Finite-size scaling of O(n) models with long-range interactions

Surjit Singh
Picosecond and Quantum Radiation Laboratory, Texas Tech University, P.O. Box 4260, Lubbock, Texas 79409-4260

R. K. Pathria
Guelph-Waterloo Program for Graduate Work in Physics, Waterloo Campus, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

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A scaling hypothesis is set up for the magnetization $m(T,H;L)$, susceptibility $\chi(T,H;L)$, and correlation length $\xi(T,H;L)$ of a finite-sized system, with O(n) symmetry ($n \geq 2$) and long-range interactions decaying as $1/r^{d+\sigma}$ ($0<\sigma<2$), confined to geometry $L^{d-d'} \times \infty^{d'}$ ($\sigma<d<2\sigma, d' \leq \sigma$) and subjected to periodic boundary conditions. Finite-size effects are predicted, in the region of a first-order phase transition ($T<T_c$) as well as in the region of a second-order phase transition ($T \simeq T_c$), for different regimes of the parameters H and L. To test these predictions, a detailed analytical study is carried out in the case of the spherical model of ferromagnetism ($n=\infty$), and all predictions based on the scaling hypothesis are seen to be fully borne out. In situations where $L \gg \xi$, finite-size corrections to standard bulk values of the various physical quantities pertaining to these models are found to vary as $(\xi/L)^{d+\sigma}$, rather than as $e^{-L/\xi}$, which is characteristic of models with short-range interactions.

I. INTRODUCTION

In recent years considerable attention has been paid to the study of finite-size effects in systems undergoing phase transitions. For the most part, this study has been based on the scaling hypothesis initially put forward by Privman and Fisher[1] for application in the region of second-order phase transitions ($T \simeq T_c$) and later generalized by Singh and Pathria[2] for application to the region of first-order phase transitions ($T<T_c$) as well. At the same time, exact calculations have been done that pertain to models with O(n) symmetry ($n \geq 2$) confined to geometry $L^{d-d'} \times \infty^{d'}$ (with $d_< < d < d_>$ and $d' \leq d_<$, where $d_<$ and $d_>$ are, respectively, the lower and upper critical dimensions of the system). Not surprisingly, calculations for general n have often been restricted to special geometries[1,3] such as the "block" ($d'=0$) or the "cylinder" ($d'=1$), while those for general d' have been restricted to models with special symmetry[2,4] such as the "mean spherical model" ($n=\infty$). Furthermore, barring a few exceptions, the work along these lines has been confined almost entirely to systems with *short-range* interactions—exceptions being the work of Fisher and Privman[5] and of Brankov and Tonchev[6,7] on the spherical model with *long-range* interactions decaying as $1/r^{d+\sigma}$, with $0<\sigma<2$. While the former authors have studied properties such as the magnetization $m(T,H;L)$ and susceptibility $\chi(T,H;L)$ of a finite-sized system in the presence of an external field H, the latter have concentrated on the "singular" part of the free energy density $f^{(s)}(T,H;L)$ and its derivatives with respect to T and H in the limit $H \to 0$; in each case, the geometry of the system is restricted to that of the "block" or at best the "cylinder". This prompted us to explore the problem of long-range models in general geometry $L^{d-d'} \times \infty^{d'}$ in a manner similar to the one adopted earlier for short-range models.[8] The results of that exploration are reported here.

In Sec. II we set up scaling hypotheses for the magnetization $m(T,H;L)$, susceptibility $\chi(T,H;L)$, and correlation length $\xi(T,H;L)$ for a system with an arbitrary interaction potential $u(\mathbf{r})$ whose Fourier transform has a long-wavelength exponent σ such that $0<\sigma \leq 2$; as is well known, the case $\sigma=2$ pertains to models with short-range interactions (including the nearest-neighbor one) whereas $\sigma<2$ pertains to models with long-range interactions decaying as $1/r^{d+\sigma}$. The system in either case is confined to geometry $L^{d-d'} \times \infty^{d'}$ ($\sigma<d<2\sigma$, $d'<\sigma$) and is subjected to periodic boundary conditions. For $T<T_c$, we adopt the scaled variables[2-4]

$$z_1 \sim \frac{M_0^2(T)}{A(T)} L^{d-\sigma}, \quad z_2 \sim \frac{m_0(T)H}{T} L^d , \tag{1}$$

where $M_0(T)$ is the spontaneous magnetization *per spin* and $A(T)$ a system-dependent coefficient, which appear in the standard expression for the field-free *bulk* correlation function[9,10]

$$G(\mathbf{R},T)=M_0^2(T)+\frac{A(T)}{R^{d-\sigma}} \quad (T<T_c) ; \tag{2}$$

the quantity $m_0(T)$ appearing in the definition of z_2 is, however, spontaneous magnetization *per unit volume* of the (bulk) system.[11] Arguing on the basis of the singularity encountered by a d'-dimensional bulk system as $T \to T_c(d')=0$, we are able to make definitive predictions for the various properties of the finite-sized system in the regime $z_2 \ll z_1$, i.e., $L \ll \xi_\infty(T,H)$, where

$$\xi_\infty(T,H) \sim (TM_0^2/m_0AH)^{1/\sigma}$$

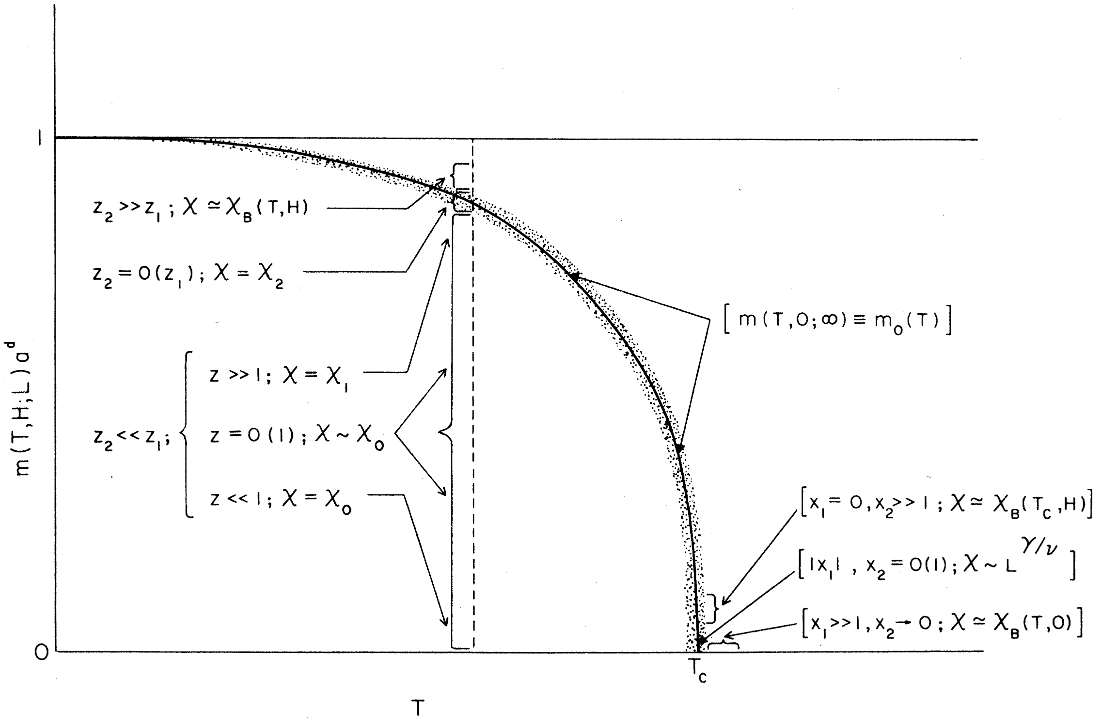

FIG. 1. Schematic plot of magnetization $m(T,H;L)$ against temperature T of a finite-sized system in geometry $L^{d-d'}\times\infty^{d'}$. The bulk phase boundary $m_0(T)$, which serves as a useful line of reference for the actual system, corresponds to a *universal* value of the ratio z_2/z_1. Different regimes, in which the susceptibility χ depends in a specific manner on the variables H and L, are shown (i) for a *fixed* $T<T_c$ as well as (ii) for $T\simeq T_c$; the quantities χ_0, χ_1, and χ_2 appearing in the sketch are given by Eqs. (13b), (18), and (26), respectively, while χ_B denotes the bulk susceptibility pertaining to the regime in question.

is the bulk correlation length at $T<T_c$ for $H>0$. This regime, we find, is governed by a special combination z of the variables z_1 and z_2, viz.,

$$z=z_2z_1^{d'/(\sigma-d')}\sim HL^{\sigma(d-d')/(\sigma-d')}\quad(d'<\sigma)\ , \tag{3}$$

which determines the scaling functions of the various quantities of interest almost single handedly. The combination z enables us to cover essentially the whole of the (m,T) plane *below* the phase boundary of the bulk system; see Fig. 1. As the boundary is approached from below, we encounter the "spin-wave region," $L=O(\xi_\infty(T,H))$, where $|m-m_0|\sim L^{-(d-\sigma)}$ and $\chi\sim L^{2\sigma-d}$. For $L\gg\xi_\infty(T,H)$, we recover the appropriate bulk result, viz., $\chi\sim H^{-(2\sigma-d)/\sigma}$, with finite-size corrections in the variable ξ_∞/L. A distinctive feature of the long-range models discovered here is that these correlations are no longer exponentially small; they vary instead as $(\xi_\infty/L)^{d+\sigma}$.

Next, we explore the region of second-order phase transition ($T\simeq T_c$) in which the variables z_1 and z_2 get replaced by the conventional variables[1]

$$x_1=C_1L^{1/\nu}t\,[t=(T-T_c)/T_c\ ,$$
$$x_2=C_2L^{\Delta/\nu}H/T_c\ ; \tag{4}$$

here, too, the nonuniversal parameters C_1 and C_2 pertain to the corresponding bulk system. Once again, we make predictions about the quantities m, χ, and ξ in various regimes of the variables x_1 and x_2—especially in the absence of the field ($x_2=0$), with x_1 of order unity or much greater than unity, and at the bulk critical point ($x_1=0$), with x_2 of order unity or much greater than unity.

In Sec. III we report the results of a detailed, analytical evaluation of the quantities $m(T,H;L)$, $\chi(T,H;L)$, and $\xi(T,H;L)$ for the spherical model of ferromagnetism ($n=\infty$), in geometry $L^{d-d'}\times\infty^{d'}$ ($\sigma<d<2\sigma$, $d'\le\sigma$), in different regimes of the parameters T, H, and L, and compare them with the predictions made in Sec. II. This requires a substantially different line of analysis from the one pertaining to models with short-range interactions; nonetheless, all the predictions based on the scaling hypotheses are fully borne out. In Sec. IV we examine the special case $d'=\sigma$ which, at temperatures below T_c, differs qualitatively from the case $d'<\sigma$. Finally, in Sec. V, we conclude the paper with some closing remarks on the problem under study.

II. SCALING PREDICTIONS IN THE REGIONS OF FIRST- AND SECOND-ORDER PHASE TRANSITIONS

Case 1: $T<T_c$

In this region the "singular" part of the free energy density of the system may be written in the form[3,4,8]

$$f^{(s)}(T,H;L)\approx(T/L^{d})W_f(z_1,z_2)\ , \tag{5}$$

where $W_f(z_1,z_2)$ is a universal function of the variables z_1 and z_2. The magnetization m and susceptibility χ per unit volume of the system are then given by

$$m(T,H;L)=-(\partial f^{(s)}/\partial H)_{T,L}\approx m_0W_m(z_1,z_2) \tag{6}$$

and

$$\chi(T,H;L)=-(\partial^2 f^{(s)}/\partial H^2)_{T,L} \approx(m_0^2L^d/T)W_\chi(z_1,z_2)\ , \tag{7}$$

where the scaling functions W_m and W_χ are again universal. Formulas (5)–(7) may be supplemented with the subsidiary hypothesis[4]

$$\xi(T,H;L)\approx LW_\xi(z_1,z_2)\ , \tag{8}$$

where W_ξ is the universal scaling function for the correlation length ξ of the system.

Now, in the zero-field limit ($z_2\to 0$), the scaling function W_m should vanish while W_χ should be such as to reproduce the "low-temperature behavior of a d'-dimensional bulk system," viz., $\chi_{T\to 0}\sim T^{-\dot\gamma}$, where $\dot\gamma=\sigma/(\sigma-d')$ for $d'<\sigma$. We may, therefore, write

$$\left.\begin{aligned} W_m(z_1,z_2)&\approx W_0z_2z_1^{\theta} \quad &(9a)\\ W_\chi(z_1,z_2)&\approx W_0z_1^{\theta} \quad &(9b)\end{aligned}\right\}\ (z_1\gg 1,\ z_2\to 0)$$

with W_0 and θ universal. To determine θ, we observe that if the coefficient $A(T)$, which appears in the definition of the variable z_1, varies, in the limit $T\to 0$, as T^q, then Eqs. (7) and (9b) yield the limiting result

$$\chi_{T\to 0}\sim T^{-(1+q\theta)}L^{d+(d-\sigma)\theta}\ . \tag{10}$$

To reproduce the desired T dependence, we obviously require $q\theta=\dot\gamma-1$, whence

$$\chi_{T\to 0}\sim T^{-\dot\gamma}L^{d+(d-\sigma)(\dot\gamma-1)/q}\ . \tag{11}$$

At this point we recall a general result derived on the basis of considerations on the "fluctuations in the order parameter of the system," with $d_<=\sigma$, namely,[12]

$$\chi_{T<T_c}\sim L^{\sigma(d-d')/(\sigma-d')}\quad (d'<\sigma)\ . \tag{12}$$

Comparing (11) and (12), we infer that $q=1$ and hence $\theta=\dot\gamma-1=d'/(\sigma-d')$. Equations (6), (7), and (9) now give

$$m\approx\chi_0H\ , \tag{13a}$$

$$\chi_0\approx W_0(m_0^2L^d/T)z_1^{d'/(\sigma-d')} \sim(m_0^2/T)(M_0^2/A)^{d'/(\sigma-d')}L^{\sigma(d-d')/(\sigma-d')} \quad (L\gg a,\ \ H\to 0)\ , \tag{13b}$$

where a denotes a microscopic length, such as the lattice constant, of the system. By similar argument, we obtain for the correlation length (in the limit $H\to 0$)

$$\xi_0\approx X_0Lz_1^{1/(\sigma-d')}\sim(M_0^2/A)^{1/(\sigma-d')}L^{(d-d')/(\sigma-d')}\ , \tag{14}$$

with X_0 universal. We readily infer that the ratio $\chi_0/\xi_0^{\sigma}\sim A/Ta^{2d}$, and is independent of L.

Equation (9a) suggests that, for a certain range of H, the variables z_1 and z_2 may appear in the combination $(z_2z_1^{\dot\gamma-1})$ only; it turns out that this indeed is the case and the relevant range is governed by the condition $z_2\ll z_1$, i.e., $L\ll\xi_\infty(T,H)$ or, in other words, $(H/T_c)\ll(a/L)^{\sigma}$. We may then write[8]

$$\left.\begin{aligned} m&\approx m_0M(z) \quad &(15a)\\ \chi&\approx\frac{m_0}{H}\left[z\frac{dM}{dz}\right] \quad &(15b)\end{aligned}\right\}\ (z=z_2z_1^{\dot\gamma-1}\propto H)\ ,$$

with the stipulation that, for vanishing z,

$$M(z)\approx W_0z\quad (z\to 0) \tag{16}$$

so that Eqs. (9) are correctly reproduced. At the other extreme, where $z\gg 1$, i.e., $z_2\gg z_1^{-d'/(\sigma-d')}$ (though z_2 is still much less than z_1), we require the scaling function $M(z)$ to reproduce the "low-temperature behavior of the d'-dimensional bulk system with $H>0$," viz., $\chi_{T\to 0}(H>0)\sim TH^{-(2\sigma-d')/\sigma}$. This, in turn, requires that

$$z(dM/dz)\approx W_\infty z^{-(\sigma-d')/\sigma}\quad (z\gg 1) \tag{17}$$

with the result that

$$\chi\sim(m_0A/TM_0^2)^{d'/\sigma}TH^{-(2\sigma-d')/\sigma}L^{-(d-d')} \times[(T/T_c)^{\dot\gamma}(a/L)^{\zeta}\ll(H/T_c)\ll(a/L)^{\sigma}]\ , \tag{18}$$

where

$$\zeta=\sigma(d-d')/(\sigma-d')\ge d>\sigma\ . \tag{19}$$

In this situation the system finds itself very close to the phase boundary of the bulk system (see Fig. 1), with

$$(m-m_0)\sim-(m_0A/TM_0^2)^{d'/\sigma}TH^{-(\sigma-d')/\sigma}L^{-(d-d')}\ . \tag{20}$$

For $z_2\gg z_1$, we expect to recover the appropriate d-dimensional bulk result, viz., $\chi_{T\to 0}\ (H>0)\sim TH^{-(2\sigma-d)/\sigma}$. The function W_χ of Eq. (7) should then behave as

$$W_\chi(z_1,z_2)\approx W_Bz_1^{-d/\sigma}z_2^{-(2\sigma-d)/\sigma}\quad (z_2\gg z_1\gg 1)\ , \tag{21}$$

with W_B universal. It follows that

$$W_m(z_1,z_2)\approx 1+\frac{\sigma W_B}{d-\sigma}z_1^{-d/\sigma}z_2^{(d-\sigma)/\sigma}\quad (z_2\gg z_1\gg 1) \tag{22}$$

Equation (22) suggests that the bulk limit for $T<T_c$ is governed by the combination

$$z_B=z_2z_1^{-d/(d-\sigma)}\sim(m_0H/T)(A/M_0^2)^{d/(d-\sigma)}\ , \tag{23}$$

and that expressions (21) and (22) apparently hold only when $z_B\ll 1$, i.e., $(H/T_c)(T/T_c)^{\sigma/(d-\sigma)}\ll 1$. It is now straightforward to see that, in this limit,

$$\chi\sim(m_0A/TM_0^2)^{d/\sigma}TH^{-(2\sigma-d)/\sigma} \tag{24}$$

and

$$(m - m_0) \sim (m_0 A / TM_0^2)^{d/\sigma} T H^{(d-\sigma)/\sigma} . \tag{25}$$

Of course, for a finite-sized system, Eqs. (24) and (25) are expected to be modified by corrections which, in the case of long-range interactions, are found to vary as an algebraic power of the variable (ξ_∞ / L).

Finally, in the spin-wave region, where $z_2/z_1 = O(1)$, i.e., $L = O(\xi_\infty(T,H))$ or $(H/T_c) = O((a/L)^\sigma)$, we find that

$$\chi \sim T(m_0 A / TM_0^2)^2 L^{2\sigma - d} \tag{26}$$

and

$$|m - m_0| / m_0 \sim A / M_0^2 L^{d-\sigma} , \tag{27}$$

regardless of whether we approach this region from below [via Eqs. (18) and (20)] or from above [via Eqs. (24) and (25)]—with H becoming of order $(TM_0^2 / m_0 A L^\sigma)$. This completes our predictions in the region *below* T_c.

Case 2: $T \simeq T_c$

The scaling predictions in this region are formally similar to the ones for systems with short-range interactions. However, since comparison with actual calculations is intended, we need to quote the main results here.

The scaling hypothesis for $f^{(s)}$ may now be written in terms of the variables x_1 and x_2 defined in Eqs. (4), that is,[1]

$$f^{(s)} \approx (T_c / L^d) Y_f(x_1, x_2) , \tag{28}$$

where $Y_f(x_1, x_2)$ is a universal function of the arguments x_1 and x_2. It follows that

$$m \approx C_2 L^{-\beta/\nu} Y_m(x_1, x_2) \tag{29}$$

and

$$\chi \approx (C_2^2 / T_c) L^{\gamma/\nu} Y_\chi(x_1, x_2) , \tag{30}$$

where the scaling functions Y_m and Y_χ are also universal. Once again, we supplement formulas (28)–(30) with the subsidiary hypothesis[1]

$$\xi \approx L Y_\xi(x_1, x_2) , \tag{31}$$

where Y_ξ is the scaling function for ξ.

Now, in the *absence* of the field and with $t > 0$ and $L \to \infty$, we expect to recover the standard bulk behavior, viz., $\chi \sim t^{-\gamma}$ and $\xi \sim t^{-\nu}$. We, therefore, require that

$$\left.\begin{aligned} Y_\chi(x_1, 0) &\approx G_+ x_1^{-\gamma} \quad (32a) \\ Y_\xi(x_1, 0) &\approx S_+ x_1^{-\nu} \quad (32b) \end{aligned}\right\} \quad (x_1 \to +\infty)$$

with G_+ and S_+ universal. It follows that

$$\left.\begin{aligned} \chi_0 &\approx G_+ (C_2^2 / C_1^\gamma T_c) t^{-\gamma} \quad (33a) \\ \xi_0 &\approx S_+ C_1^{-\nu} t^{-\nu} \quad (33b) \end{aligned}\right\} \quad (t > 0,\ L \to \infty) .$$

On the other hand, in the *presence* of the field and with $t = 0$ and $L \to \infty$, we expect to recover the behavior $m_c \sim H^{1/\delta}$ and $\xi_c \sim H^{-\nu/\Delta}$ which, in turn, requires that

$$\left.\begin{aligned} Y_m(0, x_2) &\approx M_c x_2^{1/\delta} \quad (34a) \\ Y_\xi(0, x_2) &\approx S_c x_2^{-\nu/\Delta} \quad (34b) \end{aligned}\right\} \quad (x_2 \to \infty) ,$$

with M_c and S_c universal. It now follows that

$$\left.\begin{aligned} m_c &\approx M_c C_2^{(\delta+1)/\delta} (H/T_c)^{1/\delta} \quad (35a) \\ \xi_c &\approx S_c C_2^{-\nu/\Delta} (H/T_c)^{-\nu/\Delta} \quad (35b) \end{aligned}\right\} \quad (t = 0,\ L \to \infty) .$$

For susceptibility, one readily obtains

$$\chi_c \approx (M_c / \delta)(C_2^{(\delta+1)/\delta} / T_c)(H/T_c)^{-(\delta-1)/\delta} \tag{36}$$

Finally, in the "core region," where both $|x_1|$ and x_2 are of order unity, we expect $m \sim L^{-\beta/\nu} \sim H L^{\gamma/\nu}$, $\chi \sim L^{\gamma/\nu}$, and $\xi \sim L$. This completes our predictions in the region *close to* T_c.

III. VERIFICATION OF SCALING PREDICTIONS IN THE CASE OF THE SPHERICAL MODEL OF FERROMAGNETISM

For comparison with scaling predictions we have carried out a detailed analytical study of a spherical-model system with long-range interactions decaying as $1/r^{d+\sigma}$ $(0 < \sigma < 2)$, confined to geometry $L^{d-d'} \times \infty^{d'}$ $(\sigma < d < 2\sigma,\ d' \le \sigma)$ and subjected to periodic boundary conditions. Since the derivation of the various results in this case differs substantially from the one pertaining to short-range interactions, we present some of the essential steps in Appendixes A and B. Our final results are summarized below.

Case 1: $T < T_c$

In view of the fact that the quantities $M_0(T)$ and $A(T)$ of the corresponding bulk system are of the form[9]

$$M_0(T) \sim \left[\frac{K - K_c}{K}\right]^{1/2} , \quad A(T) \sim \frac{a^{d-\sigma}}{K} \quad \left[K = \frac{J}{T}\right] \tag{37}$$

our scaled variables z_1 and z_2 must be such that, see Eqs. (1),

$$\begin{aligned} z_1 &\sim (K - K_c)(L/a)^{d-\sigma} , \\ z_2 &\sim (1 - K_c/K)^{1/2} (L/a)^d (H/T) . \end{aligned} \tag{38}$$

For consistency with the case $\sigma = 2$, we adopt

$$z_1 = 2|\tilde{x}_1| = 2(K - K_c)(L/a)^{d-\sigma} , \tag{39a}$$

$$z_2 = |\tilde{x}_1|^{1/2} \tilde{x}_2 = (1 - K_c/K)^{1/2} (L/a)^d (H/T) , \tag{39b}$$

where $\tilde{x}_1$ and $\tilde{x}_2$ are defined in Eqs. (A11). The constraint equation (A12) then takes the form

$$\left[\frac{z_2^2}{4^\sigma z_1^2 y^{2\sigma}}-1\right]z_1=\frac{y^{d-\sigma}}{\sigma 2^{\sigma-1}\pi^{d/2}}\left\{\left[\Gamma\left[\frac{d-\sigma}{\sigma}\right]\Gamma\left[\frac{2\sigma-d}{\sigma}\right]\Big/\Gamma(d/2)\right]-\sigma\mathcal{K}_\sigma\left[\frac{d-2}{2}\Big|d^*;y\right]\right\}, \tag{40}$$

where the scaled length parameter y and the function $\mathcal{K}_\sigma$ are defined by Eqs. (A8) and (A10), respectively. Equation (40) determines y as a function of z_1 and z_2. The scaling functions W_m and W_χ of Eqs. (6) and (7) turn out to be

$$W_m(z_1,z_2)=\frac{z_2}{z_1(2y)^\sigma}, \tag{41a}$$

$$W_\chi(z_1,z_2)=\left[\frac{\partial W_m}{\partial z_2}\right]_{z_1}, \tag{41b}$$

while the scaling function W_ξ of Eq. (8) is given by

$$W_\xi(z_1,z_2)\equiv\frac{\xi}{L}=\frac{a/\phi^{1/\sigma}}{L}=\frac{1}{2y}; \tag{42}$$

see Eq. (A4), which prompts a simple relationship between the correlation length ξ and the parameters a and ϕ.

For most of the region below the phase boundary of the bulk system (see Fig. 1), the parameter y is much less than unity. The function $\mathcal{K}_\sigma$ may then be approximated by the asymptotic expression (B4) which holds for $d'<\sigma$; the constraint equation then becomes

$$(W_m^2-1)z_1\approx-\Gamma\left[\frac{d'}{\sigma}\right]\Gamma\left[\frac{\sigma-d'}{\sigma}\right]\Big/\sigma 2^{\sigma-1}\pi^{d'/2}\Gamma\left[\frac{d'}{2}\right]y^{\sigma-d'}\quad(d'<\sigma). \tag{43}$$

Eliminating y between Eqs. (41a) and (43), we obtain

$$(W_m^2-1)+\frac{2}{\sigma}\left[\Gamma\left[\frac{d'}{\sigma}\right]\Gamma\left[\frac{\sigma-d'}{\sigma}\right]\Big/(4\pi)^{d'/2}\Gamma\left[\frac{d'}{2}\right]\right]\left[\frac{W_m}{z}\right]^{(\sigma-d')/\sigma}=0, \tag{44}$$

where $z=z_2z_1^{d'/(\sigma-d')}$. Thus, for $z_2\ll z_1$ the scaling function W_m is indeed a function of a single variable z which agrees with prediction (15), with $\dot{\gamma}=\sigma/(\sigma-d')$. It may be mentioned here that, for the special cases $d'=0$ and 1, Eq. (44) reduces precisely to the corresponding equations obtained by Fisher and Privman[5] for the block and cylinder geometries, respectively.

For $z\ll1$, W_m varies linearly with z which agrees with prediction (16), with the universal number W_0 given by

$$W_0=\left[\frac{\sigma}{2}(4\pi)^{d'/2}\Gamma\left[\frac{d'}{2}\right]\Big/\Gamma\left[\frac{d'}{\sigma}\right]\Gamma\left[\frac{\sigma-d'}{\sigma}\right]\right]^{\sigma/(\sigma-d')}; \tag{45}$$

this takes care of Eqs. (13a) and (13b) for the quantities m and χ_0 in the limit of zero field. The corresponding result for ξ_0, see Eq. (14), is also verified, with

$$X_0=W_0^{1/\sigma}. \tag{46}$$

For $z\gg1$ (with z_2 still much less than z_1 and hence y still much less than unity), the scaling function $W_m(z)[\equiv M(z)]$ conforms to prediction (17), with the universal number W_∞ given by

$$W_\infty=\Gamma\left[\frac{d'}{\sigma}\right]\Gamma\left[\frac{2\sigma-d'}{\sigma}\right]\Big/\sigma(4\pi)^{d'/2}\Gamma\left[\frac{d'}{2}\right]. \tag{47}$$

It is obvious that predictions (18) and (20) are also verified. In passing, we note that, in the case of "block" geometry ($d'\to0$), our results for m and χ_0 in this regime turn out to be *independent* of the interaction parameter σ.

We now consider the region $z_2\gg z_1$, and hence $y\gg1$, which lies *above* the phase boundary of the (bulk) system. The function $\mathcal{K}_\sigma$ is now small in value, so in the *zeroth* approximation it may be dropped altogether. Equation (40) then leads to a scaling function $W_m(z_1,z_2)$ in conformity with prediction (22), with

$$W_B=\Gamma\left[\frac{d}{\sigma}\right]\Gamma\left[\frac{2\sigma-d}{\sigma}\right]\Big/\sigma(4\pi)^{d/2}\Gamma\left[\frac{d}{2}\right], \tag{48}$$

which may be compared with (47). It follows that, in this limit, the quantities χ and $(m-m_0)$ would conform to Eqs. (24) and (25), respectively. At this point we observe that, since W_m in this regime is very close to unity, $y\approx\frac{1}{2}(z_2/z_1)^{1/\sigma}$. Accordingly, the correlation length ξ is given by

$$\xi=\frac{L}{2y}\approx L\left[\frac{z_1}{z_2}\right]^{1/\sigma}=a\left[\frac{2J}{H}\left[1-\frac{K_c}{K}\right]^{1/2}\right]^{1/\sigma}, \tag{49}$$

which obviously holds in the bulk limit. To study finite-size effects in this regime, we replace $\mathcal{K}_\sigma$ by the asymptotic approximation (B12) and obtain the following results:

$$m(T,H;L)-m(T,H,\infty)\approx-\sigma m_0C_{d,\sigma}(T,H;L), \tag{50a}$$

$$\chi(T,H;L)-\chi(T,H;\infty)\approx2\sigma(m_0/H)C_{d,\sigma}(T,H;L), \tag{50b}$$

$$\xi(T,H;L)-\xi(T,H;\infty)\approx-\xi_\infty C_{d,\sigma}(T,H;L), \tag{50c}$$

where

$$C_{d,\sigma}(T,H;L)=\frac{1}{4(4\pi)^{d/2}}\left[\sum_{\mathbf{q}(d^*)}{}' \frac{1}{q^{d+\sigma}}\right]\left[\Gamma\left[\frac{d+\sigma}{2}\right]\Big/\Gamma\left[\frac{2-\sigma}{2}\right]\right]z_B^{(d-\sigma)/\sigma}\left[\frac{2\xi_\infty}{L}\right]^{d+\sigma} ; \quad (51)$$

here, ξ_∞ denotes the bulk correlation length, given by Eq. (49), while

$$z_B=z_2 z_1^{-d/(d-\sigma)}=(m_0 H/T)[2(K-K_c)]^{-d/(d-\sigma)} . \quad (52)$$

It will be noted that the finite-size effects in this case vary as an algebraic power of the variable (ξ_∞/L), which is in sharp contrast to the case of short-range interactions where these effects are known to be exponentially small.[8]

Finally, in the very close vicinity of the phase boundary $m_0(T)$, y is of order unity which means that, while z_1 and z_2 are both much larger than 1, the ratio $z_2/z_1=O(1)$. Equation (40) then tells us that $|W_m-1|\sim z_1^{-1}$, with the result that

$$|m-m_0|/m_0\sim(a/L)^{d-\sigma}/(K-K_c) ,$$

which agrees with prediction (27), while

$$\chi\sim(L/a)^{2\sigma-d}/Ja^d(K-K_c) ,$$

which agrees with prediction (26). The correlation length in this region is clearly $\sim L$.

Case 2: $T\simeq T_c$

In the region of second-order phase transition, the variables $\tilde{x}_1$ and $\tilde{x}_2$ reduce to the conventional variables x_1 and x_2 of Eqs. (4), with

$$C_1=K_c a^{-(d-\sigma)}, \quad C_2=(K_c a^{d+\sigma})^{-1/2} . \quad (53)$$

Our results for m and χ now conform to the scaling forms (29) and (30), with

$$Y_m(x_1,x_2)=\frac{x_2}{2(2y)^\sigma}, \quad Y_\chi(x_1,x_2)=\left[\frac{\partial Y_m}{\partial x_2}\right]_{x_1} , \quad (54)$$

where $y(x_1,x_2)$ is determined by the constraint equation (A12).

In the absence of the field ($x_2=0$) and for $t>0$ (which makes $x_1\gg 1$), the scaling function $Y_\chi(x_1,0)$ agrees with prediction (32a), with[9]

$$G_+=\frac{1}{2}\left[\Gamma\left[\frac{d-\sigma}{\sigma}\right]\Gamma\left[\frac{2\sigma-d}{\sigma}\right]\Big/\sigma(4\pi)^{d/2}\Gamma\left[\frac{d}{2}\right]\right]^{\sigma/(d-\sigma)} , \quad \gamma=\frac{\sigma}{d-\sigma} . \quad (55)$$

Equation (33a) then gives the bulk susceptibility χ_0, while the finite-size effect in this case turns out to be

$$\frac{\delta\chi_0}{\chi_0}\approx-\frac{\sigma^2}{4}\left[\sum_{\mathbf{q}(d^*)}{}' \frac{1}{q^{d+\sigma}}\right]\left[\Gamma\left[\frac{d}{2}\right]\Gamma\left[\frac{d+\sigma}{2}\right]\Big/\Gamma\left[\frac{d}{\sigma}\right]\Gamma\left[\frac{2\sigma-d}{\sigma}\right]\Gamma\left[\frac{2-\sigma}{2}\right]\right]\left[\frac{2\xi_>}{L}\right]^{d+\sigma} , \quad (56)$$

where

$$\xi_>\simeq\xi(T,0;\infty)\sim at^{-1/(d-\sigma)} . \quad (57)$$

At the same time, the scaling function $Y_\xi(x_1,0)(=1/2y)$ agrees with prediction (32b), with[9]

$$S_+=(2G_+)^{1/\sigma}, \quad \nu=\gamma/\sigma , \quad (58)$$

while the corresponding finite-size effect is given by

$$\delta\xi_0/\xi_0\approx\frac{1}{\sigma}(\delta\chi_0/\chi_0) . \quad (59)$$

On the other hand, if the system is at the bulk critical temperature ($x_1=0$) and $H>0$ (which makes $x_2\gg 1$), then the scaling functions $Y_m(0,x_2)$ and $Y_\xi(0,x_2)$ agree with predictions (34), with[9]

$$M_c=\frac{1}{2}\left[4\Gamma\left[\frac{d-\sigma}{\sigma}\right]\Gamma\left[\frac{2\sigma-d}{\sigma}\right]\Big/\sigma(4\pi)^{d/2}\Gamma\left[\frac{d}{2}\right]\right]^{\sigma/(d+\sigma)} , \quad \delta=\frac{d+\sigma}{d-\sigma} \quad (60)$$

and

$$S_c=(2M_c)^{1/\sigma}, \quad \nu/\Delta=2/(d+\sigma) . \quad (61)$$

Equations (35) and (36) then give the bulk results for m_c, ξ_c, and χ_c, while the finite-size effects turn out to be

$$\frac{\delta m_c}{m_c}\approx-\frac{\sigma^3}{4(d+\sigma)}\left[\sum_{\mathbf{q}(d^*)}{}' \frac{1}{q^{d+\sigma}}\right]\left[\Gamma\left[\frac{d}{2}\right]\Gamma\left[\frac{d+\sigma}{2}\right]\Big/\Gamma\left[\frac{d-\sigma}{\sigma}\right]\Gamma\left[\frac{2\sigma-d}{\sigma}\right]\Gamma\left[\frac{2-\sigma}{2}\right]\right]\left[\frac{2\xi_c}{L}\right]^{d+\sigma} , \quad (62)$$

$$\delta\xi_c/\xi_c\approx\frac{1}{\sigma}(\delta m_c/m_c) , \quad (63)$$

and

$$\delta\chi_c/\chi_c \approx -[(d+3\sigma)/(d-\sigma)](\delta m_c/m_c) , \quad (64)$$

where

$$\xi_c \simeq \xi(T_c,H;\infty) \sim a(H/T_c)^{-2/(d+\sigma)} . \quad (65)$$

Finally, in the core region, where $|x_1|$ and x_2 are of order unity, i.e.,

$$|t| = O((a/L)^{1/(d-\sigma)}) , \quad (H/T_c) = O((a/L)^{(d+\sigma)/2}) , \quad (66)$$

the parameter y is also of order unity, with the result that

$$\chi \sim a^{-d} T_c^{-1} (L/a)^{\sigma} , \quad (67)$$

$$m \sim a^{-d}(H/T_c)(L/a)^{\sigma} \sim a^{-d}(a/L)^{(d-\sigma)/2} , \quad (68)$$

while

$$\xi \sim L . \quad (69)$$

Expressions (67)–(69) are indeed of the form expected on the basis of the scaling hypotheses of Sec. II.

IV. SPECIAL CASE $d'=\sigma$

The results of Sec. III, especially the ones for $T<T_c$, make it clear that the case $d'=\sigma$ merits a separate investigation. In this case, the dominant behavior of the function $\mathcal{K}_\sigma$, for $y \ll 1$, is given by Eq. (B6), rather than by (B4). The constraint equation (40) then assumes the form

$$(W_m^2-1)z_1 \approx -[\ln(1/y)+\text{const}]/2^{\sigma-1}\pi^{\sigma/2}\Gamma(\sigma/2) , \quad (70)$$

rather than (43). Now, in the absence of the field ($z_2=0$ and hence $W_m=0$), we get

$$y(z_1,0) \sim \exp[-2^{\sigma-1}\pi^{\sigma/2}\Gamma(\sigma/2)z_1] \quad (z_1 \gg 1) , \quad (71)$$

whereby

$$W_\chi(z_1,0) \sim \frac{1}{z_1}\exp[\sigma 2^{\sigma-1}\pi^{\sigma/2}\Gamma(\sigma/2)z_1] \quad (72)$$

and hence

$$\chi_0 \sim \frac{1}{Ja^d}\left[\frac{L}{a}\right]^{\sigma} \times \exp[\sigma 2^{\sigma}\pi^{\sigma/2}\Gamma(\sigma/2)(K-K_c)(L/a)^{d-\sigma}] ; \quad (73)$$

at the same time,

$$\xi_0 \sim L\exp[2^{\sigma}\pi^{\sigma/2}\Gamma(\sigma/2)(K-K_c)(L/a)^{d-\sigma}] . \quad (74)$$

In the presence of the field, so long as $z_2 \ll z_1$, y continues to be much smaller than unity. Equation (70) may then be written as

$$2^{\sigma-1}\pi^{\sigma/2}\Gamma(\sigma/2)z_1(W_m^2-1) \approx \frac{1}{\sigma}\ln(z_2/z_1W_m)+\text{const} , \quad (75)$$

with the result that

$$W_m \exp[\sigma 2^{\sigma-1}\pi^{\sigma/2}\Gamma(\sigma/2)z_1(W_m^2-1)] \sim z_2/z_1 . \quad (76)$$

For $W_m \ll 1$, we obtain

$$W_m \sim \frac{z_2}{z_1}\exp[\sigma 2^{\sigma-1}\pi^{\sigma/2}\Gamma(\sigma/2)z_1] , \quad (77)$$

which tallies with the zero-field expression (72) for W_χ. For $W_m \simeq 1$, on the other hand, we get

$$W_m \approx 1 - \frac{\ln(z_1/z_2)+\text{const}}{\sigma 2^{\sigma}\pi^{\sigma/2}\Gamma(\sigma/2)z_1} , \quad (78a)$$

$$W_\chi \approx \frac{1}{\sigma 2^{\sigma}\pi^{\sigma/2}\Gamma(\sigma/2)z_1z_2} , \quad (78b)$$

so that

$$\frac{m-m_0}{m_0} \approx -\frac{1}{\sigma 2^{\sigma+1}\pi^{\sigma/2}\Gamma(\sigma/2)(K-K_c)} \times \left[\frac{a}{L}\right]^{d-\sigma} \left\{\ln\left[\frac{J}{H}\left[1-\frac{K_c}{K}\right]^{1/2} \times \left[\frac{a}{L}\right]^{\sigma}\right]+\text{const}\right\} , \quad (79)$$

while

$$\chi \approx \frac{m_0}{\sigma 2^{\sigma+1}\pi^{\sigma/2}\Gamma(\sigma/2)(K-K_c)H}\left[\frac{a}{L}\right]^{d-\sigma} . \quad (80)$$

Equations (79) and (80) may be compared with predictions (18) and (20), respectively.

In all other cases, where the parameter y is either of order unity or much greater than unity, no qualitative differences appear between geometries $d'<\sigma$ and $d'=\sigma$.

V. CONCLUDING REMARKS

In this investigation we have examined the consequences of the finite-size scaling hypothesis for a magnetic system, with $O(n)$ symmetry ($n \geq 2$) and long-range interactions decaying as $1/r^{d+\sigma}$ ($0<\sigma<2$), confined to geometry $L^{d-d'}\times\infty^{d'}$ ($\sigma<d<2\sigma, d'\leq\sigma$) and subjected to periodic boundary conditions, in the presence of an external field H. Our attention has mostly been devoted to the study of magnetization $m(T,H;L)$, susceptibility $\chi(T,H;L)$, and correlation length $\xi(T,H;L)$ of the system in the region of both first-order ($T<T_c$) and second-order ($T\simeq T_c$) phase transitions. The predictions following from the scaling hypothesis are then verified in the case of a finite-sized spherical model ($n=\infty$) by carrying out a detailed analysis of the quantities m, χ, and ξ of the system in different regimes of T, H, and L.

As in the case of models with short-range interactions, the situation in the region of first-order phase transition with $L \ll \xi_\infty(T,H)$ depends crucially on whether $d'<\sigma$

or $d'=\sigma$, while in the former case quantities such as χ and ξ approach their standard bulk behavior through power laws in L, in the latter case they do so exponentially instead. On the other hand, in situations where $L \gg \xi_\infty(T,H)$, whether at $T<T_c$ or $T \simeq T_c$, finite-size corrections to standard bulk values of the various quantities of interest depend crucially on whether the interactions operating in the system are short-range ($\sigma=2$) or long-range ($\sigma<2$). While in the former case these corrections are known to vary essentially as $\exp[-L/\xi_\infty(T,H)]$, in the latter case they are found to vary as $[\xi_\infty(T,H)/L]^{d+\sigma}$ instead; this is clearly a reflection of the manner in which the bulk correlation function $G(\mathbf{R},T,H)$ of the system decays with R for $R \gg \xi$.[9] Though derived here for the special case of the *spherical model* ($n=\infty$) with long-range interactions decaying as $1/r^{d+\sigma}$ ($0<\sigma<2$), this result may as well hold for all O(n) models with $n \geq 2$.

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APPENDIX A: CONSTRAINT EQUATION FOR THE SPHERICAL MODEL

In standard notation,[5,13] the constraint equation for an N-spin spherical model under periodic boundary conditions is given by

$$2K(1-M^2)=\frac{1}{N}\sum_{\mathbf{k}}\frac{1}{\phi+\Omega(\mathbf{k})}, \tag{A1}$$

where $K=J/T$, J being the interaction parameter, M is the magnetization per spin, while ϕ is the (appropriately shifted and scaled) spherical field; the qualitative nature of the interactions operating in the system enters through the function $\Omega(\mathbf{k})$, while the vector sum runs over the eigenvalues

$$k_j=\frac{2\pi n_j}{aN_j} \quad (n_j=0,1,\ldots,N_j-1; j=1,\ldots,d; \prod_j N_j=N). \tag{A2}$$

The summation over $\{n_j\}$ can be facilitated with the help of the *Poisson summation formula*[14] which, in view of the periodic character of the function $\Omega(\mathbf{k})$, gives

$$2K(1-M^2)=\frac{1}{N}\sum_{\{q_j\}=-\infty}^{\infty}\int_{-(1/2)N_1}^{(1/2)N_1}\cdots\int_{-(1/2)N_d}^{(1/2)N_d}\frac{\cos(2\pi\mathbf{q}\cdot\mathbf{n})}{\phi+\Omega(\mathbf{k})}d\mathbf{n}. \tag{A3}$$

In the region of phase transition ($\phi \ll 1$), the function $\Omega(\mathbf{k})$ may be replaced by its long-wavelength approximation $(ka)^\sigma$, where $0<\sigma\leq 2$. The constraint equation then takes the form

$$2K(1-M^2)=\frac{1}{(2\pi)^d}\sum_{\{q_j\}=-\infty}^{\infty}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\frac{\cos(\boldsymbol{\gamma}\cdot\mathbf{k}a)}{\phi+(ka)^\sigma}d(\mathbf{k}a) \quad (\gamma_j=N_jq_j;\ j=1,\ldots,d). \tag{A4}$$

The only term that contributes in the bulk limit is the one with $\boldsymbol{\gamma}=0$; that leads to the standard result[9]

$$2K(1-M^2)\approx 2K_c-\left[\Gamma\left[\frac{d-\sigma}{\sigma}\right]\Gamma\left[\frac{2\sigma-d}{\sigma}\right]\Big/\sigma 2^{d-1}\pi^{d/2}\Gamma\left[\frac{d}{2}\right]\right]\phi^{(d-\sigma)/\sigma} \quad (\sigma<d<2\sigma). \tag{A5}$$

With $M=H/2J\phi$, Eq. (A5) determines the singular behavior of the bulk system in all essential details.

To study finite-size effects we need to include terms with $\boldsymbol{\gamma}\neq 0$. To put these terms in a tractable form we transform the integral over $d\mathbf{k}$ into its polar form by using the volume element[15]

$$d\mathbf{k}=k^{d-1}(\sin\theta_1)^{d-2}\cdots(\sin\theta_{d-2})^1 dk\, d\theta_1\cdots d\theta_{d-2}d\phi \tag{A6}$$

and taking polar axis in a direction parallel to $\boldsymbol{\gamma}$. The angular integrations are then readily carried out, and we are left with the expression

$$\frac{\phi^{(d-\sigma)/\sigma}}{(2\pi)^{d/2}}\sum_{\mathbf{q}}{}'\frac{1}{(\phi^{1/\sigma}\gamma)^{(d-2)/2}}\int_0^\infty\frac{x^{d/2}J_{(d-2)/2}(\phi^{1/\sigma}\gamma x)}{1+x^\sigma}dx, \tag{A7}$$

where $J_\nu(z)$ is the ordinary Bessel function, while $\gamma=|\boldsymbol{\gamma}|>0$; it is not difficult to see that, for a system in geometry $L^{d^*}\times\infty^{d'}$, it is only the components $q_1,\ldots,q_{d^*}$ of the vector $\mathbf{q}$ that contribute to the sum in (A7).

At this point it seems imperative to introduce the *scaled length parameter*[14] y, defined by

$$y=\tfrac{1}{2}(L/a)\phi^{1/\sigma} \quad (L=N_j a;j=1,\ldots,d^*)\ . \tag{A8}$$

Expression (A7) then takes the form

$$\frac{\phi^{(d-\sigma)/\sigma}}{2^{d-1}\pi^{d/2}}\mathcal{K}_\sigma\left[\frac{d-2}{2}\Big|d^*;y\right], \tag{A9}$$

where

$$\mathcal{K}_\sigma(v|d^*;y)=\sum_{\mathbf{q}(d^*)}{}' \frac{1}{(yq)^v}\int_0^\infty \frac{x^{v+1}J_v(2yqx)}{1+x^\sigma}dx\ . \tag{A10}$$

Combining (A5) and (A9), and expressing parameters T, H, and L in terms of the scaled variables

$$\tilde{x}_1=(K_c-K)(L/a)^{d-\sigma},\quad \tilde{x}_2=(H/TK^{1/2})(L/a)^{(d+\sigma)/2}\ , \tag{A11}$$

we finally obtain

$$\tilde{x}_1+\frac{\tilde{x}_2^2}{4^{\sigma+1}y^{2\sigma}}=\frac{y^{d-\sigma}}{\sigma 2^\sigma \pi^{d/2}}\left\{\left[\Gamma\left[\frac{d-\sigma}{\sigma}\right]\Gamma\left[\frac{2\sigma-d}{\sigma}\right]\Big/\Gamma\left[\frac{d}{2}\right]\right]-\sigma\mathcal{K}_\sigma\left[\frac{d-2}{2}\Big|d^*;y\right]\right\}. \tag{A12}$$

It seems important to emphasize here that the constraint equation (A12) applies to $\sigma=2$ as well as $\sigma<2$. In the former case, the radial integral in (A10) is precisely equal to the modified Bessel function[16] $K_v(2yq)$ and we recover the constraint equation pertaining to short-range interactions.[8] We may also remark that the only other case where the integral in (A10) can be expressed in terms of standard mathematical functions is the one with $\sigma=1$; however, the resulting expression, which involves Struve functions as well as Bessel functions, is not particularly illuminating.

APPENDIX B: ASYMPTOTIC BEHAVIOR OF THE FUNCTION $\mathcal{K}_\sigma(v|d^*;y)$

For analyzing finite-size effects we need to know the behavior of the function $\mathcal{K}_\sigma(v|d^*;y)$ for $y\to 0$ and for $y\to\infty$; not surprisingly, these two limits require quite different procedures of analysis.

Case 1: $y\to 0$

To obtain the *dominant* behavior in this limit, we replace the summation over $\mathbf{q}(d^*)$ in Eq. (A10) by an integration over $d^{d^*}q$; we thus get

$$\mathcal{K}_\sigma(v|d^*;y)\approx\frac{2\pi^{d^*/2}}{\Gamma(d^*/2)y^v}\int_0^\infty\int_0^\infty\frac{x^{v+1}J_v(2yqx)}{1+x^\sigma}q^{d^*-v-1}dq\,dx\ . \tag{B1}$$

To avoid any problems of convergence, we make use of the representation

$$\frac{1}{q^{2v+2-d^*}}=\frac{1}{\Gamma(2v+2-d^*)}\int_0^\infty e^{-qu}u^{2v+1-d^*}du \quad [(2v+2)>d^*] \tag{B2}$$

and write (B1) in the form

$$\mathcal{K}_\sigma(v|d^*;y)\approx\frac{2\pi^{d^*/2}}{\Gamma(d^*/2)\Gamma(2v+2-d^*)y^v}\int_0^\infty\int_0^\infty\int_0^\infty e^{-qu}u^{2v+1-d^*}\frac{x^{v+1}J_v(2yqx)}{1+x^\sigma}q^{v+1}du\,dq\,dx\ . \tag{B3}$$

Integrations may now be performed over dq, du, and dx—*in that order;* no problems of convergence are encountered and we obtain,[17] for $v=(d-2)/2$,

$$\mathcal{K}_\sigma\left[\frac{d-2}{2}\Big|d^*,y\right]\approx\left[\pi^{d^*/2}\Gamma\left[\frac{d'}{\sigma}\right]\Gamma\left[\frac{\sigma-d'}{\sigma}\right]\Big/\sigma\Gamma\left[\frac{d'}{2}\right]\right]y^{-d^*}\ , \tag{B4}$$

where $d'=d-d^*$ and $0<d'<\sigma$.

For $d'\to 0$, which corresponds to the case of block geometry, we obtain the simple result

$$\mathcal{K}_\sigma\left[\frac{d-2}{2}\Big|d;y\right]\approx\tfrac{1}{2}\pi^{d/2}y^{-d}\ ; \tag{B5}$$

for $d'\to\sigma$, on the other hand,

$$\mathcal{K}_\sigma\left[\frac{d-2}{2}\Big|d-\sigma;y\right]\approx\frac{\pi^{(d-\sigma)/2}}{\Gamma(\sigma/2)}y^{-(d-\sigma)}\times\left[\ln\left[\frac{1}{y}\right]+\text{const}\right]. \tag{B6}$$

Case 2: $y \to \infty$

In this case we employ the customary representation

$$\frac{1}{1+x^{\sigma}} = \int_0^\infty e^{-u(1+x^{\sigma})} du \tag{B7}$$

and write the integral appearing in (A10) in the form

$$I_\sigma(\nu;yq) = \int_0^\infty e^{-u} \left[\int_0^\infty x^{\nu+1} J_\nu(2yqx) \times e^{-ux^{\sigma}} dx \right] du = \frac{1}{(2yq)^{\nu+2}} \int_0^\infty e^{-u} \left[\int_0^\infty w^{\nu+1} J_\nu(w) \times e^{-u\eta w^{\sigma}} dw \right] du \,, \tag{B8}$$

where $\eta = (2yq)^{-\sigma}$. We shall evaluate this integral in the limit $\eta \to 0$. For this we follow a procedure due to Montroll and West,[18] which suggests writing the integral over dw in the form

$$L_\sigma(\nu;u\eta) = \int_0^\infty w^{\nu+1} J_\nu(w) e^{-u\eta w} e^{-u\eta(w^{\sigma}-w)} dw = \int_0^\infty w^{\nu+1} J_\nu(w) e^{-u\eta w} [1 - u\eta w^{\sigma} + u\eta w + O(\eta^2)] dw \,. \tag{B9}$$

The first and third integrals in (B9) turn out to be[19]

$$\frac{2^{\nu+1}\Gamma(\nu+\frac{3}{2}) u\eta}{\pi^{1/2}(1+u^2\eta^2)^{\nu+3/2}}$$

and

$$-\frac{2^{\nu+1}\Gamma(\nu+\frac{3}{2}) u\eta}{\pi^{1/2}(1+u^2\eta^2)^{\nu+5/2}} [1-(2\nu+2)u^2\eta^2] \,,$$

respectively; to order η, they cancel one another and leave a remainder of order η^3. The second integral turns out to be[20]

$$-\frac{\Gamma(2\nu+2+\sigma) u\eta}{(1+u^2\eta^2)^{(\nu+2+\sigma)/2}} P^{-\nu}_{\nu+1+\sigma} \left[\frac{u\eta}{(1+u^2\eta^2)^{1/2}} \right] ,$$

where $P^{\alpha}_{\beta}(z)$ is the associated Legendre function of the first kind; to order η, we get[21]

$$L_\sigma(\nu;u\eta) \approx \left[\sigma 2^{\nu+\sigma} \Gamma\left[\frac{2\nu+2+\sigma}{2}\right] \Big/ \Gamma\left[\frac{2-\sigma}{2}\right] \right] u\eta \,. \tag{B10}$$

Substituting (B10) into (B8) and integrating over du, we get

$$I_\sigma(\nu;yq) \approx \left[\sigma \Gamma\left[\frac{2\nu+2+\sigma}{2}\right] \Big/ 4\Gamma\left[\frac{2-\sigma}{2}\right] \right] (yq)^{-(\nu+2+\sigma)} \,. \tag{B11}$$

Equation (10) then gives, for $\nu = (d-2)/2$,

$$\mathcal{K}_\sigma \left[\frac{d-2}{2} \middle| d^*; y \right] \approx \left[\sigma \Gamma\left[\frac{d+\sigma}{2}\right] \Big/ 4\Gamma\left[\frac{2-\sigma}{2}\right] \right] \left[\sum_{\mathbf{q}(d^*)}{}' \frac{1}{q^{d+\sigma}} \right] y^{-(d+\sigma)} \quad (\sigma < 2) \,. \tag{B12}$$

It will be noted that the foregoing expression holds *only* for $\sigma < 2$, in which case it leads to finite-size corrections $\sim L^{-(d+\sigma)}$. For $\sigma = 2$, the integral $I_\sigma(\nu;yq)$ is precisely equal to the modified Bessel function $K_\nu(2yq)$, which leads to exponentially small corrections instead.

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