RAPID COMMUNICATIONS

PHYSICAL REVIEW B VOLUME 40, NUMBER 1 1 JULY 1989

Scaling laws in fracture

L. de Arcangelis
Service de Physique Théorique de Saclay, F-91191 Gif-sur-Yvette CEDEX, France

A. Hansen
Institut für Theoretische Physik, Universität zu Köln, 5 Köln 41, West Germany

H. J. Herrmann
Service de Physique Théorique de Saclay, F-91191 Gif-sur-Yvette CEDEX, France
and Laboratoire d'Hydrodynamique et de Mécanique Physique, École Supérieure de Physique et de Chimie Industrielles de la Ville de Paris, 10 Rue Vauquelin, 75231 Paris 05 CEDEX, France

S. Roux
Laboratoire d'Hydrodynamique et de Mécanique Physique, École Supérieure de Physique et de Chimie Industrielles de la Ville de Paris, 10 Rue Vauquelin, 75231 Paris 05 CEDEX, France
(Received 4 May 1988; revised manuscript received 28 December 1988)

We studied numerically the fracture of three types of disordered media: a scalar, a central-force, and a beam model. We discovered the following novel, universal laws: in an initial regime, force and displacement both scale as $L^{0.75}$ with the system size L; the number of bonds that break scales during the whole process as $L^{1.7}$, and the distribution of local forces is multifractal just before the system breaks, whereas it has constant-gap scaling when catastrophic breaking sets in.

The laws that govern the fracture of disordered media are technologically of outstanding interest since they cover phenomena ranging from the cracking of glass to the tearing of cloth, from the burning of fuse networks to the aging of concrete. Despite the large effort that has been undertaken for decades by material scientists[1] many fundamental questions about fracture have not been answered yet. One of them is the question of whether there exists some universal behavior in the breaking characteristics of a disordered system independent of the details of the model. In this paper we will present for the first time universal scaling laws of the breaking characteristics (i.e., force versus displacement) that we discovered using techniques developed in statistical physics in recent years and that we applied to various models and different distributions of disorder.

Many models similar to the one we use have been considered before: electrical fuse models with dilution[2] or disorder in the conductivities,[3] electric breakdown models with dilution[4] or with disorder in the conductivities[5] or thresholds,[6] and elastic central-force models with dilution[7] and thermal[8] or probabilistic[9] activation. These papers focus mainly on the fractal dimensions of the cracks, the values of the external force necessary to break the system apart, and its probability distribution.[2,7] We quantitatively investigate for the first time the breaking characteristics and the local distribution of strain just before the system breaks, two quantities that are particularly accessible experimentally.

We consider a finite two-dimensional lattice $L \times L$ with periodic boundary conditions in the horizontal direction and fixed bus bars on top and bottom. On these bars the external strain (elongation or shear) will be applied. Each bond of the lattice should model the material on a mesoscopic level. Its behavior is supposed to be ideally fragile; i.e., to have a linear elastic dependence between force f and displacement δ with unit elastic constant up to a certain threshold force f_c where it breaks (see inset of Fig. 1). The thresholds are randomly distributed according to some probability distribution $P(f_c)$. Once a force beyond f_c is applied to a bond, this is irreversibly removed from the system. As the external strain is increased one can watch bonds breaking one by one until the system falls

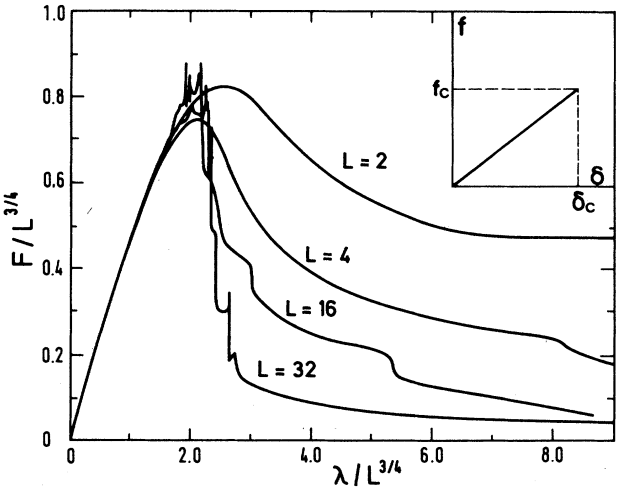

FIG. 1. Breaking characteristics of the beam model with $x=0.5$, $r=1$ with both axis scaled by $L^{-3/4}$ for different sizes L. The data have been smoothed to reduce statistical fluctuations (Ref. 6). The inset shows the characteristics of a single beam.

apart altogether. Plasticity and nonlinearity induced by large deformations are not taken into account in our model.

Three different models for fracture, corresponding to different physical situations, have been investigated. In the *scalar model*[10,11]—which is actually a network of electrical fuses, replacing force f by current i and displacement δ by voltage v—there is only one scalar variable on each site, namely the electrical potential. In the *central-force model*[12]—which for stability reasons must be implemented on a triangular lattice[13]—the bonds are springs that can freely rotate around the sites and the two variables on each site i are the x_i and y_i coordinates of its displacement. In the *beam model*[14]—which contains full bond-bending elasticity[15,16]—there are three variables on each site: the coordinates x_i and y_i of the displacement and an angle θ_i. The bonds are elastic beams soldered at the two ends in such a way that they must enter site i tangentially at the angle θ_i (measured with respect to the undistorted square lattice). The elastic energy of a beam is the sum of elongation, shear, and flexural energy, and all three terms are expressed as quadratic functions of the x_i, y_i, and θ_i of the two adjacent sites (linear elasticity).

In the scalar model, a fuse burns if $|i|/i_c \geq 1$ where we have considered for i_c the Weibull distribution $P(i_c) \propto i_c^{m-1}\exp[-(i_c/i_0)^m]$ which for $2 \leq m \leq 10$ describes empirically a variety of real materials, and the power-law distribution $P(i_c) \propto i_c^{-x}$ with $0 < i_c \leq 1$ which is the uniform distribution for $x=0$. We take $x < 1$ so that the distribution is always normalized. In the central-force model, a bond breaks if $|f|/f_c \geq 1$ and we only considered a uniform distribution for f_c. For the beam model the breaking happens if[17]

$$(f/f_c)^2 + \max(|m_i|, |m_j|)/m_c \geq 1 , \quad (1)$$

where f is the longitudinal force acting along the beam, and m_i and m_j are the moments applied on the two adjacent sites. f_c and m_c are two random thresholds. Therefore, two mechanisms contribute to the breaking of a beam, elongation and flexion. Let us call r the ratio between the strength of these two effects. The thresholds are then distributed according to $P(f_c) = (1-x) f_c^{-x}$ with $0 < f_c \leq 1$ and $P(m_c) = (1-x) r^{x-1} m_c^{-x}$ with $0 < m_c \leq r$. We note that all these criteria are symmetric with respect to the sign of the displacement.

We simulated the breaking of the above models in a constant-displacement (voltage) ensemble using about 200 h on an IBM 3090 and 100 h on a Cray XMP. In order to know which bond has to be broken one must calculate the distribution of local forces (and moments in the case of beams) by minimizing the energy of the system. This is done using recently developed conjugate-gradient relaxation techniques.[18] Each time a bond is removed the distribution of local forces changes and must be recalculated. For this reason the procedure is quite time consuming. Typically we averaged over 50000 samples for $L=4$ and over 50 samples (1 sample) for $L=64$ for the scalar (beam) model.

Each time a bond is broken we monitor the external force F and the external displacement λ both averaged over fixed number n of bonds cut. The relation between the two gives the breaking characteristics of the entire system as shown in Fig. 1 for the beam model. We see that, unlike the single bond that was ideally brittle, the macroscopic characteristic is "ductile," by that we mean that after a maximum force F_b has been applied, the system can still be extensively elongated before becoming disconnected, a regime accessible experimentally only if a displacement and not a force is imposed. After reaching the maximum, the breaking characteristic is subject to strong statistical fluctuations and for weak disorder ($m > 2$) it seems to bend back; i.e., both F and λ decrease. Before the maximum is reached, there is an initial regime with fewer statistical fluctuations that is dominated by the disorder, so that it shrinks for decreasing disorder. In this regime Fig. 1 verifies the scaling law

$$F = L^{\alpha}\phi(\lambda L^{-\beta}) , \quad (2)$$

with $\alpha \approx \beta \approx 0.75$, where we have considered α and β as two independent parameters. We checked this law for all three models,[10,12,14] all distributions ($0.8 \leq x \leq -1$, $2 \leq m \leq 5$), and for both external extension and shear (in the elastic case) with exponents that agree with $\frac{3}{4}$ within 5–10%. For the same range of forces we found the scaling law for the number n of bonds cut

$$n = L^{\gamma}\psi(\lambda L^{-\beta}) , \quad (3)$$

with $\gamma \approx 1.7$ and the same universal range of validity as for Eq. (2).

The number of bonds n_b that have been cut when the force reaches the maximum scales again for most cases such as $n_b \sim L^{1.7}$, as seen in Fig. 2(a). Only when the disorder becomes very small, i.e., for $m=5$ and 10, does there seem to be a crossover to $n_b \sim L^0$, which is expected. Force and displacement at the maximum do not seem to

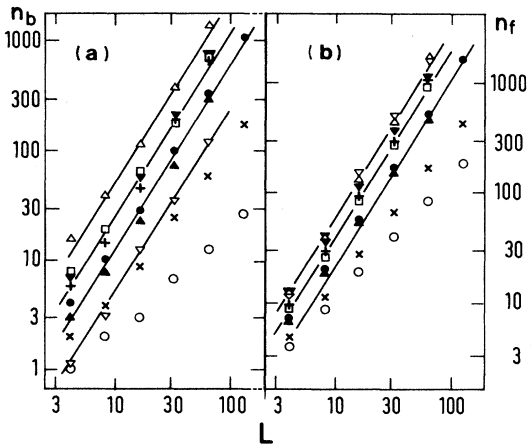

FIG. 2. Number of bonds cut when (a) the maximum force is reached and (b) when the system falls apart in a log-log plot against L for the scalar model with $x=0$ (□), $x=0.5$ (△), $m=2$ (●), $m=5$ (×), and $m=10$ (○); the beam model ($r=1$) for an external elongation with $x=0$ (+), $x=0.5$ (▽), and $x=-1$ (▲); and for an external shear with $x=0$ (▼). The solid lines are guides to the eye of slope 1.7.

obey a power-law relation, at least for the small sizes we consider.

Finally, after n_f bonds are cut, the system breaks apart altogether. Again, a behavior $n_f \sim L^{1.7}$ is reasonably well followed by the data, except for small disorder where for $m=5$ and 10 a crossover to the expected $n_f \sim L$ is observed [see Fig. 2(b)]. For the scalar model we also verified that the length of the largest crack, which causes the failure of the system, scales proportional to L for all distributions considered.

The scaling relation for the number of bonds cut, valid during the whole breaking up to its end with an exponent of about 1.7, is an unexpected result since Ref. 3 predicted $n \sim L$ for the case $x=0$. Moreover, this finding is universal with respect to the three models, the type of randomness, and the external boundary conditions within our error bars (5–10%, depending on the model). Considering that the dual of our scalar model is a Laplacian dielectric-breakdown model,[4–6] we conjecture that the value 1.7 of the exponent, at least in the last regime, equals the fractal dimension of diffusion-limited aggregation.[19]

At the point when the last bond is cut before the system breaks apart, we also analyzed the distribution $n(i)$ of local currents (local forces, shears, and moments in the elastic case). The moments of this distribution are defined as $M_q = \sum_{\text{bonds}} i^q n(i)$. In Fig. 3(b), M_0 and the quantities $m_q = (M_q/M_0)^{1/q}$ are plotted as functions of L for the scalar model. We see that with varying q the m_q scale as $m_q \sim L^{y_q}$, with different exponents y_q. This is in sharp contrast to what happens if the same analysis is made for the $n(i)$ at the maximum of the breaking characteristics [see Fig. 3(a)]. Here, the m_q fall on parallel straight lines for different q, and so all y_q are the same (constant-gap scaling).

The phenomenon of y_q varying with q, as shown in Fig. 3(b), is a manifestation of *multifractality* that has recently been observed in various contexts.[20] For each value of q, we have derived by graphical extrapolation to $L \to \infty$, the asymptotic value of the exponents $p(q)$ for the current moments $M_q \sim L^{-p(q)}$. Figure 4 shows the set of these critical exponents as functions of q. The dependence is clearly nonlinear due to non-constant-gap exponent scaling. For $q \to +\infty$, the curve goes asymptotically to a straight line of slope ~ 0.25 and vertical intercept at zero, consistent with the picture that, just before the last bond breaks, there is only a finite number of bonds ($\sim L^0$) that carry the total current in the system [$\sim L^{p(2)}$ where $p(2) \sim 0.3$]. Various other analysis, such as the investigation of the $f(\alpha)$ spectrum,[21] also lead to the conclusion that $n(i)$ is multifractal, just before the last bond is cut, for the central-force model[12] and the beam model.[14]

The multifractality of the distribution of local strains just before the system breaks physically means that the regions with highest variation in local strains, i.e., the regions that are finally responsible for rupture, lie on a fractal subset of the system. The fractal dimension of this subset depends on the strength of the local variations. In practical terms, this means that the larger the system the more pronounced the contrast becomes between highly strained and practically unstrained regions. This effect is particularly spectacular since it only occurs just before the system breaks and not during the whole process, as seen in Fig. 3(a). Our data permit us to quantify this statement. The appearance of multifractality is more astonishing if one considers that only a negligible number of bonds ($n \sim L^{1.7}$) has been cut, in contrast to the case of percolation where multifractality only appears at the percolation threshold p_c ($n = p_c L^2$). Local strains can be studied by photoelasticity, and this might also be the best method to verify the multifractality properties.

In conclusion, we have investigated the dependence of fracture on the system size from a novel point of view: We first proposed scaling laws between external force, total displacement, number n of bonds cut, and the size of the system [Eqs. (2) and (3)]. We have verified numeri-

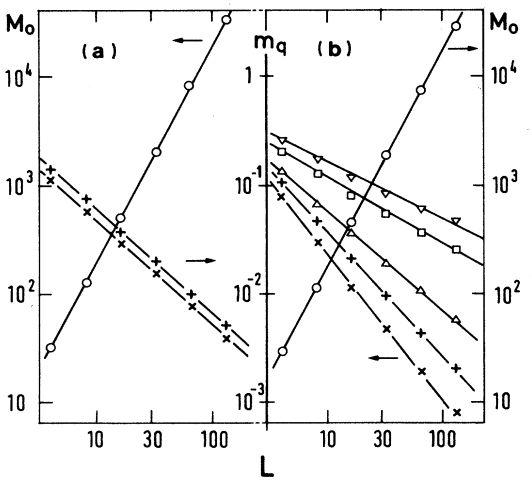

FIG. 3. Moments of the current distribution for $m=2$ (a) at the maximum of the characteristics and (b) just before the system falls apart in a log-log plot against L. The symbols are ○ for M_0, × for m_1, + for m_2, △ for m_3, □ for m_6, and ▽ for m_9, where $m_q = (M_q/M_0)^{1/q}$.

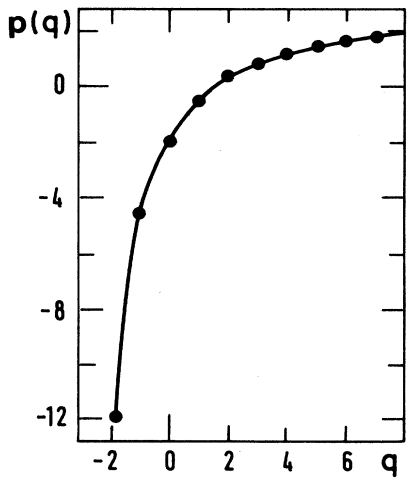

FIG. 4. $p(q)$ against q for the scalar model with $x=0$.

cally that these laws hold for three different models that correspond to three different physical situations.[11,13,15] Second, we found that $n \propto L^{1.7}$, which is different from the prediction of some approximations[11] but within their exact bounds.[11] Finally, we found striking evidence that the distribution of local strains becomes multifractal at the very end of the breaking process but still has constant-gap scaling at the maximum of the breaking characteristics.

Other statistical approaches to fracture[2-9] of random materials have been investigated in different contexts and it would be interesting to see if similar scaling laws also apply in those cases. It would also be interesting to investigate next the experimentally more relevant three-dimensional model. Size dependence of fracture has been studied experimentally for a long time[22] but to our knowledge very precise quantitative data are not yet available. More experiments will be done in the near future.[23]

We thank E. Guyon for discussions and M. Novotny and the IBM Bergen Scientific Center for their hospitality in Bergen where part of the computation was performed. S.R. acknowledges the support of the Centre National de la Recherche Scientifique and of the Ecole Superieure de Physique et Chimie Industrielle. A.H. acknowledges Sonderforschungsbereich 125. Service de Physique Théorique de Saclay is a "Laboratoire de l'Institut de Recherche Fondamentale du Commissariat à l'Energie Atomique."

[1] *Fracture*, edited by H. Liebowitz (Academic, New York, 1984), Vols. I–VII.

[2] P. M. Duxbury, B. D. Beale, and P. L. Leath, Phys. Rev. Lett. **57**, 1052 (1986); P. M. Duxbury, P. L. Leath, and P. D. Beale, Phys. Rev. B **36**, 367 (1987); L. de Arcangelis, S. Redner, and H. J. Herrmann, J. Phys. (Paris) Lett. **46**, L585 (1985); A. Gilabert, C. Vanneste, D. Sornette, and E. Guyon, J. Phys. (Paris) **48**, 763 (1987).

[3] H. Takayasu, in *Fractals in Physics,* edited by L. Pietronero and E. Tosatti (Elsevier, Amsterdam, 1986), p. 181.

[4] B. D. Beale and P. M. Duxbury, Phys. Rev. B **37**, 2785 (1988).

[5] H. Takayasu, Phys. Rev. Lett. **54**, 1099 (1985).

[6] F. Family, Y. C. Zhang, and T. Vicsek, J. Phys. A **19**, L733 (1986).

[7] B. D. Beale and D. J. Srolovitz, Phys. Rev. B **37**, 5500 (1988); M. Sahimi and J. D. Goddard, *ibid.* **33**, 7848 (1986).

[8] P. Meakin, Thin Solid Films **151**, 165 (1987); Y. Termonia, P. Meakin, and P. Smith, Macromolecules **18**, 2246 (1985).

[9] E. Louis and F. Guinea, Europhys. Lett. **3**, 871 (1987).

[10] L. de Arcangelis and H. J. Herrmann, Phys. Rev. B **39**, 2678 (1988).

[11] B. Kahng, G. G. Batrouni, S. Redner, L. de Arcangelis, and H. J. Herrmann, Phys. Rev. B **37**, 7625 (1988); for the case of infinite disorder see, S. Roux, A. Hansen, H. J. Herrmann, and E. Guyon, J. Stat. Phys. **52**, 251 (1988).

[12] A. Hansen, S. Roux, and H. J. Herrmann (unpublished).

[13] S. Feng and P. N. Sen, Phys. Rev. Lett. **52**, 216 (1984).

[14] H. J. Herrmann, A. Hansen, and S. Roux, Phys. Rev. B **39**, 637 (1988).

[15] S. Roux and E. Guyon, J. Phys. (Paris) Lett. **46**, L999 (1985).

[16] Y. Kantor and I. Webman, Phys. Rev. Lett. **52**, 1891 (1984).

[17] Under the general assumption that the beam is made out of a homogeneous, fragile solid whose rupture criterion is of von Mises or the Tresca type Eq. (1) follows, see, e.g., S. P. Timoshenko and J. N. Goodier, *Theory of Elasticity* (McGraw-Hill, New York, 1951).

[18] G. G. Batrouni, A. Hansen, and M. Nelkin, Phys. Rev. Lett. **57**, 1336 (1986); G. G. Batrouni and A. Hansen, J. Stat. Phys. **52**, 747 (1988).

[19] T. A. Witten and L. M. Sander, Phys. Rev. Lett. **47**, 1400 (1981); L. Niemeyer, L. Pietronero, and H. J. Wiemann, *ibid.* **52**, 1033 (1984).

[20] G. Paladin and A. Vulpiani, Phys. Rep. **156**, 147 (1987); L. de Arcangelis, in *Disorder and Mixing,* Proceedings of the Summer School Cargése, 1987, edited by E. Guyon and Y. Pomeau (Martinus Nijhoff, Hingham, MA, 1988).

[21] R. Benzi, G. Paladin, G. Parisi, and A. Vulpiani, J. Phys. A **17**, 3521 (1984); T. C. Halsey, P. Meakin, and I. Procaccia, Phys. Rev. Lett. **56**, 854 (1986).

[22] F. M. Beremin, Metall. Trans. A **14**, 2277 (1983).

[23] A. Gilabert, C. Vanneste, D. Sornette, and E. Guyon, J. Phys. (Paris) **48**, 763 (1987); A. Gilabert and C. Vanneste (unpublished).