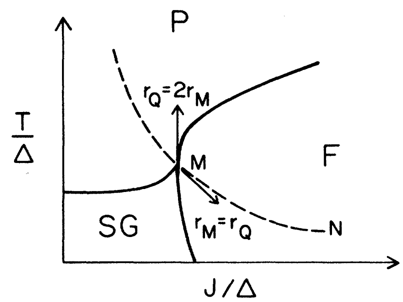

FIG. 1. Topology of the multicritical point where ferromagnetic (F), paramagnetic (P), and spin-glass (SG) phases coexist. The vertical axis is temperature T and the horizontal axis is disorder Δ. The dashed line is the Nishimori line. The slope of the F phase boundary at the multicritical point is infinite as required by general arguments. (Refs. 1 and 4).

model differs from the coupling constants at the fixed point by potentials which are irrelevant in the renormalization-group sense. At the multicritical fixed point, although not over the entire multicritical surface, the coupling constants obey the Nishimori gauge symmetry of Eq. (5). In other words, the potentials for models like that of Eq. (19) which do not obey Nishimori symmetry at the multicritical point are irrelevant. Thus the multicritical behavior of systems which do not have a Nishimori manifold are nevertheless asymptotically controlled by that remarkable symmetry.

To conclude we may summarize our results. One can look at the renormalization group fixed point corresponding to simultaneous criticality of ferromagnetic order M and spin-glass order Q. For simple models, this fixed point is known[1,4] to lie in the Nishimori manifold, and at the corresponding field point the values of the coupling constants and the critical point exponents η_M and η_Q are found to obey the relations obtained by general arguments[1,4] due to the gauge symmetry of the Nishimori manifold. For systems for which the multicritical point is in the Nishimori manifold it is found that the eigendirections correspond as expected: one lies along the temperature axis the other in the Nishimori manifold. The eigenvalue corresponding to the Nishimori manifold is the larger one, so that the crossover exponent ϕ as usually defined [see Eq. (15)] is greater than unity. In addition, the scaling exponent corresponding to the Nishimori manifold is related to that of the thermal spin-glass order-disorder fixed point,[10] again as expected from general arguments.[4] The asymptotic behavior near the multicritical point displays Nishimori gauge symmetry even when the bare Hamiltonian lacks this symmetry.

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[7]If we set $x_1=w_1\sqrt{K_d/\epsilon}$ and $x=w\sqrt{K_d/\epsilon}$, then the fixed point values (for $x_1\neq 0$) are determined by

$$[4x_1^2(1-n)-1][4x_1^4(2n^2+55n-121)+x_1^2(5n-22)-3]=0$$

and

$$x=x_1\pm\tfrac{1}{2}[(176x_1^2-112nx_1^2-12)/(2-n)]^{1/2} .$$

The only solutions we will analyze are those written in Eq. (10). The fixed point with $x_1=0$ and $x=1/(6\sqrt{2-n})$ describes the spin-glass ordering transition.

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[9]It is interesting to note that $w_{c,\mathrm{SG}}$, the fixed point value of w for the spin-glass order-disorder fixed point, given as $w_{c,\mathrm{SG}}=\frac{1}{6}\sqrt{\epsilon/[K_d(2-n)]}$, can be obtained from w_c of Eq. (10a) by replacing n by $n-1$ as might be suggested by Eq. (7).

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