Quantum mechanics of the fractional-statistics gas: Hartree-Fock approximation

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The two-dimensional ideal gas of particles obeying ν fractional statistics is transformed to the Fermi representation and studied in the Hartree-Fock approximation. The extremal ground state is shown to be composed of Landau orbitals. When the filling factor $(1-\nu)^{-1}$ of the ground state is an integer, a logarithmically large energy gap appears in the single-particle excitation spectrum, and the particle and hole states are charged vortices with circulation $\pm(1-\nu)h/m$. The linear dependence of the total energy on the density, together with the presence of this gap, suggests that the true ground state at these fractions is a superfluid.

I. INTRODUCTION

It was recently proposed by one of us[1,2] that the charge carriers in high-temperature superconductors obey one-half fractional statistics[3,4] and that this might account for the superconductivity. In this paper and the two planned to follow, we shall provide some computational detail supporting this point of view. The issue we wish to address is whether fractional statistics can, as a matter of principle, cause superconductivity, as opposed to whether such things occur in the real materials. While we believe that they do, this matter is controversial and must ultimately be dealt with by experiment. In this first paper, we define the problem, identify the appropriate representation for solving it, and work out the mean-field solution in detail. We conclude that $\nu=\frac{1}{2}$ fractional statistics is a special case in which an energy gap opens up in the "fermionic" excitation spectrum while the bulk modulus remains finite. This suggests very strongly that the ground state is a fluid supporting longitudinal sound waves that cannot decay, i.e., superfluid flow. The broken symmetry of the superfluid state is not manifested in this calculation, as the solution is prevented by construction from being degenerate. However, this is a well-known feature[5] of variational descriptions of quantum liquids remedied by hybridizing macroscopic numbers of phonons into the mean-field ground state. The formal development of this degeneracy and the excitation spectrum associated with it is the subject of the next two papers.

By a gas of particles obeying ν fractional statistics, we mean a set of N spinless particles described by a wave function Ψ of the form

$$\Psi(z_1,\dots,z_N) = \left[\prod_{j<k}\frac{(z_j-z_k)^{1-\nu}}{|z_j-z_k|^{1-\nu}}\right]\Phi(z_1,\dots,z_N)\ , \qquad (1.1)$$

where $z_j=x_j+iy_j$ is a complex number representing the location of the jth particle in the x-y plane, and Φ is a spinless fermion wave function, satisfying the Schrödinger equation

$$\mathcal{H}\Psi=E\Psi\ , \qquad (1.2)$$

where $\mathcal{H}$ is the free-particle Hamiltonian

$$\mathcal{H}=\sum_{i=1}^{N}\frac{|\mathbf{P}_i|^2}{2m}\ . \qquad (1.3)$$

Equation (1.2) may also be written

$$\mathcal{H}'\Phi=E\Phi\ , \qquad (1.4)$$

where

$$\mathcal{H}'=\sum_{i=1}^{N}\frac{1}{2m}\left|\mathbf{P}_i+\frac{e}{c}\mathbf{A}_i\right|^2\ , \qquad (1.5)$$

with

$$\mathbf{A}_i=\sum_{j\neq i}^{N}\mathbf{A}_{ij}=(1-\nu)\frac{\hbar c}{e}\hat{\mathbf{z}}\times\sum_{j\neq i}^{N}\frac{(\mathbf{r}_i-\mathbf{r}_j)}{|\mathbf{r}_i-\mathbf{r}_j|^2}\ . \qquad (1.6)$$

The fractional-statistics gas may this be thought of as a collection of spinless fermions acting as though each carried a solenoid containing a fraction $(1-\nu)$ of a flux quantum. Since we can also write

$$\mathcal{H}''\chi=E\chi\ , \qquad (1.7)$$

where

$$\Psi(z_1,\dots,z_N)=\left[\prod_{j<k}\frac{(z_j-z_k)^{\nu}}{|z_j-z_k|^{\nu}}\right]\chi(z_1,\dots,z_N) \qquad (1.8)$$

and

$$\mathcal{H}''=\sum_{i=1}^{N}\frac{1}{2m}\left|\mathbf{P}_i+\frac{e}{c}\mathbf{A}_i^{\text{Bose}}\right|^2 , \tag{1.9}$$

with

$$\mathbf{A}_i^{\text{Bose}}=\sum_{j\neq i}^{N}\mathbf{A}_{ij}^{\text{Bose}}=\nu\frac{\hbar c}{e}\hat{\mathbf{z}}\times\sum_{j\neq i}^{N}\frac{(\mathbf{r}_i-\mathbf{r}_j)}{|\mathbf{r}_i-\mathbf{r}_j|^2} , \tag{1.10}$$

it may also be thought of as a gas of bosons carrying solenoids containing a fraction ν of a flux quantum. Equations (1.4)–(1.6), which constitute the Fermi representation for the fractional-statistics gas problem, are completely equivalent to Eqs. (1.7)–(1.10).

Very little is currently known about the quantum mechanics of the fractional-statistics gas. It was originally pointed out by Wilczek[6] that fractional statistics could exist physically in two-dimensional systems and that it interpolated continuously between Bose ($\nu=0$) and Fermi ($\nu=1$) statistics. Arovas *et al.*[7] subsequently calculated the second virial coefficient of the noninteracting fractional-statistics gas and found it to interpolate sensibly between the Bose and Fermi limits, up to a slope discontinuity at $\nu=0$. At about the same time it was pointed out by Halperin[8] that the fractionally charged quasiparticles in the fractional quantum Hall effect probably obeyed fractional statistics, and that this accounted for the experimentally observed hierarchical states. The correctness of this latter idea has led us to consider the possibility that fractional statistics also occurs in high-temperature superconductors. We must emphasize that the quantum Hall context of fractional statistics is very different from the one being considered here, in that the background magnetic field in the quantum Hall problem strongly suppresses density fluctuations, and thus precludes superfluidity.

There are several reasons to suspect the $\nu=\frac{1}{2}$ fractional-statistics gas of being a charge-2 superconductor. The first is that $\frac{1}{2}$ fractional statistics is reliably present if the analogy[3,4] between the "spin-liquid state" and the fractional quantum Hall state is correct, which has been argued by one of us[2] on general grounds to be necessary. The second is that statistical transmutation is a big effect. Given that the interpolation between the Bose and Fermi limits is continuous, which is supported by the work of Arovas *et al.*,[7] the $\frac{1}{2}$ fractional-statistics gas is expected to have about half the degeneracy pressure of fermions at the same density. Thus, if we were to mistakenly assume the particles to be fermions, we would conclude that an enormous attractive potential, comparable in scale to the Fermi energy, was present. Large attractive potentials are, of course, required to account for high-temperature superconductivity in the Bardeen-Cooper-Schrieffer (BCS) framework. Similarly, if we assumed the particles to be bosons, we would conclude that an enormous repulsive potential was present. Since repulsive potentials are actually required for achieving ordinary charge-1 superfluidity, the observation of such potentials does not rule out ordinary Bose condensation. What rules it out is the third reason to suspect charge-2 superfluidity, namely that the particles are not bosons, while *pairs* of them are. As illustrated in Fig. 1, one can understand this by imagining a "Berry phase" experiment, in which a pair of wells containing trapped particles are adiabatically interchanged. If each well contains one particle, the wave function evolves back to itself up to a phase $\pi/2$, for clockwise evolution. If each well contains two particles, the wave function evolves back to itself up to four times this amount, or 2π, because one gets $\pi/2$ for each pairing of a particle in the first well with a particle in the second.

Because the fractional-statistics gas has a nonzero degeneracy pressure, it can be argued that it is more "like" the Fermi sea than the noninteracting Bose fluid. It is primarily for this reason that we have approached the problem in the Fermi representation. Another consideration is that perturbation theory is usually better controlled in Fermi systems because the number of low-lying excitations is smaller. However, this representation is not fundamentally superior, and calculations using different ones should produce the same results. Use of the Fermi representation has the side effect of forcing the number of particles in the exchange-correlation hole to be exactly 1, which complicates the analysis of the collective-mode spectrum.

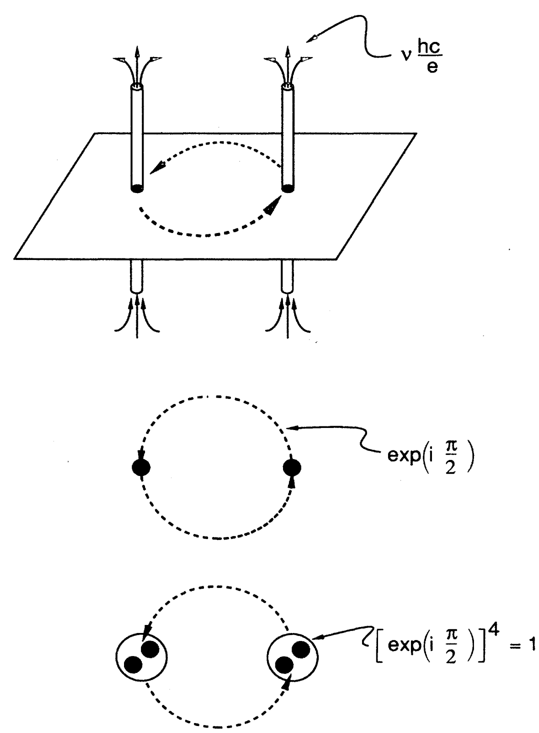

FIG. 1. Top: Particles obeying ν fractional statistics act as though they were bosons carrying a magnetic solenoid containing a fraction ν of a magnetic flux quantum. Middle: When $\nu=\frac{1}{2}$, adiabatic exchange of two wells trapping the particles returns the wave function to itself up to a phase of $\pi/2$. Bottom: If pairs of particles are similarly exchanged, the wave function acquires phase $\pi/2$ for each of the four possible associations of a particle in the first well with a particle in the second. Pairs of such particles are thus bosons.

The first step in any fermion calculation, and the subject of this paper, is solution of the problem in the Hartree-Fock approximation. We adopt a variational ground-state wave function of the form

$$\Phi(z_1, \ldots, z_N) = \frac{1}{\sqrt{N!}} \sum_{\sigma}^{N!} \mathrm{sgn}(\sigma) \varphi_{\sigma(1)}(z_1) \cdots \varphi_{\sigma(N)}(z_N) , \tag{1.11}$$

where σ denotes a permutation, $\mathrm{sgn}(\sigma)$ is its sign, and the φ_i are single-particle orbitals, and then vary the orbitals to minimize the expected energy $\langle \Phi | \mathcal{H}' | \Phi \rangle$. This is appropriate formally because it provides a stable point about which to do perturbation theory. However, it is also the case that such calculations tend generally to be physically correct. Well-known examples of this include the energy levels of atoms and molecules,[9] the properties of itinerant ferromagnets,[10] the properties of charge-density-wave[11] and spin-density-wave[12] materials, and, in an abstract sense, the properties of ordinary superconductors.[13]

That a Hartree-Fock calculation should be physically correct in the present case is suggested by elementary considerations. As illustrated in Fig. 2, the uniform sea of magnetic solenoids seen by each particle is equivalent to a uniform magnetic field of the form

$$\mathbf{B} = (1-\nu) \frac{hc}{e} \rho \hat{\mathbf{z}} , \tag{1.12}$$

where ρ is the particle density. This may be idealized with a single-electron Hamiltonian of the form

$$\overline{\mathcal{H}} = \frac{1}{2m} \left| \mathbf{P} + \frac{e}{c} \overline{\mathbf{A}} \right|^2 , \tag{1.13}$$

where

$$\overline{\mathbf{A}} = (1-\nu) \frac{hc}{2e} \hat{\mathbf{z}} \times \mathbf{r} , \tag{1.14}$$

The eigenstates of $\overline{\mathcal{H}}$, which satisfy

$$\overline{\mathcal{H}} \varphi_{nk}(z) = (n + \tfrac{1}{2}) \hbar\omega_c \varphi_{nk}(z) , \tag{1.15}$$

where

$$\hbar\omega_c = \hbar \frac{eB}{mc} = 2\pi(1-\nu) \frac{\hbar^2}{m} \rho = (1-\nu) E_F , \tag{1.16}$$

with E_F the Fermi energy of spinless fermions at density ρ, are Landau orbitals. Written out in units of the magnetic length a_0, defined in the manner

$$a_0^2 = \frac{\hbar c}{eB} = \frac{1}{2\pi(1-\nu)\rho} , \tag{1.17}$$

these are

$$\varphi_{nk}(z) = \frac{(\frac{1}{2}z - 2\partial_z^*)^n}{(2^n n!)^{1/2}} \frac{(\frac{1}{2}z^* - 2\partial_z)^k}{(2^k k!)^{1/2}} \frac{e^{-(1/4)|z|^2}}{(2\pi)^{1/2}} . \tag{1.18}$$

The set of orbitals with the same value of n is the nth Landau level. Since the density associated with a filled Landau level is $(2\pi a_0^2)^{-1}$, occupying the orbitals with fermions causes the number of Landau levels filled to be $(1-\nu)^{-1}$. Thus, the fractions $\nu = 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots$ are special cases in which an integral number of Landau levels becomes filled, and an energy gap opens in the "fermionic" spectrum. Since these are precisely the cases for which this integral number of particles is a boson, the presence of this gap would seem to signal occurrence of Bose condensation. This identification is supported by the fact that $\nu = 0$ corresponds to an ordinary Bose fluid.

The association of superfluidity with the presence of an energy gap, while unusual, is also reasonable. The superfluid state is characterized[14] by the presence of macroscopic density fluctuations, which allow an order-parameter phase ϕ to be defined by satisfying the uncertainty relation $\Delta N \, \Delta\phi \geq 1$. This, in turn, implies the existence of a linearly dispersing collective mode[15] (in the absence of long-range Coulomb forces) which may be identified physically with compressional sound. The long-wavelength limit of this collective mode is rectilinear flow. It is "super" because the collective mode has no decay channel and thus has an infinite lifetime. Let us now reason backward. Suppose it is established that a fluid has a "ground state" $|0\rangle$ and bosonic "excitations" $a_q^\dagger |0\rangle$ which cannot decay except by scattering off each

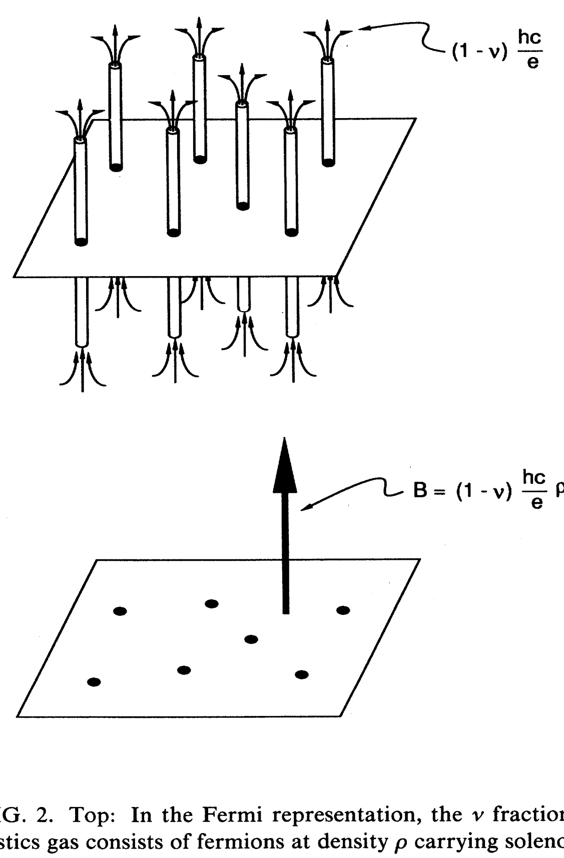

FIG. 2. Top: In the Fermi representation, the ν fractional-statistics gas consists of fermions at density ρ carrying solenoids containing a fraction $(1-\nu)$ of a magnetic flux quantum. Bottom: On the average, each particle sees a uniform magnetic field of $(1-\nu)\rho$ flux quanta per unit area.

other, so that the Hamiltonian may be idealized[5] in the manner

$$\mathcal{H}_{\text{fluid}} = \sum_{\mathbf{k}} [E_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \tfrac{1}{2}\Delta_{\mathbf{k}}(a_{\mathbf{k}} a_{-\mathbf{k}} + a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger)] \ . \tag{1.19}$$

Then the Bogoliubov transformation

$$b_{\mathbf{k}} = u_{\mathbf{k}} a_{\mathbf{k}} + v_{\mathbf{k}} a_{-\mathbf{k}}^\dagger \ , \tag{1.20}$$

where

$$u_k = \left[\frac{E_{\mathbf{k}} + \varepsilon_{\mathbf{k}}}{2\varepsilon_{\mathbf{k}}} \right]^{1/2} ,$$

$$v_k = \left[\frac{E_{\mathbf{k}} - \varepsilon_{\mathbf{k}}}{2\varepsilon_{\mathbf{k}}} \right]^{1/2} , \tag{1.21}$$

$$\varepsilon_k = (E_{\mathbf{k}}^2 - \Delta_{\mathbf{k}}^2)^{1/2}$$

removes the scattering terms, in the manner

$$\mathcal{H}_{\text{fluid}} = \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}} b_{\mathbf{k}}^\dagger b_{\mathbf{k}} \ . \tag{1.22}$$

effectively making a new ground state of the form

$$|\text{vac}\rangle = \prod_{\mathbf{k}} \exp\left[-\frac{v_{\mathbf{k}}}{u_{\mathbf{k}}} a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger \right] |0\rangle \ . \tag{1.23}$$

If the $k \to 0$ limit of $v_{\mathbf{k}}/u_{\mathbf{k}}$ is 1, as is commonly the case,[5] then $|\text{vac}\rangle$ has macroscopic density fluctuations and the requisite broken symmetry of the superfluid state. Thus, a gap in the fermionic spectrum is quite consistent with superfluidity, provided that a soft collective mode is present.

II. EXPECTED ENERGY

Let us begin by evaluating the expected energy

$$\langle \mathcal{H}' \rangle = \frac{\langle \Phi | \mathcal{H}' | \Phi \rangle}{\langle \Phi | \Phi \rangle} , \tag{2.1}$$

where $\mathcal{H}'$ and $|\Phi\rangle$ are given by Eqs. (1.5) and (1.11). Decomposing the Hamiltonian into one-body, two-body, and three-body pieces, in the manner

$$\mathcal{H}' = \mathcal{H}_1 + \mathcal{H}_{2a} + \mathcal{H}_{2b} + \mathcal{H}_3 \ , \tag{2.2}$$

where

$$\mathcal{H}_1 = \sum_{i=1}^{N} \frac{\mathbf{P}_i^2}{2m} , \tag{2.3}$$

$$\mathcal{H}_{2a} = \frac{e}{mc} \sum_{i=1}^{N} \sum_{j \neq i}^{N} \mathbf{A}_{ij} \cdot \mathbf{P}_i \ , \tag{2.4}$$

$$\mathcal{H}_{2b} = \frac{e^2}{2mc^2} \sum_{i=1}^{N} \sum_{j \neq i}^{N} \mathbf{A}_{ij} \cdot \mathbf{A}_{ij} \ , \tag{2.5}$$

$$\mathcal{H}_3 = \frac{e^2}{2mc^2} \sum_{i=1}^{N} \sum_{j \neq i}^{N} \sum_{k \neq i,j}^{N} \mathbf{A}_{ij} \cdot \mathbf{A}_{ik} \ , \tag{2.6}$$

we have

$$\langle \mathcal{H}_1 \rangle = \sum_{l=1}^{N} \int dl \, \varphi_l^*(1) \left[\frac{\mathbf{P}_1^2}{2m} \right] \varphi_l(1) \ , \tag{2.7}$$

where the sum is over all occupied orbitals, the numeral 1 stands for the coordinate $\mathbf{r}_1$, and $d1 = d^2 r_1$ denotes integration with respect to $\mathbf{r}_1$. We have similarly

$$\langle \mathcal{H}_{2a} \rangle = \frac{e}{mc} \sum_{l=1}^{N} \sum_{m=1}^{N} \int d1 \int d2 \, \varphi_l^*(1) \varphi_m^*(2) \, \mathbf{A}_{12} \cdot \mathbf{P}_1 [\varphi_l(1)\varphi_m(2) - \varphi_l(2)\varphi_m(1)] \ , \tag{2.8}$$

$$\langle \mathcal{H}_{2b} \rangle = \frac{e^2}{2mc^2} \sum_{l=1}^{N} \sum_{m=1}^{N} \int d1 \int d2 \, \varphi_l^*(1) \varphi_m^*(2) |\mathbf{A}_{12}|^2 [\varphi_l(1)\varphi_m(2) - \varphi_l(2)\varphi_m(1)] \ , \tag{2.9}$$

$$\langle \mathcal{H}_3 \rangle = \frac{e^2}{2mc^2} \sum_{l=1}^{N} \sum_{m=1}^{N} \sum_{n=1}^{N} \int d1 \int d2 \int d3 \, \varphi_l^*(1) \varphi_m^*(2) \varphi_n^*(3) \, \mathbf{A}_{12} \cdot \mathbf{A}_{13} \left[\sum_{\sigma}^{3!} \text{sgn}(\sigma) \varphi_l(\sigma(1)) \varphi_m(\sigma(2)) \varphi_n(\sigma(3)) \right] . \tag{2.10}$$

The quantity $\overline{\mathbf{A}}$ defined in Eq. (1.14) appears repeatedly in these expressions and is naturally associated with $\mathbf{P}$ in the form of the dynamic momentum $[\mathbf{P} + (e/c)\overline{\mathbf{A}}]$. The computation is greatly simplified if this association is made formally. Thus, picking energy and length units for which $\hbar\omega_c$ and a_0 as defined in Eqs. (1.16) and (1.17) are unity, so that

$$\mathbf{A}_{ij} = (1-\nu)\hat{\mathbf{z}} \times \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^2} \ , \tag{2.11}$$

and

$$\overline{\mathbf{A}}_1 = \int d2 \, \mathbf{A}_{12} \left[\sum_l |\varphi_l(2)|^2 \right] = \tfrac{1}{2} \hat{\mathbf{z}} \times \mathbf{r}_1 \ , \tag{2.12}$$

we rewrite Eq. (2.2) in the manner

$$\langle \mathcal{H}' \rangle = \langle \mathcal{H}_a \rangle + \langle \mathcal{H}_b \rangle + \langle \mathcal{H}_c \rangle + \langle \mathcal{H}_d \rangle + \langle \mathcal{H}_e \rangle + \langle \mathcal{H}_f \rangle \ , \tag{2.13}$$

where

$$\langle \mathcal{H}_a \rangle = \sum_{l=1}^{N} \int d1 \, \varphi_l^*(1) [\tfrac{1}{2}(\mathbf{P}_1 + \overline{\mathbf{A}}_1)^2] \varphi_l(1) \ , \tag{2.14}$$

$$\langle \mathcal{H}_b \rangle = -\sum_{l=1}^{N}\sum_{m=1}^{N} \int d1 \int d2\, \varphi_l^*(1)\varphi_m^*(2)[\mathbf{A}_{12}\cdot(\mathbf{P}_1+\overline{\mathbf{A}}_1)]\varphi_l(2)\varphi_m(1) , \tag{2.15}$$

$$\langle \mathcal{H}_c \rangle = \tfrac{1}{2}\sum_{l=1}^{N}\sum_{m=1}^{N} \int d1 \int d2 |\varphi_l(1)|^2 |\varphi_m(2)|^2 |\mathbf{A}_{12}|^2 , \tag{2.16}$$

$$\langle \mathcal{H}_d \rangle = -\tfrac{1}{2}\sum_{l=1}^{N}\sum_{m=1}^{N} \int d1 \int d2\, \varphi_l^*(1)\varphi_m^*(2) |\mathbf{A}_{12}|^2 \varphi_l(2)\varphi_m(1) , \tag{2.17}$$

$$\langle \mathcal{H}_e \rangle = \sum_{l=1}^{N}\sum_{m=1}^{N}\sum_{n=1}^{N} \int d1 \int d2 \int d3\, \varphi_l^*(1)\varphi_m^*(2)\varphi_n^*(3)\, \mathbf{A}_{12}\cdot \mathbf{A}_{13}\varphi_l(3)\varphi_m(1)\varphi_n(2) , \tag{2.18}$$

$$\langle \mathcal{H}_f \rangle = -\tfrac{1}{2}\sum_{l=1}^{N}\sum_{m=1}^{N}\sum_{n=1}^{N} \int d1 \int d2 \int d3\, |\varphi_l(1)|^2 \varphi_m^*(2)\varphi_n^*(3)\, \mathbf{A}_{12}\cdot \mathbf{A}_{13}\varphi_m(3)\varphi_n(2) . \tag{2.19}$$

Note that Eq. (2.12) has been used to cancel several terms.

III. EXTREMAL CONDITION

Let us now evaluate the extremal condition. If we minimize the total energy with respect to variations in the φ_l^*, subject to the normalization condition

$$\langle \varphi_l | \varphi_l \rangle = 1 , \tag{3.1}$$

we obtain a set of equations of the form

$$\mathcal{H}_{\mathrm{HF}} | \varphi_l \rangle = \varepsilon_l | \varphi_l \rangle , \tag{3.2}$$

where $\mathcal{H}_{\mathrm{HF}}$ is a nonlocal single-body Hamiltonian and ε_l is a Lagrange multiplier. $\mathcal{H}_{\mathrm{HF}}$ decomposes naturally into 12 pieces, in the manner

$$\mathcal{H}_{\mathrm{HF}} = \sum_n \mathcal{H}_{\mathrm{HF}}^{(n)} . \tag{3.3}$$

From the first variation of Eq. (2.14), we obtain

$$\mathcal{H}_{\mathrm{HF}}^{(1)}\varphi_l(1) = \tfrac{1}{2}(\mathbf{P}_1+\overline{\mathbf{A}}_1)^2\varphi_l(1) , \tag{3.4}$$

$$\mathcal{H}_{\mathrm{HF}}^{(2)}\varphi_l(1) = \sum_{m=1}^{N} \int d2\, \varphi_m^*(2)[\mathbf{A}_{21}\cdot(\mathbf{P}_2+\overline{\mathbf{A}}_2)]\varphi_l(1)\varphi_m(2) . \tag{3.5}$$

From the first variation of Eq. (2.15), we obtain

$$\mathcal{H}_{\mathrm{HF}}^{(3)}\varphi_l(1) = -\sum_{m=1}^{N} \int d2\, \varphi_m^*(2)[\mathbf{A}_{12}\cdot(\mathbf{P}_1+\overline{\mathbf{A}}_1)]\varphi_l(2)\varphi_m(1) , \tag{3.6}$$

$$\begin{aligned}\mathcal{H}_{\mathrm{HF}}^{(4)}\varphi_l(1) &= -\sum_{m=1}^{N} \int d2\, \varphi_m^*(2)[\mathbf{A}_{21}\cdot(\mathbf{P}_2+\overline{\mathbf{A}}_2)]\varphi_l(2)\varphi_m(1) \\ &= -\sum_{m=1}^{N} \int d2\, \varphi_l(2)\varphi_m(1)[\mathbf{A}_{21}\cdot(-\mathbf{P}_2+\overline{\mathbf{A}}_2)]\varphi_m^*(2) ,\end{aligned} \tag{3.7}$$

$$\mathcal{H}_{\mathrm{HF}}^{(5)}\varphi_l(1) = -\sum_{m=1}^{N}\sum_{n=1}^{N} \int d2 \int d3\, \varphi_m^*(2)\varphi_n^*(3)\, \mathbf{A}_{32}\cdot \mathbf{A}_{31} \times \varphi_l(1)\varphi_m(3)\varphi_n(2) . \tag{3.8}$$

From the first variation of Eq. (2.16), we obtain

$$\mathcal{H}_{\mathrm{HF}}^{(6)}\varphi_l(1) = \sum_{m=1}^{N} \int d2 |\varphi_m(2)|^2 |\mathbf{A}_{12}|^2 \varphi_l(1) . \tag{3.9}$$

From the first variation of Eq. (2.17), we obtain

$$\mathcal{H}_{\mathrm{HF}}^{(7)}\varphi_l(1) = -\sum_{m=1}^{N} \int d2\, \varphi_m^*(2) |\mathbf{A}_{12}|^2 \varphi_l(2)\varphi_m(1) . \tag{3.10}$$

From the first variation of Eq. (2.18), we obtain

$$\mathcal{H}_{\mathrm{HF}}^{(8)}\varphi_l(1) = \sum_{m=1}^{N}\sum_{n=1}^{N} \int d2 \int d3\, \varphi_m^*(3)\varphi_n^*(2)\, \mathbf{A}_{12}\cdot \mathbf{A}_{13}\varphi_l(2)\varphi_m(1)\varphi_n(3) , \tag{3.11}$$

$$\mathcal{H}_{\mathrm{HF}}^{(9)}\varphi_l(1) = \sum_{m=1}^{N}\sum_{n=1}^{N} \int d2 \int d3\, \varphi_m^*(3)\varphi_n^*(2)\, \mathbf{A}_{21}\cdot \mathbf{A}_{23}\varphi_l(2)\varphi_m(1)\varphi_n(3) , \tag{3.12}$$

$$\mathcal{H}_{\mathrm{HF}}^{(10)}\varphi_l(1) = \sum_{m=1}^{N}\sum_{n=1}^{N} \int d2 \int d3\, \varphi_m^*(3)\varphi_n^*(2)\, \mathbf{A}_{31}\cdot \mathbf{A}_{32}\varphi_l(2)\varphi_m(1)\varphi_n(3) . \tag{3.13}$$

From the first variation of Eq. (2.19), we obtain

$$\mathcal{H}_{\rm HF}^{(11)}\varphi_l(1)=-\tfrac{1}{2}\sum_{m=1}^{N}\sum_{n=1}^{N}\int d2\int d3\,\varphi_m^*(2)\varphi_n^*(3)\,\mathbf{A}_{12}\cdot\mathbf{A}_{13}\varphi_l(1)\varphi_m(3)\varphi_n(2) \tag{3.14}$$

$$\mathcal{H}_{\rm HF}^{(12)}\varphi_l(1)=-\sum_{m=1}^{N}\sum_{n=1}^{N}\int d2\int d3|\varphi_n(3)|^2\varphi_m^*(2)\,\mathbf{A}_{31}\cdot\mathbf{A}_{32}\varphi_m(1)\varphi_l(2)\ . \tag{3.15}$$

IV. LANDAU LEVEL PROJECTORS

It is possible to solve Eq. (3.2) exactly because the orbitals guessed in Eq. (1.18) are correct. The demonstration of this requires that we first establish two key properties of the projector onto the nth Landau level, defined in the manner

$$\Pi_n=\sum_k|\varphi_{nk}\rangle\langle\varphi_{nk}|\ . \tag{4.1}$$

Π_n is a nonlocal operator similar in behavior to the Hartree-Fock Hamiltonian defined in Eq. (3.2). From Eq. (1.18) it can be seen that the nth projector may be generated from the zeroth, given explicitly by

$$\Pi_0(z_1,z_2)=\prod_{k=0}^{\infty}\varphi_{0k}(z_1)\varphi_{0k}^*(z_2)$$
$$=\frac{1}{2\pi}e^{-(1/4)|z_1|^2+|z_2|^2)}e^{(1/2)z_1^*z_2}\ , \tag{4.2}$$

in the manner

$$\Pi_n(z_1,z_2)=\frac{1}{2^n n!}(\tfrac{1}{2}z_1-2\partial_{z_1^*})^n(\tfrac{1}{2}z_2^*-2\partial_{z_2})^n\Pi_0(z_1,z_2)$$
$$=L_n(\tfrac{1}{2}|z_1-z_2|^2)\Pi_0(z_1,z_2)\ , \tag{4.3}$$

where $L_n(t)$ is the nth Laguerre polynomial.[16] However, from the orthonormality of these polynomials it also follows that

$$F(|z_1-z_2|)\Pi_0(z_1,z_2)$$
$$=\sum_{n=0}^{\infty}\left[\int_0^{\infty}F(r)L_n(\tfrac{1}{2}r^2)e^{-(1/2)r^2}r\,dr\right]\Pi_n(z_1,z_2)\ , \tag{4.4}$$

for any function F. Thus the set of Π_n is complete over the set of nonlocal operators consisting of a function of separation times Π_0. Proving that the Landau orbitals constitute an extremal solution therefore reduces to showing that $\mathcal{H}_{\rm HF}$ is of this form and then computing its projector expansion coefficients.

V. BOSE FLUID

Since the formalism used in the paper is new, its predictions need to be tested in limits for which the answer is known. The limit of $\nu=1$ is the noninteracting Fermi gas, and thus is trivially correct. The other limit for which we know the answer is $\nu=0$, which corresponds to the noninteracting Bose fluid. If the case can be made that the formalism describes this limit properly, there is good reason to believe its predictions for $\nu=\frac{1}{2}$. Let us therefore solve Eq. (3.2) for the case of $\nu=0$.

We shall proceed by assuming that the orbitals φ_l appearing in Eq. (3.2) are of the form of Eq. (1.18) and that only the lowest Landau level is occupied. This solution is extremal if the Hartree-Fock Hamiltonian $\mathcal{H}_{\rm HF}$ generated from the filled orbitals using Eqs. (3.3)–(3.15) may be expressed as a sum of Landau level projectors Π_n defined by Eq. (4.1). From Eqs. (3.4)–(3.7) we obtain

$$\mathcal{H}_{\rm HF}^{(1)}=\sum_{n=0}^{\infty}(n+\tfrac{1}{2})\Pi_n\ , \tag{5.1}$$

$$\mathcal{H}_{\rm HF}^{(2)}=\lim_{3\to2}(\mathbf{P}_2+\overline{\mathbf{A}}_2)\Pi_0(2,3)=0\ , \tag{5.2}$$

$$\mathcal{H}_{\rm HF}^{(3)}=-[\,\mathbf{A}_{12}\cdot(\mathbf{P}_1+\overline{\mathbf{A}}_1)]\Pi_0(1,2)=-\tfrac{1}{2}\Pi_0\ , \tag{5.3}$$

$$\mathcal{H}_{\rm HF}^{(4)}=-\{[\,\mathbf{A}_{21}\cdot(\mathbf{P}_2+\overline{\mathbf{A}}_2)]\Pi_0(2,1)\}^*=-\tfrac{1}{2}\Pi_0\ . \tag{5.4}$$

Using a number of algebraic steps that are described in detail in Sec. VI, we obtain from Eq. (3.8)

$$\mathcal{H}_{\rm HF}^{(5)}=-\tfrac{1}{4}\ . \tag{5.5}$$

From Eq. (3.9) we obtain

$$\mathcal{H}_{\rm HF}^{(6)}=\frac{1}{2\pi}\int d2|\,\mathbf{A}_{12}|^2=\lim_{\epsilon\to0}\int_0^R dr\frac{1}{r+\epsilon}\ . \tag{5.6}$$

where R is the sample radius. This expression is formally infinite, but combines with Eq. (5.7) to yield a finite result. From Eq. (3.10) we obtain

$$\mathcal{H}_{\rm HF}^{(7)}=-\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{2^k k!}\lim_{\epsilon\to0}\int_0^R dr(r+\epsilon)^{2k-1}e^{-(1/2)r^2}\right]\Pi_n$$
$$=-\lim_{\epsilon\to0}\int_0^R dr\frac{1}{r+\epsilon}e^{-(1/2)r^2}+\tfrac{1}{2}\sum_{n=1}^{\infty}\left[\sum_{k=1}^{n}\frac{1}{k}\right]\Pi_n\ , \tag{5.7}$$

where we have used Eq. (A1) in the last line. Again using algebraic methods discussed in Sec. VI, we obtain from Eqs. (3.11)–(3.15)

$$\mathcal{H}_{\rm HF}^{(8)}=\tfrac{1}{4}\Pi_0\ , \tag{5.8}$$

$$\mathcal{H}_{\rm HF}^{(9)}=\tfrac{1}{4}\Pi_0\ , \tag{5.9}$$

$$\mathcal{H}_{\rm HF}^{(10)}=\tfrac{1}{4}\Pi_0+\tfrac{1}{4}\sum_{n=1}^{\infty}\left[\frac{1}{n+1}+\frac{1}{n}\right]\Pi_n\ , \tag{5.10}$$

$$\mathcal{H}_{\rm HF}^{(11)}=-\tfrac{1}{2}E_R\ , \tag{5.11}$$

$$\mathcal{H}_{\rm HF}^{(12)}=-E_R\Pi_0-\tfrac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\Pi_n\ , \tag{5.12}$$

where

$$E_R=\int_0^R dr\frac{1}{r}(1-e^{-(1/2)r^2})=\ln(R)+\tfrac{1}{2}[\gamma-\ln(2)]\ , \tag{5.13}$$

with $\gamma=0.577\ldots$ denoting Euler's constant. Combining these results, we obtain finally

$$\mathcal{H}_{\rm HF}=(-\tfrac{1}{2}E_R)\Pi_0+\sum_{n=1}^{\infty}\left[n+\tfrac{1}{4}-\frac{1}{4n(n+1)}+\tfrac{1}{2}\sum_{k=1}^{n}\frac{1}{k}+\tfrac{1}{2}E_R\right]\Pi_n\ . \tag{5.14}$$

This expression, which supersedes Eq. (6) of Ref. 1, demonstrates the extremal nature of the ground state.

Let us now consider the eigenvalues of $\mathcal{H}_{\rm HF}$, the first few of which are listed in Table I and sketched in Fig. 3. According to Koopmans's theorem,[17] these should roughly approximate the energies of excited states associated with the injection of a "particle" or a "hole," measured with the chemical potential set to zero. The largest factor in this energy is the logarithmically large quantity $\frac{1}{2}E_R$, which is the same for particles and holes. The large size of this energy is consistent with the fact that the Bose fluid has no low-lying fermionic excitations. Also, the equality of the particle and hole energies indicates that the state is stable, since otherwise it could gain a logarithmically large energy by bringing charge in from infinity. The fact that this energy is logarithmic suggests strongly that the excitations to which the particle and hole correspond are charged vortices. This will be discussed further in Sec. VIII.

TABLE I. Eigenvalues of $\mathcal{H}_{\rm HF}$, as given by Eq. (5.14), for $\nu=0$.

Landau level (n)	Eigenvalue (ε_n)
0 (filled)	$-\frac{1}{2}E_R$
1 (empty)	$\frac{13}{8}+\frac{1}{2}E_R$
2 (empty)	$\frac{71}{24}+\frac{1}{2}E_R$
3 (empty)	$\frac{199}{48}+\frac{1}{2}E_R$

Regardless of their physical nature, the particles and holes form a natural basis for describing the true ground state and elementary excitations perturbatively. The practicality of this depends on whether ways can be found to handle the large interactions between them. The presence of such interactions is easily appreciated by noting that if they were absent the lowest excited state energy would be $\frac{13}{8}+E_R$, which is absurd. The system has no preferred density, and thus its low-lying excitations must be soft compressional sound waves.

Let us now pursue the question of compressional sound by calculating the ground-state energy, summing Eqs. (2.14)–(2.19) over a single filled Landau level. From Eq. (2.14) using Eq. (5.1) we obtain

$$\langle\mathcal{H}_a\rangle=\sum_{l=1}^{N}\left\langle\varphi_l\left|\frac{1}{2}+\sum_{n=1}^{\infty}n\Pi_n\right|\varphi_l\right\rangle=\tfrac{1}{2}N\ . \tag{5.15}$$

From Eq. (2.15), using Eq. (5.3) we obtain

$$\langle\mathcal{H}_b\rangle=\sum_{l=1}^{N}\langle\varphi_l|-\tfrac{1}{2}\Pi_0|\varphi_l\rangle=-\tfrac{1}{2}N\ . \tag{5.16}$$

From Eq. (2.16), using Eq. (5.6) we obtain

$$\langle\mathcal{H}_c\rangle=\tfrac{1}{2}\sum_{l=1}^{N}\left\langle\varphi_l\left|\lim_{\epsilon\to 0}\int_0^R dr\frac{1}{r+\epsilon}\right|\varphi_l\right\rangle=\tfrac{1}{2}N\lim_{\epsilon\to 0}\int_0^R dr\frac{1}{r+\epsilon}\ , \tag{5.17}$$

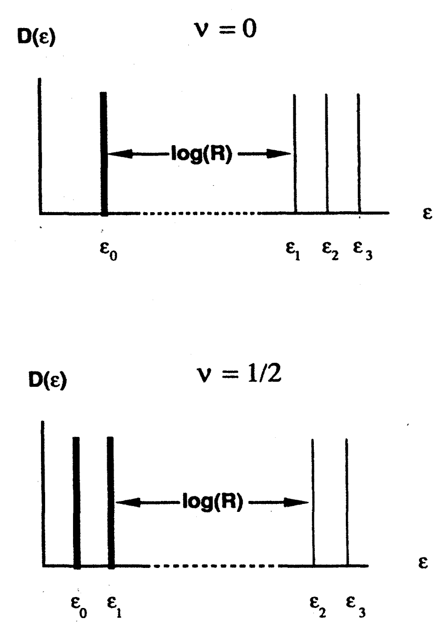

FIG. 3. Illustration of Hartree-Fock density of states for Bose (top) and $\nu=\frac{1}{2}$ fractional statistics (bottom) fluids, as given by Eqs. (5.14) and (7.13). In each case, the number of occupied Landau-levels is $(1-\nu)^{-1}$, and an energy gap that grows as the logarithm of the sample size R appears between the highest occupied and lowest empty Landau level. The logarithmic nature of this gap is due to the fact that particles and holes are charged vortices.

which is formally infinite, but combines with Eq. (5.18) to yield a finite result. From Eq. (2.17), using Eq. (5.7) we obtain

$$\langle \mathcal{H}_d \rangle = -\tfrac{1}{2}\sum_{l=1}^{N}\left\langle \varphi_l \left| \lim_{\epsilon\to 0}\int_0^R dr\frac{1}{r+\epsilon}e^{-(1/2)r^2} - \sum_{n=1}^{\infty}\left[\sum_{k=1}^{n}\frac{1}{2k}\right]\Pi_n \right| \varphi_l \right\rangle = -\tfrac{1}{2}N\lim_{\epsilon\to 0}\int_0^R dr\frac{1}{r+\epsilon}e^{-(1/2)r^2} . \quad (5.18)$$

From Eq. (2.18), using Eq. (5.8) we obtain

$$\langle \mathcal{H}_e \rangle = \sum_{l=1}^{N}\langle \varphi_l | \tfrac{1}{4}\Pi_0 | \varphi_l \rangle = \tfrac{1}{4}N . \quad (5.19)$$

From Eq. (2.19), using Eq. (5.11) we obtain

$$\langle \mathcal{H}_f \rangle = \sum_{l=1}^{N}\langle \varphi_l | -\tfrac{1}{2}E_R | \varphi_l \rangle = -\tfrac{1}{2}NE_R . \quad (5.20)$$

Evaluating Eq. (2.13) by summing Eqs. (5.15)–(5.20) we obtain finally

$$\langle \mathcal{H}' \rangle = \frac{N}{4} . \quad (5.21)$$

This result is midway between the true Bose gas energy of zero and the energy $N/2$ of a spinless two-dimensional Fermi gas. In light of Eq. (1.16), this energy is proportional to the number density ρ, and gives the finite sound speed

$$v_s = \left[\frac{\partial}{\partial\rho}\left[\rho^2\frac{\partial}{\partial\rho}\frac{E}{N}\right]\right]^{1/2} = \sqrt{1/2} , \quad (5.22)$$

in units of $\omega_c a_0$. Note that it is stated erroneously in Ref. 1 that the sound speed is zero.

Since the problem we are actually solving is the noninteracting Bose gas, the finite sound speed of Eq. (5.22) is at least partially a pathology. Its origin may be understood by considering the Bose wave function to which the Hartree-Fock ground state we have constructed corresponds. From Eq. (1.8) we have

$$\Psi(z_1,\ldots,z_N) = \prod_{j<k}^{N}|z_j - z_j|\exp\left[-\tfrac{1}{4}\sum_{l=1}^{N}|z_l|^2\right] , \quad (5.23)$$

where we have used the Vandermonde determinant expansion of Eq. (1.11). Thus, use of the Fermi representation forces the Bose wave function to have nodes at particle coincidences. While the Fermi wave functions have sufficient variational freedom to describe the true ground state, in that the wave function may rise from zero to a constant over a short distance without significant variational energy cost, high orders in perturbation theory are required to accomplish this. However, a wave function of this form is quite reasonable for a Bose fluid with strong short-range repulsive potentials, such as helium.[5]

VI. CONTOUR INTEGRATION

The evaluation of $\mathcal{H}_{\rm HF}^{(5)}$ and $\mathcal{H}_{\rm HF}^{(8)}-\mathcal{H}_{\rm HF}^{(13)}$ in the preceding section, which derive from the three-body interactions in the problem, involves a number of algebraic steps which require discussion. In doing so we shall make frequent use of the identity

$$\mathbf{A}_{12}\cdot\mathbf{A}_{13} = \tfrac{1}{2}(1-\nu)^2\left[\frac{1}{(z_1-z_2)^*(z_1-z_3)} + \frac{1}{(z_1-z_2)(z_1-z_3)^*}\right] . \quad (6.1)$$

Let us first consider $\mathcal{H}_{\rm HF}^{(5)}$. From Eq. (3.8) we have

$$\mathcal{H}_{\rm HF}^{(5)} = -\int d2\int d3\ \mathbf{A}_{32}\cdot\mathbf{A}_{31}|\Pi_0(2,3)|^2 = -\frac{1}{8\pi^2}\int d^2y\int d^2z\left[\frac{1}{z(y+z/2)^*} + \frac{1}{z^*(y+z/2)}\right]e^{-(1/2)|z|^2} , \quad (6.2)$$

where $z = z_2 - z_3$ and $y = \tfrac{1}{2}(z_2+z_3)+z_1$. Writing $z = re^{i\theta}$ and performing the angular integration as a contour integral in the manner

$$\int_0^{2\pi}\left[\frac{1}{z(y+z/2)^*}+\frac{1}{z^*(y+z/2)}\right]d\theta = \frac{1}{i}\oint_{|u|=1}\left[\frac{1}{ry^*}\frac{1}{u(u+r/2y^*)} + \frac{2}{r^2}\frac{1}{(u+2y/r)}\right]du = \begin{cases} 0, & r<|2y| \\ \dfrac{8\pi}{r^2}, & r\ge|2y| , \end{cases} \quad (6.3)$$

we obtain

$$\mathcal{H}_{\rm HF}^{(5)} = -\frac{1}{\pi}\int d^2y\int_{|2y|}^{\infty}dr\frac{1}{r}e^{-(1/2)r^2} = -\tfrac{1}{4} . \quad (6.4)$$

Note that care must be taken in choosing the change of variables to obtain the correct result given by Eq. (6.4). With our choice (y,z) of integration variables, Eq. (6.4) agrees with the requirement that the integrals of the first line and last two lines of Eq. (6.2) with respect to z_1 be equal. From Eq. (3.11) we have

$$\mathcal{H}_{\rm HF}^{(8)} = \int d3\ \mathbf{A}_{12}\cdot\mathbf{A}_{13}\Pi_0(1,3)\Pi_0(3,2) = \frac{1}{4\pi}\Pi_0(1,2)\int d^2z\left[\frac{1}{zy^*}+\frac{1}{z^*y}\right]\times e^{-(1/2)|z|^2}e^{(1/2)yz^*} , \quad (6.5)$$

where $z = z_3 - z_1$ and $y = z_2 - z_1$. Defining $z = re^{i\theta}$ we have for the angular integral

$$\int_0^{2\pi}\left[\frac{1}{zy^*}+\frac{1}{z^*y}\right]e^{(1/2)yz^*}d\theta$$
$$=\sum_{k=0}^{\infty}\frac{1}{k!}\int_0^{2\pi}\left[\frac{e^{-i\theta}}{ry^*}+\frac{e^{i\theta}}{ry}\right]\left[\frac{rye^{-i\theta}}{2}\right]^k d\theta=\pi\ , \tag{6.6}$$

and thus

$$\mathcal{H}_{\rm HF}^{(8)}=\tfrac{1}{4}\Pi_0\ . \tag{6.7}$$

From Eq. (3.12) we have

$$\mathcal{H}_{\rm HF}^{(9)}=\int d3\ \mathbf{A}_{21}\cdot\mathbf{A}_{23}\Pi_0(1,3)\Pi_0(3,2)\ . \tag{6.8}$$

Since this is the complex conjugate of Eq. (6.5) under the interchange $1\leftrightarrow 2$, we have

$$\mathcal{H}_{\rm HF}^{(9)}=\tfrac{1}{4}\Pi_0\ . \tag{6.9}$$

From Eq. (3.13) we have

$$\mathcal{H}_{\rm HF}^{(10)}=\int d3\ \mathbf{A}_{31}\cdot\mathbf{A}_{32}\Pi_0(1,3)\Pi_0(3,2)=\frac{1}{4\pi}\Pi_0(1,2)\int d^2z\left[\frac{1}{z^*(z+y)}+\frac{1}{z(z+y)^*}\right]e^{-(1/2)|z|^2}e^{-(1/2)yz^*}\ , \tag{6.10}$$

where $z=z_3-z_1$ and $y=z_1-z_2$. Defining $z=re^{i\theta}$ and performing the angular integration as a contour integral, in the manner

$$\int_0^{2\pi}\left[\frac{1}{z^*(z+y)}+\frac{1}{z(z+y)^*}\right]e^{-(1/2)yz^*}d\theta=\frac{1}{i}\oint_{|u|=1}\left[\frac{1}{ry}\frac{1}{u(u+r/y)}+\frac{2}{r^2}\frac{1}{(u+y^*/r)}\right]e^{-(1/2)yru}du$$
$$=\frac{2\pi}{r^2}+\frac{2\pi}{r^2}\begin{cases}\exp(-\tfrac{1}{2}r^2), & r<|y|\\ \exp(\tfrac{1}{2}|y|^2), & r\ge|y|\ ,\end{cases} \tag{6.11}$$

we obtain

$$\mathcal{H}_{\rm HF}^{(10)}=F(|1-2|)\Pi_0(1,2)\ , \tag{6.12}$$

where

$$F(s)=\tfrac{1}{2}\left[-\int_0^s\frac{1}{r}(1-e^{-(1/2)r^2})dr+(1+e^{(1/2)s^2})\int_s^{\infty}\frac{1}{r}e^{-(1/2)r^2}dr\right]\ . \tag{6.13}$$

In light of Eq. (4.4) this may also be written

$$\mathcal{H}_{\rm HF}^{(10)}=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(-1)^k}{2^k k!}\int_0^{\infty}ds\ s^{2k+1}e^{-(1/2)s^2}F(s)\right]\Pi_n\ . \tag{6.14}$$

The coefficient of Π_n in this expression may be evaluated simply by reversing the order of the r and s integrations. We have specifically

$$\int_0^{\infty}ds\ s^{2k+1}e^{-(1/2)s^2}F(s)=\lim_{\epsilon\to 0}\tfrac{1}{2}\int_0^{\infty}ds\ s^{2k+1}e^{-(1/2)s^2}\left[\int_{\epsilon}^{\infty}\frac{1}{r}e^{-(1/2)r^2}dr+e^{(1/2)s^2}\int_s^{\infty}\frac{1}{r}e^{-(1/2)r^2}dr-\int_{\epsilon}^{s}\frac{1}{r}dr\right]$$
$$=\lim_{\epsilon\to 0}\frac{1}{2}\left[2^k k!\int_{\epsilon}^{\infty}\frac{1}{r}e^{-(1/2)r^2}dr+\int_{\epsilon}^{\infty}\frac{1}{r}e^{-(1/2)r^2}\frac{r^{2k+2}}{2k+2}dr\right.$$
$$\left.-2^k k!\sum_{p=0}^{k}\frac{1}{p!}\int_{\epsilon}^{\infty}\frac{1}{r}e^{-(1/2)r^2}(\tfrac{1}{2}r^2)^p dr\right]$$
$$=\frac{2^k k!}{4}\left[\frac{1}{k+1}-(1-\delta_{k0})\sum_{p=1}^{k}\frac{1}{p}\right]\ , \tag{6.15}$$

and thus

$$\mathcal{H}_{\rm HF}^{(10)}=\tfrac{1}{4}\Pi_0+\tfrac{1}{4}\sum_{n=1}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}(-1)^k\left[\frac{1}{k+1}-(1-\delta_{k0})\sum_{p=1}^{k}\frac{1}{p}\right]\Pi_n\ . \tag{6.16}$$

Using Eqs. (A2) and (A3), we find that this simplifies to

$$\mathcal{H}_{\rm HF}^{(10)}=\tfrac{1}{4}\Pi_0+\tfrac{1}{4}\sum_{n=1}^{\infty}\left[\frac{1}{n+1}+\frac{1}{n}\right]\Pi_n \; . \tag{6.17}$$

From Eq. (3.14) we have

$$\begin{aligned}\mathcal{H}_{\rm HF}^{(11)}&=-\tfrac{1}{2}\int d2\int d3\ \mathbf{A}_{12}\cdot\mathbf{A}_{13}|\Pi_0(2,3)|^2\\&=\frac{-1}{16\pi^2}\int d^2z\int d^2y\left[\frac{1}{z^*(z+y)}+\frac{1}{z(z+y)^*}\right]e^{-(1/2)|y|^2} ,\end{aligned} \tag{6.18}$$

where $z=z_3-z_1$ and $y=z_2-z_3$. Defining $y=se^{i\theta}$ and performing the angular part of the y integration as a contour integral, in the manner

$$\begin{aligned}\int_0^{2\pi}\left[\frac{1}{z^*(z+y)}+\frac{1}{z(z+y)^*}\right]d\theta&=\frac{1}{i}\oint_{|u|=1}\left[\frac{1}{z^*s}\frac{1}{u(u+z/s)}+\frac{1}{|z|^2}\frac{1}{(u+s/z^*)}\right]du\\&=\begin{cases}0, & s>|z|\\ \dfrac{4\pi}{|z|^2}, & s\le|z| ,\end{cases}\end{aligned} \tag{6.19}$$

and thus

$$\mathcal{H}_{\rm HF}^{(11)}=-\tfrac{1}{2}\int_0^{\infty}dr\frac{1}{r}\left[\int_0^{r}ds\ se^{-(1/2)s^2}\right]=-\tfrac{1}{2}E_R\ , \tag{6.20}$$

where E_R is defined as in Eq. (5.13). From Eq. (3.15) we have

$$\begin{aligned}\mathcal{H}_{\rm HF}^{(12)}&=\frac{-1}{2\pi}\Pi_0(1,2)\int d3\ \mathbf{A}_{31}\cdot\mathbf{A}_{32}\\&=-\frac{1}{4\pi}\Pi_0(1,2)\int d^2z\left[\frac{1}{z^*(z+y)}+\frac{1}{z(z+y)^*}\right] ,\end{aligned} \tag{6.21}$$

where $z=z_3-z_1$ and $y=z_1-z_2$. Writing $z=re^{i\theta}$ and performing the angular integration as a contour integral, in the manner

$$\begin{aligned}\int_0^{2\pi}\left[\frac{1}{z^*(z+y)}+\frac{1}{z(z+y)^*}\right]d\theta&=\frac{1}{i}\oint_{|u|=1}\left[\frac{1}{ry}\frac{1}{u(u+r/y)}+\frac{1}{r^2}\frac{1}{(u+y^*/r)}\right]du\\&=\begin{cases}0, & r>|y|\\ \dfrac{4\pi}{r^2}, & r\le|y| ,\end{cases}\end{aligned} \tag{6.22}$$

and thus

$$\mathcal{H}_{\rm HF}^{(12)}=F(|1-2)\Pi_0(1,2)\ , \tag{6.23}$$

where

$$F(s)=-\int_s^R dr\frac{1}{r}\ . \tag{6.24}$$

In light of Eq. (4.4), this may also be written

$$\mathcal{H}_{\rm HF}^{(12)}=-\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}\frac{(-1)^k}{2^k k!}\int_0^{\infty}ds\ s^{2k+1}e^{-(1/2)s^2}\int_s^R dr\frac{1}{r}\right]\Pi_n\ . \tag{6.25}$$

The coefficient of Π_n in this expression may again be evaluated by reversing the order of the r and s integrations. We have

$$\int_0^{\infty}ds\ s^{2k+1}e^{-(1/2)s^2}F(s)=\int_0^R dr\frac{1}{r}\left[1-e^{-(1/2)r^2}\sum_{p=0}^{k}\frac{(\frac{1}{2}r^2)^p}{p!}\right] , \tag{6.26}$$

which gives

$$\mathcal{H}_{\rm HF}^{(12)}=\left[-\int_0^R ds\frac{1}{s}(1-e^{-(1/2)s^2})\right]\Pi_0+\sum_{n=1}^{\infty}\left[\sum_{k=1}^{n}\binom{n}{k}(-1)^k\tfrac{1}{2}\sum_{p=1}^{k}\frac{1}{p}\right]\Pi_n\ . \tag{6.27}$$

Using Eq. (A3), we find this simplifies to

$$\mathcal{H}_{\rm HF}^{(12)}=-E_R\Pi_0-\tfrac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}\Pi_n\ . \tag{6.28}$$

VII. $\frac{1}{2}$ FRACTIONAL-STATISTICS FLUID

Let us now reevaluate the expressions of Sec. V for the case of two filled ($n=0$ and $n=1$) Landau levels, which corresponds to a two-dimensional gas of particles obeying one-half fractional statistics. These are slightly more complex, due to the necessity of evaluating integrals with factors of $\Pi_0+\Pi_1$ rather than Π_0 only, but ultimately quite similar. From Eqs. (3.4)–(3.7) we obtain

$$\mathcal{H}_{\rm HF}^{(1)}=\sum_{n=0}^{\infty}(n+\tfrac{1}{2})\Pi_n\ , \tag{7.1}$$

$$\mathcal{H}_{\rm HF}^{(2)}=0\ , \tag{7.2}$$

$$\mathcal{H}_{\rm HF}^{(3)}=-\tfrac{1}{4}\Pi_0-\tfrac{1}{4}\Pi_1\ , \tag{7.3}$$

$$\mathcal{H}_{\rm HF}^{(4)}=-\tfrac{1}{4}\Pi_0-\tfrac{1}{4}\Pi_1\ . \tag{7.4}$$

Using algebraic steps described in Sec. VI, we obtain from Eq. (3.8)

$$\mathcal{H}_{\mathrm{HF}}^{(5)} = -\tfrac{1}{8} \ . \tag{7.5}$$

From Eq. (3.9) we obtain

$$\mathcal{H}_{\mathrm{HF}}^{(6)} = \tfrac{1}{2} \lim_{\epsilon \to 0} \int_0^R dr \frac{1}{r+\epsilon} \ , \tag{7.6}$$

where R is the sample radius. This expression is formally infinite, but combines with Eq. (7.7) to yield a finite result. From Eq. (3.10) we obtain

$$\mathcal{H}_{\mathrm{HF}}^{(7)} = -\tfrac{1}{2} \lim_{\epsilon \to 0} \int_0^R dr \frac{1}{r+\epsilon} e^{-(1/2)r^2} + \tfrac{1}{8}\Pi_0 + \tfrac{1}{4}\Pi_1 + \tfrac{1}{4} \sum_{n=2}^{\infty} \left[\sum_{k=1}^{n} \frac{1}{k} \right] \Pi_n \ , \tag{7.7}$$

where we have used Eq. (A1) in the last line. Again using algebraic methods discussed in Sec. VI, we obtain from Eqs. (3.11)–(3.15)

$$\mathcal{H}_{\mathrm{HF}}^{(8)} = \tfrac{3}{16}\Pi_0 + \tfrac{1}{16}\Pi_1 \ , \tag{7.8}$$

$$\mathcal{H}_{\mathrm{HF}}^{(9)} = \tfrac{3}{16}\Pi_0 + \tfrac{1}{16}\Pi_1 \ , \tag{7.9}$$

$$\mathcal{H}_{\mathrm{HF}}^{(10)} = \tfrac{1}{16}\Pi_0 + \tfrac{3}{16}\Pi_1 + \tfrac{1}{4} \sum_{n=2}^{\infty} \left[\frac{1}{n} + \frac{1}{n+1} \right] \Pi_n \ , \tag{7.10}$$

$$\mathcal{H}_{\mathrm{HF}}^{(11)} = -\tfrac{1}{16} - \tfrac{1}{4}E_R \ , \tag{7.11}$$

$$\mathcal{H}_{\mathrm{HF}}^{(12)} = (-\tfrac{1}{4} - \tfrac{1}{2}E_R)\Pi_0 - \tfrac{1}{2}E_R\Pi_1 - \tfrac{1}{4} \sum_{n=2}^{\infty} \left[\frac{1}{n} + \frac{1}{n-1} \right] \Pi_n \ , \tag{7.12}$$

where E_R is defined as in Eq. (5.13). Combining these results, we obtain finally

$$\mathcal{H}_{\mathrm{HF}} = (\tfrac{1}{8} - \tfrac{1}{4}E_R)\Pi_0 + (\tfrac{11}{8} - \tfrac{1}{4}E_R)\Pi_1 + \sum_{n=2}^{\infty} \left[n + \tfrac{5}{16} - \frac{1}{2(n^2-1)} + \tfrac{1}{4} \sum_{k=1}^{n} \frac{1}{k} + \tfrac{1}{4}E_R \right] \Pi_n \ . \tag{7.13}$$

This result, which supersedes Eq. (8) of Ref. 1, demonstrates the extremal nature of the ground state.

Let us now consider the eigenvalues of $\mathcal{H}_{\mathrm{HF}}$, the first few of which are listed in Table II and sketched in Fig. 3. As is the case in Table I, the largest factor in this energy is the logarithmically large quantity $\frac{1}{4}E_R$, which is the same for particles and holes. This energy is actually $\frac{1}{4}$ of the corresponding quantity for the Bose gas because the energy unit is half as large [cf. Eq. (1.16)]. This is due to the fact that the vortex associated with this excitation has $\frac{1}{2}$ the vorticity, and thus $\frac{1}{4}$ the kinetic energy, of the Bose fluid vortex. This will be discussed further in Sec. VIII.

TABLE II. Eigenvalues of $\mathcal{H}_{\mathrm{HF}}$, as given by Eq. (7.13), for $\nu = \frac{1}{2}$.

Landau level (n)	Eigenvalue (ε_n)
0 (filled)	$\frac{1}{8} - \frac{1}{4}E_R$
1 (filled)	$\frac{11}{8} - \frac{1}{4}E_R$
2 (empty)	$\frac{121}{48} + \frac{1}{4}E_R$
3 (empty)	$\frac{89}{24} + \frac{1}{4}E_R$

Let us now calculate the ground-state energy, summing Eqs. (2.14)–(2.19) over two ($n=0$ and $n=1$) filled Landau levels. From Eq. (2.14), using Eq. (7.1) we obtain

$$\langle \mathcal{H}_a \rangle = \sum_{l=1}^{N} \left\langle \varphi_l \left| \frac{1}{2} + \Pi_1 + \sum_{n=1}^{\infty} n\Pi_n \right| \varphi_l \right\rangle = N \ . \tag{7.14}$$

From Eq. (2.15), using Eq. (7.3) we obtain

$$\langle \mathcal{H}_b \rangle = \sum_{l=1}^{N} \langle \varphi_l | -\tfrac{1}{4}(\Pi_0 + \Pi_1) | \varphi_l \rangle = -\tfrac{1}{4}N \ . \tag{7.15}$$

From Eq. (2.16), using Eq. (7.5) we obtain

$$\langle \mathcal{H}_c \rangle = \tfrac{1}{2} \sum_{l=1}^{N} \left\langle \varphi_l \left| \frac{1}{2} \int_0^{\infty} dr \frac{1}{r} \right| \varphi_l \right\rangle = \tfrac{1}{4}N \lim_{\epsilon \to 0} \int_0^R dr \frac{1}{r+\epsilon} \ , \tag{7.16}$$

which is formally infinite, but combines with Eq. (7.17) to yield a finite result. From Eq. (2.17), using Eq. (7.7) we obtain

$$\langle \mathcal{H}_d \rangle = \tfrac{1}{2} \sum_{l=1}^{N} \left\langle \varphi_l \left| -\tfrac{1}{2} \lim_{\epsilon \to 0} \int_0^R dr \frac{1}{r+\epsilon} e^{-(1/2)r^2} + \tfrac{1}{8}\Pi_0 + \tfrac{1}{4}\Pi_1 \right| \varphi_l \right\rangle = -\tfrac{1}{4}N \lim_{\epsilon \to 0} \int_0^R dr \frac{1}{r+\epsilon} e^{-(1/2)r^2} + \tfrac{3}{32}N \ . \tag{7.17}$$

From Eq. (2.18), using Eq. (7.8) we obtain

$$\langle \mathcal{H}_e \rangle = \sum_{l=1}^{N} \langle \varphi_l | \tfrac{3}{16}\Pi_0 + \tfrac{1}{16}\Pi_1 | \varphi_l \rangle = \tfrac{1}{8}N \ . \tag{7.18}$$

From Eq. (2.19), using Eq. (7.11) we obtain

$$\langle \mathcal{H}_f \rangle = \sum_{l=1}^{N} \langle \varphi_l | -\tfrac{1}{16} - \tfrac{1}{4}E_R | \varphi_l \rangle = -\tfrac{1}{16}N - \tfrac{1}{4}NE_R \ . \tag{7.19}$$

Evaluating Eq. (2.13) by summing Eqs. (7.14)–(7.19) we obtain finally

$$\langle \mathcal{H}' \rangle = \tfrac{29}{32}N \ . \tag{7.20}$$

This result is only slightly smaller than the ground-state energy N of a two-dimensional Fermi gas in an external vector potential $\overline{\mathbf{A}}_1$. In light of Eq. (1.16), this energy is proportional to the number density ρ, and gives a finite sound speed of

$$v_s = \left[\frac{\partial}{\partial \rho} \left[\rho^2 \frac{\partial}{\partial \rho} \frac{E}{N} \right] \right]^{1/2} = \sqrt{29/16} \ , \tag{7.21}$$

in units of $\omega_c a_0$.

Unlike the result of Eq. (5.22), this finite sound speed is not an artifact of the formalism, but is rather mandated by the nodes in the fractional-statistics wave function at particle coincidences. The value of v_s is not expected to be accurate, however, because the Hartree-Fock wave function need not give the correct functional form of the node.

VIII. CHARGED VORTICES

As we previously remarked, the logarithmic nature of the energy cost to make particle and hole excitations in the Bose and $\frac{1}{2}$ fractional-statistics fluids indicates that they are actually charged vortices. Let us now demonstrate this explicitly by calculating the expectation of the current density operator, given by

$$\mathbf{J}(\mathbf{r})=\tfrac{1}{2}\sum_{i=1}^{N}\left\{\mathbf{P}_i+\sum_{j\neq i}\mathbf{A}_{ij},\delta(\mathbf{r}-\mathbf{r}_i)\right\}, \tag{8.1}$$

where $\mathbf{A}_{ij}$ is given by Eq. (1.6). For any single Slater determinant wave function $|\Psi\rangle$ comprised of orbitals φ_l, the expected current density at $\mathbf{r}_1$ is given by

$$\langle\Psi|\mathbf{J}(1)|\Psi\rangle=\sum_{l=1}^{N}\int d2\,\varphi_l^*(2)\tfrac{1}{2}\{\mathbf{P}_2,\delta(1-2)\}\varphi_l(2) + \sum_{l=1}^{N}\sum_{m=1}^{N}\int d2\int d3\,\varphi_l^*(2)\varphi_m^*(3)\,\mathbf{A}_{23}\delta(1-2)[\varphi_l(2)\varphi_m(3)-\varphi_l(3)\varphi_m(2)]\ . \tag{8.2}$$

The change to $\langle\mathbf{J}\rangle$ resulting from adding a particle in orbital φ is therefore

$$\delta\mathbf{J}(1)=\int d2\,\varphi^*(2)\tfrac{1}{2}\{\mathbf{P}_2+\overline{\mathbf{A}}_2,\delta(1-2)\}\varphi(2) -\int d2\,\mathbf{A}_{12}[\varphi^*(1)\Pi(1,2)\varphi(2)+\varphi^*(2)\Pi(2,1)\varphi(1)] +\rho\int d2\,\mathbf{A}_{12}|\varphi(2)|^2\ , \tag{8.3}$$

where ρ denotes the particle density and where

$$\Pi=\sum_{\substack{m\\ \text{filled}}}\Pi_m\ , \tag{8.4}$$

with Π_m defined as in Eq. (4.3). The current density associated with a hole is the negative of this expression. Let us now specialize to the case of a particle (hole) in a Gaussian orbital in the nth Landau level centered at the origin, given by

$$\varphi(z)=\varphi_{n0}(z)=\frac{1}{(2^{n+1}\pi n!)^{1/2}}z^n e^{-(1/4)|z|^2}\ . \tag{8.5}$$

Elementary considerations give, for $\mathbf{r}_1=\mathbf{r}$,

$$\int d2\,\varphi^*(2)\tfrac{1}{2}\{\mathbf{P}_2+\overline{\mathbf{A}}_2,\delta(1-2)\}\varphi(2) =\frac{\hat{\mathbf{z}}\times\hat{\mathbf{r}}}{2\pi r}e^{-(1/2)r^2}\frac{(\frac{1}{2}r^2)^n}{n!}(n+\tfrac{1}{2}r^2)\ , \tag{8.6}$$

and

$$\rho\int d2\,\mathbf{A}_{12}|\varphi(2)|^2=\frac{\hat{\mathbf{z}}\times\hat{\mathbf{r}}}{2\pi r}\left[1-e^{-(1/2)r^2}\sum_{p=0}^{n}\frac{1}{p!}(\tfrac{1}{2}r^2)^p\right]. \tag{8.7}$$

Using methods outlined in Sec. VI, we also obtain

$$\int d2\,\mathbf{A}_{12}[\varphi^*(1)\Pi_m(1,2)\varphi(2)+\varphi^*(2)\Pi_m(2,1)\varphi(1)] =(1-\nu)\frac{\hat{\mathbf{z}}\times\hat{\mathbf{r}}}{2\pi r}e^{-(1/2)r^2}\frac{(\frac{1}{2}r^2)^n}{n!}\left[-\Theta(n-m)+\frac{1}{n+1}(\tfrac{1}{2}r^2)\Theta(n-m+1)\right], \tag{8.8}$$

where

$$\Theta(n-m)=\begin{cases}1 & m<n\\ 0 & m\geq n\ .\end{cases}$$

Combining these results, we obtain finally

$$\delta\mathbf{J}_{\text{particle}}=\frac{\hat{\mathbf{z}}\times\hat{\mathbf{r}}}{2\pi r}\left\{\frac{1}{n!}(\tfrac{1}{2}r^2)^n e^{-(1/2)r^2}\left[(n+1)+\left[\frac{n}{n+1}\right](\tfrac{1}{2}r^2)\right]+\left[1-e^{-(1/2)r^2}\sum_{p=0}^{n}\frac{1}{p!}(\tfrac{1}{2}r^2)^p\right]\right\} \tag{8.9}$$

and

$$\delta\mathbf{J}_{\text{hole}}=-\frac{\hat{\mathbf{z}}\times\hat{\mathbf{r}}}{2\pi r}\left[\frac{1}{n!}(\tfrac{1}{2}r^2)^n e^{-(1/2)r^2}[n(2-\nu)+(\tfrac{1}{2}r^2)\nu]+\left[1-e^{-(1/2)r^2}\sum_{p=0}^{n}\frac{1}{p!}(\tfrac{1}{2}r^2)^p\right]\right]. \tag{8.10}$$

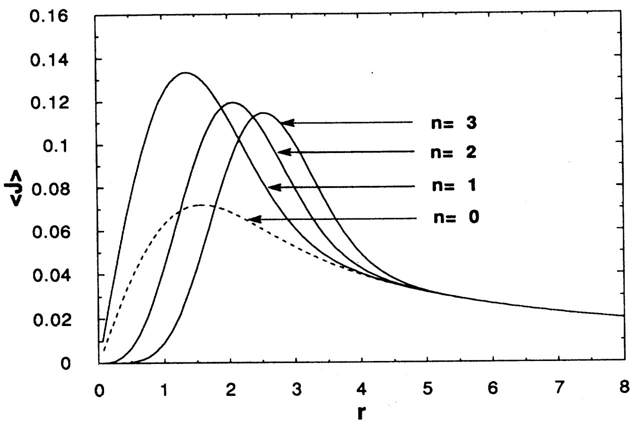

FIG. 4. Expected current density versus r for particles (solid) and holes (dashed) in the Bose fluid ($\nu=0$), as given by Eqs. (8.9) and (8.10). The hole current is the negative of the plotted value.

Plots of the first few of these expressions for $\nu=0$ and $\nu=\frac{1}{2}$ are shown in Figs. 4 and 5. Far from the origin, the core contributions in these expressions become negligible, and we have

$$J \cong \pm \frac{\hat{\mathbf{z}}\times\hat{\mathbf{r}}}{2\pi r} , \tag{8.11}$$

which is the desired result.

Let us now calculate the circulation κ of this vortex. Dividing Eq. (8.11) by ρ, which is related to the magnetic length by Eq. (1.17), to obtain the particle velocity, and integrating this around the vortex, we obtain, with the dimensions reinstated,

$$\kappa=(1-\nu)\frac{h}{m}\oint dl\cdot\delta\mathbf{J}=(1-\nu)\frac{h}{m} . \tag{8.12}$$

While the existence of vortices with this value of circulation is consistent with charge-$(1-\nu)^{-1}$ superfluidity, the attachment of charge to the vortex is highly unusual. This result is clearly pathological in the case of $\nu=0$ because the underlying physical system, the Bose fluid, conserves parity and thus cannot associate positive charge with a particular handedness of circulation. In the case of $\nu=\frac{1}{2}$, however, it is quite reasonable, particularly in light of the recent work of March-Russell and Wilczek[18] showing in a continuum version of this problem that charges are vortices and vortices charged as a matter of principle. Thus the result should be taken as an indication that the charge of the order parameter will then be $(1-\nu)^{-1}$ when the superfluid state is correctly described, and as predicting that a $\nu=\frac{1}{2}$ superconducting vortex carries charge $\pm e$.

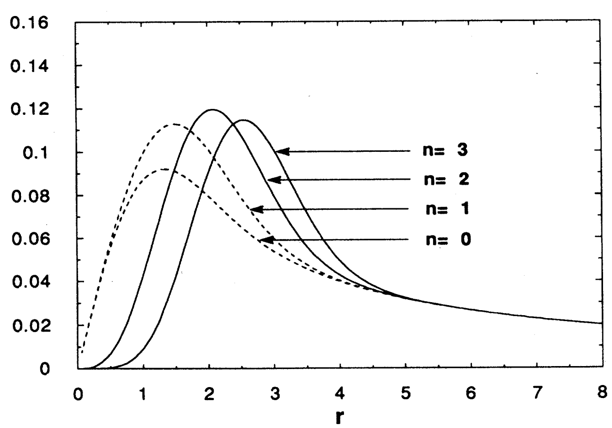

FIG. 5. Expected current density versus r for particles (solid) and holes (dashed) for the fractional-statistics fluid ($\nu=\frac{1}{2}$), as given by Eqs. (8.9) and (8.10). The hole current is the negative of the plotted value.

IX. DISCUSSION

The results presented in this paper are consistent with the assertion that the $\nu=\frac{1}{2}$ fractional-statistics gas is a charge-2 superfluid. The key step in our reasoning is the observation that a gap in the fermionic excitation spectrum, particularly one that grows with sample size, is highly unusual in a system possessing ordinary compressional sound and indicates that this collective mode cannot dissipate except by interacting with itself. This, in turn, suggests that hybridizing macroscopic numbers of these "phonons" into the "ground state" per Eq. (1.23) produces a true ground state with the broken symmetry expected of a superfluid. Since the gaplessness of the collective mode is central to the argument, the next logical step is to develop a formal description of this mode. While Eqs. (5.22) and (7.21) do not constitute such a description, they do indicate that the correct description must be gapless.[19]

The properties of the $\nu=\frac{1}{2}$ fractional-statistics fluid implied by our results are very similar to those of an interacting Bose fluid at the same density. In particular, the superconducting transition temperature is roughly the Bose condensation temperature, which is of order 1 in our units [cf. Eq. (1.16)], and the thermodynamic properties near the transition are similar to those of helium-4 near its λ point. When Coulomb interactions are added to the problem, the superfluid transition, Meissner screening, and critical fields are expected to be those associated with a charged Bose fluid, unless these repulsions are sufficiently strong to induce crystallization. Since the experimental properties of high-temperature superconductors are quite BCS-like,[2] and thus inconsistent with this result, it is important to make clear that they are not expected to be consistent. In order to demonstrate convincingly that fractional statistics can cause superconductivity, we have elected to remove the spin degrees of freedom from the problem. As a result, the excitation generated in a tunneling experiment, the energy gap of which regulates the transition temperature in an ordinary superconductor, does not exist in our calculation. Since these critical low-energy excitations are missing, many of the low-temperature properties are expected to be given incorrectly. It should be remarked that the Bose condensation temperature forms a natural upper bound to the transition temperature of any superfluid, including an ordinary superconductor. It may be considered an accident of nature that the thermodynamic collapse of the gap in an ordinary superconductor occurs at a temperature lower than that required to destroy the long-range coherence of the order parameter by thermally exciting phonons.

We remark finally that the behavior we ascribe to the fractional-statistics gas, namely the binding of integral multiples of particles into bosons which then condense, is very reminiscent of the mechanism of "oblique confinement,"[20] thought to be a generic feature of simple gauge theories with parity-violating terms in their Lagrangians.[21] Since the confining phase, which may be thought of as a superfluid of "monopole" excitations of the gauge field,[22] occurs at any rational fraction of the parity-violating parameter, the possibility is raised that the fractional-statistics gas is actually a charge-m superfluid for any rational fraction $\nu=n/m$ of statistics. An appropriate analogy would be the existence of a fractional quantum Hall state at any "rational" value of the electron density.[8] It is also significant that the most commonly discussed precedent for this behavior is three dimensional. Thus, it is possible that a three-dimensional version of the behavior in this paper may exist and be relevant to the three-dimensional aspects of high-temperature superconductivity.

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APPENDIX

The following sums are used to simplify the expressions for the Hartree-Fock self-energies. In Eqs. (5.7) and (7.7) we have

$$\sum_{k=1}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(-1)^k}{k}=\sum_{k=1}^{n}\begin{bmatrix}n\\k\end{bmatrix}\int_0^1(-x)^k\frac{1}{x}dx$$
$$=\int_0^1\frac{1}{x}[(1-x)^n-1]dx=\int_0^1\frac{y^n-1}{1-y}dy$$
$$=-\sum_{k=0}^{n-1}\int_0^1 y^k dy=-\sum_{k=1}^{n}\frac{1}{k}\ . \tag{A1}$$

In Eq. (6.16) we have

$$\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\frac{(-1)^k}{k+1}=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}\int_0^1(-x)^k dx$$
$$=\int_0^1(1-x)^n dx=\frac{1}{n+1}\ . \tag{A2}$$

In Eqs. (6.16) and (6.27) we have, for $n\ge 1$ and $n>m$,

$$\sum_{k=m}^{n}\begin{bmatrix}n\\k\end{bmatrix}\begin{bmatrix}k\\m\end{bmatrix}(-1)^k\lim_{\epsilon\to 0}\left[\sum_{p=0}^{k}\frac{1}{p+\epsilon}-\frac{1}{\epsilon}\right]$$
$$=\lim_{y\to 1}\left[\frac{\partial}{\partial y}\right]^m\sum_{k=1}^{n}(-y)^k\sum_{p=0}^{k-1}\int_0^1 x^p dx$$
$$=\lim_{y\to 1}\left[\frac{\partial}{\partial y}\right]^m\sum_{k=1}^{n}(-y)^k\int_0^1\frac{1-x^k}{1-x}dx$$
$$=\lim_{y\to 1}\left[\frac{\partial}{\partial y}\right]^m\int_0^1\frac{(1-y)^n-(1-xy)^n}{1-x}dx$$
$$=\frac{(-1)^{m+1}m!}{n-m}\ . \tag{A3}$$

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