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Kapitza resistance between a solid wall and superfluid ^{4}He near T_λ

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The effective Kapitza resistance near T_λ, between ^{4}He II and a solid wall, is calculated employing general boundary conditions and the hydrodynamic Ψ theory of Ginzburg and Sobyanin. A comparison of the results to the experimental data is used to narrow the possible range of existing boundary conditions. Further experiments to clarify the situation are suggested.

I. INTRODUCTION

If an entropy current f is applied across any boundary, a temperature discontinuity $\Delta T \propto f$ occurs, and the constant $R_k^0 \equiv \Delta T/(Tf)$ is referred to as the Kapitza resistance.[1] For clean surfaces, it is characteristic of the pair of materials forming the boundary.[2] If one of the materials is a superfluid, there is an additional contribution R_k^s, characteristic of the superfluid alone. It arises from temperature gradients $\nabla T \propto f$, which exist in the superfluid close to the wall.[3] In ^{4}He II, the characteristic length λ of the temperature gradients is, far away from T_λ, very small, $\lambda \simeq 10^{-6}$ cm. And one can only measure the total, or effective, Kapitza resistance

$$R_k = R_k^0 + R_k^s = \left[\Delta T + \int \nabla T(\text{helium})\right] \Big/ (Tf) \ .$$

General considerations show that neither R_k^0 nor R_k^s has to be positive, but $R_k > 0$ must always hold.[4] Approaching the phase transition from below, both λ and R_k^s are expected to diverge,[5,6] and in fact, a singular contribution to R_k was recently measured.[7] Assuming that R_k^0 is noncritical, the experimental data are in qualitative agreement with the theoretical results,[8] $R_k^s = A_0\lambda/\kappa$, where A_0 is a weakly temperature-dependent coefficient and κ denotes the thermal conductivity. (This result was in fact displayed as $R_k^s \propto \xi/\kappa$ in Ref. 8, but the healing or correlation length of ρ^s, ξ, diverges essentially with the same exponent as λ.)

The question we seek to answer here concerns the choice of boundary conditions and its effects. In Refs. 5 and 6, and in part of Ref. 8, R_k^s was calculated using

$$\rho^s = 0, \quad j^s \equiv \rho^s(v^s - v^n) = 0 \ . \tag{1}$$

Both equations appear plausible, but in fact represent consequential assumptions. They are usually justified by the argument that, with the amplitude of the condensate wave function, both ρ^s and j^s ought to be zero at the solid interface. Yet, with the coarseness of a macroscopic description, all we really can say is that ρ^s is probably small.[9] Then $j^s = 0$ is an independent assumption, and by no means a preferred one: Far from T_λ, $j^s \propto f$ has a prescribed value for a given boundary.[4] If ρ^s decreases close to the wall, $v^s - v^n$ increases to compensate, or rather it does so until the critical velocity is reached. Hence,

$$j^s = \rho^s(0) v_c \rho/\rho^n \approx \rho^s(0) v_c$$

appears as the correct boundary condition to be concluded from the preceding argument. Generally, therefore, we should have two regions of heat transfer: The first one is subcritical, where j^s and $\Delta T \propto f$; it exists further away from T_λ and at a lower rate of heat transfer. The second one is critical and should entail some kind of nonlinearity.

Our strategy is to employ generally valid boundary conditions, calculate R_k^s, and compare it with the experiment of Ref. 7 to see what the realistic boundary conditions in fact are. In equilibrium and close to T_λ, the general boundary conditions are[5]

$$\rho^s(0) = l_s \partial_x \rho^s(0), \quad j^s(0) = 0 \ , \tag{2}$$

where $\rho^s(0)$ and $j^s(0)$ denote their respective value at the boundary and l_s is the extrapolation or slip length, characteristic of the surface. (The x axis is assumed to be normal to the surface.) Off equilibrium and away from T_λ, we have[14]

$$\partial_x \rho^s(0) = 0, \quad j^s(0) \propto f \ . \tag{3}$$

Obviously, for our purpose, we need

$$\rho^s(0) = l_s \partial_x \rho^s(0), \quad j^s(0) \propto f \ . \tag{4}$$

(A dynamic modification of the static relation $\rho^s \propto \partial_x \rho^s$ would only lead to nonlinear correction which we neglect.) The proportionality constant j^s/f, as well as the bare Kapitza resistance R_k^0, depend on three surface Onsager coefficients a, b, and c, introduced in Ref. 4. Retaining the original assumption that the critical contribution to the total Kapitza resistance arises only within the helium liquid, we take a, b, and c as independent of $t \equiv (T_\lambda - T)/T_\lambda$. Employing Eqs. (4), we obtain

$$R_k^s = (A_0\lambda/\kappa)(1 + A_0 B_0 \lambda/\kappa)^{-1} \ , \tag{5}$$

where B_0 depends on a, b, c, and l_s/ξ. The temperature dependence of l_s is not known, but one may plausibly assume that it either scales with the healing length $\xi=\xi_0 t^{-2/3}$, or remains of order ξ_0.[5] In the first case, B_0 is independent of t, and the detected divergent behavior of $R_k^s \propto \lambda/\kappa$ in the experimental range of Ref. 7 shows it must be small. More specifically, we obtain the upper bound (in units of $K\,s^3/\mathrm{g}$)

$$B_0 < 10^{-3} . \tag{6}$$

In the second case, B_0 depends on t. But more importantly $\rho^s(0)/\rho^s(\infty)$ approaches zero rapidly: With

$$\rho^s(0)=l_s \partial_x \rho^s(0)$$

and

$$\rho^s(\infty)-\rho^s(0)\approx \xi\partial_x \rho^s(0) ,$$

we have

$$\rho^s(0)/\rho^s(\infty)\approx l_s/\xi .$$

We therefore need to pay attention to the constraint $j^s \le \rho^s(0)v_c$ that supplements Eqs. (4). For those temperatures where $j^s=\rho^s(0)v_c$ is valid, we obtain

$$R_k^s = A_0\lambda/\kappa(1+C)\approx A_0\lambda/\kappa . \tag{7}$$

Although the nonlinear term

$$C \propto \rho^s(0)v_c/f \propto \rho^s(\infty)v_c/(\xi f) \propto t^2/f$$

may have been too small to be observable, we note that a sudden rise in the resistance should have been observed at the transition to the region of critical velocity. It was not reported in Ref. 7. So if the second case $l_s \propto \xi_0$ holds, we may conclude that the heat transfer was critical at the surface for all temperatures of the experiment. Amazingly, this implies

$$\rho^s(0)/\rho^s(\infty) < 10^{-5} j^s(0)/j^s(\infty) \quad \text{at } t=10^{-2} ,$$

and decreasing with $t^{2/3}$. [Critical transfer implies

$$\rho^s(0)v_c/(f/\sigma) < j^s(0)/j^s(\infty) .$$

Taking $t=10^{-2}$, $\rho^s(\infty)v_c \approx 5t^{4/3}$, $f/\sigma \approx 1.5\times 10^{-7}$, in cgs units, we have $\rho^s(0)/\rho^s(\infty) < 10^{-5} j^s(0)/j^s(\infty)$.] Of course, other t dependence of l_s cannot be ruled out rigorously. But it does not modify Eq. (5) or Eq. (7), only the temperature at which the transition to the surface critical heat transfer occurs. So except for the fact that $j^s(0)$ is critical, i.e., the existence of a transition from Eq. (5) to Eq. (7) and the small nonlinearity in C of Eq. (7), the experimental evidence obviously points back to Eq. (1) as a fairly realistic approximation.

For further studies, the experiments of obvious interest are (i) measurement of B_0, Eq. (6), looking (ii) for the surface critical transition, or (iii) the nonlinearity in C for supercritical heat transfer. Also, we believe that the following experiment has crucial importance: The weakest point on the theoretical side is the assumption that a, b, and c, and therefore R_k^0, do not display any critical behavior. If they do, although Eqs. (5) and (7) remain valid, the comparison between theory and experiment becomes quite unfounded. An experiment on a helium film between two parallel plates should clarify this point. As our results in Sec. V indicate, if the foregoing assumption is correct, the singular contribution would not diverge with $t\to 0$, but rather undergoes a maximum, at $t=10^{-4}$ for the film thickness of $d=10^{-4}$ cm. The reason for this behavior is that λ becomes comparable to d, and the gradients in the temperature lack space to develop. (λ diverges with the same exponent as the correlation length ξ and with a coefficient about an order of magnitude larger. Therefore the depression of ρ^s or T_λ is not a problem in this experiment.)

Finally, a note of caution with respect to the generalized Ψ theory that we employ in this paper. It is a theory that is valid to first order in $k\xi$. Our results are therefore correct only if $\lambda \gg \xi$, with perhaps qualitative predictive value until $\lambda \gtrsim \xi$. This is a true assumption, since both lengths scale approximately with $t^{-2/3}$, cf. comments following Eq. (15). On the other hand, if $\lambda \gg \xi$ is indeed correct, our results are rigorously valid. Especially fluctuations of wavelengths shorter than ξ are fully accounted for.[5] This is in contrast to the mean-field-like Ginzburg-Landau theory.

In Secs. II and III, we review the hydrodynamic Ψ theory of Ginzburg and Sobyanin and the derivation of the general boundary conditions, respectively; in Sec. IV, we derive Eqs. (5) and (7) and provide a summary in Sec. V.

II. THE GENERALIZED Ψ THEORY OF GINZBURG AND SOBYANIN

In this section we give a summary of the generalized hydrodynamics for ^{4}He II near the λ point[10,11] and discuss earlier results for the effective Kapitza resistance.[5] We start with the free energy $\Phi_{\mathrm{II}}(p,T,\Psi)$, where Ψ is the complex order parameter $\Psi=\eta^s e^{i\varphi}$ of ^{4}He II ($|\Psi|^2=\rho^s/m$). We have to write down the Ginzburg-Landau functional in the form

$$\Phi_{\mathrm{II}}=\Phi_{\mathrm{I}}+A_2|\Psi|^2+\tfrac{1}{2}A_4|\Psi|^4+\tfrac{1}{3}A_6|\Psi|^6+\cdots \tag{8}$$

with Φ_{I} being the thermodynamic potential of ^{4}He I. The expansion up to $|\Psi|^6$ is necessitated by the t dependence of the order parameter and the specific heat.[12] The coefficients A_2, A_4, A_6 are given as $A_2 \propto t^{4/3}$, $A_4 \propto t^{2/3}$, and $A_6 \propto t^0$.[5] This leads, e.g., to $\rho^s \propto t^{2/3}$. Now we examine the generalized hydrodynamic equations.[10,11] The main change lies in the addition of an equation for the variable ρ^s. It takes the form

$$\dot{\rho}^s+\partial_x(\rho^s v^s)=-\frac{2\Lambda m\rho^s}{\hbar}\left[\mu^s-\frac{\hbar^2}{2m^2\eta^s}\partial_x^2\eta^s\right] , \tag{9}$$

where the tranport coefficient $\Lambda\simeq\Lambda_0 t^{-1/3}$, $\Lambda_0\approx 0.3$; μ^s is an additional chemical potential defined as $\mu^s\equiv(1/m)\partial\Phi_{\mathrm{II}}/\partial|\Psi|^2$. Equation (9) leads to a relaxation of ρ^s with the characteristic time

$$\hbar/(2\Lambda m\rho^s)\partial\rho^s/\partial\mu^s .$$

We only consider one-dimensional variations, along x, appropriate for the experiment.[7] The superfluid is on the right ($x>0$) and the solid on the left. Interested in stationary solutions, we set all time derivatives to zero. The hydrodynamic equations are then given as[10,11]

$$\partial_x g = 0 \,, \tag{10a}$$

$$\partial_x \left[\rho\mu + sT + \frac{1}{\rho} \left[\frac{4}{3}\eta + \zeta_2 - \rho\zeta_1 + \frac{\hbar\rho^n}{2\Lambda m} \right] \partial_x j^s \right] = 0 \,, \tag{10b}$$

$$\partial_x \left[\mu - \rho^{-2} \left[\rho^2\zeta_3 - \rho\zeta_4 + \frac{\rho^2}{\rho^s}\frac{\hbar\Lambda}{2m} - \frac{\hbar\rho}{2\Lambda m}\frac{\rho^n}{\rho^s} \right] \partial_x j^s \right] = 0 \,, \tag{10c}$$

$$\partial_x \left[-\sigma j^s - \frac{\kappa}{T}\partial_x T \right] = 0 \,, \tag{10d}$$

where $g = \rho^s v^s + \rho^n v^n$ is the mass current and $j^s = \rho^s(v^s - v^n)$ the counterflow. Note that Eqs. (10a)–(10d) could have been formally obtained from the hydrodynamic equations far away from T_λ by substituting

$$\tfrac{4}{3}\eta + \zeta_2 - \rho\zeta_1 + \hbar\rho^n/(2\Lambda m)$$

and

$$\rho^2\zeta_3 - \rho\zeta_4 + \rho^2 h\Lambda/(\rho^s 2m) - \hbar\rho\rho^n/(2\Lambda m\rho^s)$$

for $\frac{4}{3}\eta + \zeta_2 - \rho\zeta_1$ and $\rho^2\zeta_3 - \rho\zeta_4$, respectively. Now we solve Eqs. (9) and (10a)–(10d) to obtain the temperature profile in helium. Our notation, e.g.,

$$\eta^s = \eta^s_{00} + \delta\eta^s_0 + \delta\eta^s \,, \tag{11}$$

is the same as in Ref. 5, double zero denotes the bulk equilibrium value, single zero the equilibrium deviation that occurs at the boundary, while the unindexed deviation only occurs off equilibrium. With the dimensionless variable $\zeta \equiv 1 + \delta\eta^s_0/\eta^s_{00}$, Eq. (9) reads

$$-(1+2p)\zeta + \zeta^3 + 2p\zeta^5 = \tfrac{1}{2}\xi^2\partial_x^2\zeta \,, \tag{12}$$

the dimensionless parameter $p \equiv 2A_6\eta^{s2}_{00}/A_4$ of order 1 is t independent, while

$$\xi \equiv mA_4\eta^{s2}_{00}/\hbar^2 = \xi_0 t^{-2/3} \,,$$

$\xi_0 \approx 10^{-8}$ cm.[5,13] Equation (12) is completely decoupled from the rest of the hydrodynamic equations and determines the equilibrium shape of

$$\rho^s = \rho^s_{00} + \delta\rho^s_0 = \rho^s_{00}\zeta^2 \,.$$

Equation (12) has been solved exactly[5] with the following boundary conditions: (i) ρ^s should reach a constant value far away from the interface, i.e., $\rho^s(x=\infty) = \rho^s_{00}$ or $\zeta(\infty) = 1$. (ii) $\rho^s(0) = \zeta(0) = 0$. With (i) and (ii) the solution of (12) is

$$\zeta(x) = \frac{\tanh[(x/\xi)\sqrt{1+p}\,]}{\{1+[p/(3+2p)]\cosh^{-2}[(x/\xi)\sqrt{1+p}\,]\}^{1/2}} \,. \tag{13}$$

With this equilibrium profile of ρ^s the rest of the hydrodynamic equations (10a)–(10d) can be solved. It is possible to reduce them to one equation for the dimensionless temperature τ ($\tau \equiv T/T_\infty$, where T_∞ is the temperature in the bulk):

$$\partial_x\tau - \partial_x(\lambda^2\partial_x^2\tau) = 0 \,. \tag{14}$$

λ is a characteristic length, mentioned in the Introduction. The behavior of the viscosities η and ζ_i near the λ point is not known in detail, but they show no divergences.[5] With $\rho^s \propto t^{2/3}$, the most divergent part of λ takes the form

$$\lambda = \lambda_0/\zeta(x) \quad \text{with} \quad \lambda_0^2 = \frac{1}{2}\frac{\hbar}{m}\frac{\rho^2\kappa\Lambda}{s^2 T\rho^s_{00}} \,. \tag{15}$$

While κ has never been measured in ^{4}He II directly, a rough estimate from scaling considerations[14] or measurements above T_λ (Refs. 15 and 16) shows that κ behaves similarly to Λ. Assuming $\kappa \simeq 10^4 t^{-1/3}$ erg/(cm K s) we have $\lambda_0 \simeq \lambda_{00} + 10^{-7}t^{-2/3}$ cm, where the nondivergent part of λ is given by $\lambda_{00} \approx 10^{-6}$ cm. Therefore our estimate of λ_0 is (as are all other estimates in this paper) only true for $t < 10^{-2}$. (Note, however, that $\kappa = [1222 + 70.5(2t)^{-0.48}]$ erg/(s cm k) describes the data of Refs. 15 and 16 best.) The solution of Eq. (14) is difficult because λ, due to its dependence on ρ^s, shows a variation in space as given by (13). However, if the parameter p is not too large, ρ^s can be approximated by

$$\rho^s = \rho^s_{00}\tanh^2(x/\xi) \,; \tag{16}$$

then, an analytic solution becomes possible. By applying the boundary condition that the temperature should reach a constant value far away from the interface, i.e., $\tau(\infty) = 1$ *and* $\partial_x\tau|_{x=\infty} = 0$, the result is[5]

$$\partial_x\tau = A\cosh^\mu\frac{x}{\xi}F(\alpha,\beta,\gamma,\cosh^{-2}(x/\xi)) \,, \tag{17}$$

with $\mu \equiv -\xi/\lambda_0$. The first three agruments of the hypergeometric function F are given as

$$\gamma = 1 - \mu \,,$$

$$4\alpha = -(1+2\mu) \mp (1+4\mu^2)^{1/2} \,,$$

and

$$4\beta = -(1+2\mu) \pm (1+4\mu^2)^{1/2} \,.$$

The corresponding Kapitza resistance

$$R^s_k \equiv f^{-1}\int_\infty^0 dx\, \partial_x\tau \tag{18}$$

is obtained by integrating Eq. (17):

$$R^s_k = -\frac{1}{2}\frac{A}{f}\xi\frac{\Gamma(-\mu/2)\Gamma(1/2)}{\Gamma((1-\mu)/2)}\,{}_3F_2 \,. \tag{19}$$

[${}_3F_2$ is an abbreviation for ${}_3F_2(\alpha,\beta,-\mu/2;\gamma,1/2-\mu/2;1)$.] The constant A in (19) or (17) is determined by the boundary condition $f = -(\kappa/T)\partial_x T = -\kappa\partial_x\tau$,

$$\frac{A}{f} = -\frac{1}{\kappa}\frac{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)} \,. \tag{20}$$

Combining (20) with (19) we see that the divergence in R^s_k

is essentially given by ξ/κ, since α, β, γ, and μ depend only weakly on t.

III. THE GENERAL BOUNDARY CONDITIONS

In this section we discuss the general boundary conditions, valid far away from T_λ.[4] We have the following situation: a vessel wall at $x=0$ and a superfluid on the right ($x>0$). We need two boundary conditions: one for the determination of the temperature jump $\Delta T = T_{\text{vessel}} - T_{\text{superfluid}}$ at $x=0$ and one to decide how the heat current f divides up into its reactive $[=-\sigma j^s]$ and its dissipative $[=(\kappa/T)\partial_x T]$ parts at the interface. [In Sec. II, it was assumed that $f=-(\kappa/T)\partial_x T$.] The structure of the boundary conditions is essentially obtained by applying the laws of irreversible thermodynamics. We start with the expression of the surface entropy production rate R_S, calculated from the continuity of the energy current,

$$R_S = f\Delta T + j^s(\Pi^D/\rho - Z^D) . \tag{21}$$

The entropy current f and the counterflow $j^s = \rho^s(v^s - v^n)$ are known from the preceding chapter. Π^D and Z^D are the dissipative parts of the momentum conservation and Josephson equation of the superfluid, respectively.[10] Equation (21) implies

$$\begin{aligned} f &= a\Delta T + c(\Pi^D/\rho - Z^D) \\ j^s &= c\Delta T + b(\Pi^D/\rho - Z^D) \end{aligned} \tag{22}$$

where a, b, and c are the surface Onsager coefficients with $a,b \ge 0$ and $ab-c^2 \ge 0$. In our case the entropy current f is given, while the temperature jump ΔT and the temperature profile inside the superfluid should be calculated. Far away from the transition point the temperature varies as

$$\tau = T/T_\infty = -A\lambda e^{-x/\lambda} , \tag{23}$$

where λ is given far away from the λ point by

$$\lambda^2 = [\tfrac{4}{3}\eta - (\zeta_1+\zeta_4)\rho + \zeta_2 + \rho^2\zeta_3]\kappa s^{-2} T^{-1} ,$$

and Eq. (14) remains valid. To determine ΔT and A if f is given, we rewrite Eqs. (22) in the form

$$-(a+\sigma c)f = a\kappa\partial_x\tau - (ab-c^2)\sigma^2 T_\infty \lambda^2 \partial_x^2 \tau , \tag{24}$$

$$\Delta T = \frac{b+c/\sigma}{ab-c^2} f + \frac{c/\sigma}{ab-c^2}\kappa\partial_x\tau , \tag{25}$$

which yield:[4]

$$R_k^s = \frac{\tau(0)-\tau(\infty)}{f} = \frac{\lambda}{\kappa}\,\frac{1+\sigma c/a}{1+(b-c^2/a)\dfrac{\lambda}{\kappa}\sigma^2 T_\infty} , \tag{26}$$

$$R_k^0 = \frac{\Delta T}{T_\infty f} = \frac{1}{aT_\infty}\,\frac{\kappa/\lambda + c\sigma + b\sigma^2}{\kappa/\lambda + (b-c^2/a)\sigma^2} . \tag{27}$$

Equations (26) and (27) are valid far away from T_λ. Due to the formal equivalence mentioned in Sec. II following Eqs. (10), however, they are also correct for $t\to 0$ after the appropriate substitution, and ρ^s is assumed to have no *independent* spatial variation $\delta\rho_0^s = 0$ (i.e., if $l_s = \infty$). If a, b, and c are noncritical, R_k^0 and R_k^s remain regular for $t\to 0$ despite the divergence of λ/κ. This is correct except if b vanishes (with $ab > c^2$ this implies $c\to 0$ also). The case $b=c=0$ is equivalent to the boundary condition, $j^s(0)=0$, as employed in Ref. 5 and the preceding section. Generally, $R_k^s = \lambda/\kappa$ and $R_k^0 = (aT)^{-1}$ hold only if b goes faster to zero than κ/λ.

In the next section we shall apply the general boundary conditions, Eq. (22), to the hydrodynamics near the λ point. The question is then why they remain valid for temperatures near T_λ. This is because the new quasihydrodynamic variable ρ^s does not contribute to the energy current in the stationary case, and therefore Eqs. (21), (22), (24), and (25) remain valid. All we need is the additional boundary condition for ρ^s,

$$\rho^s(0) = l_s \partial_x \rho^s(0) ,$$

or equivalently, the value of ρ^s at its boundary. They are related by

$$\rho^s(0)/\rho^s(\infty) \approx l_s/(\xi + l_s) ,$$

since ξ characterizes the length scales of variations in ρ^s: $\rho^s(\infty) - \rho^s(0) \approx \xi\partial_x\rho^s(0)$. In contrast, we took $\rho^s(0)=0$ in the preceding section.

IV. THE GENERAL FORM OF THE KAPITZA RESISTANCE

Now we want to combine Secs. II and III, i.e., to solve the hydrodynamic equations for ^{4}He II near the λ point[5,10,11] with the general boundary conditions.[4] This is a simple task now except for critical velocity effects. Close to T_λ, $v_c \propto t^{2/3}$ (Ref. 17) becomes small and the superfluid velocity required at the boundary $v^s \approx j^s/\rho^s \propto t^{-2/3} j^s$ becomes large. So it becomes more and more difficult to assure $|v^s| \le v_c$. However, because of the linear relation $j^s \propto v^s \propto f$, $|v^s| \le v_c$ always holds if f is chosen small enough. Therefore we divide this section into two subsections, the undercritical and the critical case.

A. Undercritical heat transfer

We calculate the equilibrium profile of $\rho^s = \rho_{00}^s \zeta^2$ with Eq. (12) that remains valid. In its solution we do not assume $\rho^s(0) = \zeta(0) = 0$. This changes the solution only slightly: One has to substitute x^* for x on the rhs in Eq. (13), with

$$x^* = x + \frac{\xi}{\sqrt{1+p}} \times \operatorname{arctanh}\left[\zeta(0)\left[\frac{1+p/(3+2p)}{1+\zeta^2(0)p/(3+2p)}\right]^{1/2}\right] . \tag{28}$$

$\zeta(0)=0$ leads to $x^*=x$ and $\zeta(0)=1$ implies $\zeta(x)\equiv 1$ as it should.

Next we calculate the temperature profile by solving Eq. (14). It was not possible to do this exactly in Sec. II; neither is it here. By using the same approximation

$\zeta(x)=\tanh(x^*/\xi)$ [cf. Eq. (16)] an analytic solution again becomes possible. It has the same structure as in Sec. II; one only has to substitute x^* for x on the rhs of Eq. (17). Now we apply Eq. (18) to calculate R_k^s:

$$R_k^s=-\frac{A}{f}\xi I$$

with (29)

$$I=\frac{1}{2}\int_0^{1-\zeta^2(0)} dz\, z^{-(2+\mu)/\mu}(1-\zeta)^{1/2}F(\alpha,\beta,\gamma,z)\ .$$

We cannot solve integral I analytically. If $l_s \propto \xi$ its dependence on $\zeta(0)$ does not lead to any variation with the temperature. The constant A can be calculated with the help of the boundary condition Eq. (24). The results are

$$\frac{A}{f}=-\frac{K}{\kappa}\left[1+\frac{\xi}{\kappa}B\right]^{-1}$$

with

$$K=\frac{1+\sigma c/a}{[1-\zeta^2(0)]^{-\mu/2}F} \tag{30}$$

and

$$B=(b-c^2/a)\frac{\sigma^2 T_\infty}{\zeta(0)\mu^2}\left[-\mu+[1-\zeta^2(0)]\frac{\alpha\beta}{\gamma}F'/F\right]\ .$$

F and F' are abbreviations for $F(\alpha,\beta,\gamma,1-\zeta^2(0))$ and $F(\alpha+1,\beta+1,\gamma+1,1-\zeta^2(0))$, respectively. Note the constants K and B depend on a, b, c, and $\zeta(0)$. Combining Eqs. (29) and (30) we find

$$R_k^s=\frac{\xi}{\kappa}KI\left[1+\frac{\xi}{\kappa}B\right]^{-1}\ . \tag{31}$$

With the boundary condition (25), $R_k^0=\Delta T/(T_\infty f)$ is also easily calculated:

$$R_k^0=\frac{1}{aT_\infty}\left[\frac{1+c/(\sigma b)}{1-c^2/(ab)}-\frac{Kc/(\sigma b)[1-\zeta^2(0)]^{-\mu/2}F}{1+(\xi/\kappa)B}\right]\ , \tag{32}$$

where the second term vanishes for $t\to 0$. Note that any t dependence of R_k^0 and R_k^s can be easily constructed by assuming a corresponding t dependence of a, b, c, and $\zeta(0)$. Taking them as t independent (first case of the introduction), however, the recent measurements[7] clearly indicate $B\xi/\kappa \ll 1$ in the temperature range of the experiment ($10^{-2}>t>10^{-6}$). Even if $B\xi/\kappa\simeq 1$ at $t=10^{-6}$ the measured divergency gives the upper bound $B<10^{13}$ K s/erg. This is a hint that the depression of j^s at $x=0$ is large in view of the fact that

$$j^s(0)/j^s(\infty)=(B\xi/\kappa-\sigma c/a)/(B\xi/\kappa+1)\ .$$

B. The supercritical heat transfer

As discussed in the introduction and the beginning of Sec. IV, the value that $j^s(0)\propto f$ ought to have becomes, for a number of combined reasons, eventually supercritical, $j^s(0)>\rho^s(0)v_c$. With $v_c\propto t^{2/3}$,[17] $\rho^s(\infty)\propto t^{2/3}$, $\rho^s(0)/\rho^s(\infty)\approx l_s/\xi$, this happens, for a given f, with a fairly high power of t. If $l_s\simeq\xi_0$, $\rho^s(0)v_c\propto t^2$; if $l_s\propto\xi$, $\rho^s(0)v_c\propto t^{4/3}$. Then we need to substitute

$$j^s(0)=\rho^s(0)v_c=\zeta^2(0)\rho_{00}^s v_c \tag{33}$$

for Eq. (24), $j^s(0)\propto f$. Of course, one can always avoid the critical region by lowering f.

There is an additional complication that we need to comment on. Although $\rho^s(x)$ is a monotonic function, $j^s(x)$ is one too only if $\lambda_0 \gg \xi$. (Our rough estimate in Sec. II yields $\lambda_0\simeq 10\xi$.) If λ_0 is comparable to ξ, the oscillatory behavior in $j^s(x)$ will lead to islands or critical currents at about the same temperature when j^s exceeds the critical current at the boundary. However, these islands would give rise to an f-dependent contribution in R_k^s. To see this, assume $|v^s(x)|\le v_c$ is violated between $x=x_1$ and $x=x_2$. Then $j_c^s(x)=\rho^s(x)v_c$ for $x\in[x_1,x_2]$. The Kapitza resistance $R_{k,x_1x_2}^s$ caused by the temperature variation between $x=x_1$ and $x=x_2$ is given by

$$R_{k,x_1x_2}^s=f^{-1}\int_{x_2}^{x_1}dx\,\partial_x\tau=-\frac{1}{\kappa f}\int_{x_2}^{x_1}dx\,[\sigma j^s(x)+f]\ .$$

Therefore the change in the measured Kapitza resistance $\Delta R_{k,x_1x_2}^s$ is given by

$$\Delta R_{k,x_1x_2}^s=\frac{\sigma}{\kappa f}\int_{x_2}^{x_1}dx\,[j^s(x)-j_c^s(x)]\ .$$

Since both x_2-x_1 *and* $j^s(x)-j_c^s(x)$ are proportional to f, we have $\Delta R_{k,x_1x_2}^s\propto f$. This contribution to R_k should have been observed, yet was not, substantiating our estimate $\lambda_0\gg\xi$.

Now we calculate R_k^s for the critical case. By using (33) instead of (24) the constant A becomes

$$\frac{A}{f}=-\frac{K_c}{\kappa}\left[1+\frac{\sigma\zeta^2(0)\rho_{00}^s v_c}{f}\right]\ , \tag{34}$$

where $K_c=[1-\zeta^2(0)]^{\mu/2}/F$. The parenthesis in (34) approaches 1 for $t\to 0$, and we obtain

$$R_k^s=K_c I\frac{\xi}{\kappa}\ . \tag{35}$$

It is important to realize that the expression of the surface entropy production R_S, Eq. (21), is no longer valid. If it would, a relation $j^s(0)\propto f$ necessarily follows, and not $j^s(0)=\text{const}$. Consequently, Eq. (32) for R_k^0 becomes invalid. One may think of the supercritical heat transport as a situation with two surface entropy productions, R_S and R_V, where the latter accounts for a narrow vortex producing region, in which $j^s\propto f$ is brought down to $j^s=\rho^s v_c$, while ΔT increases correspondingly to maintain a constant f. As a net result, R_k^0 is increased.

V. SUMMARY AND CONCLUSIONS

We have calculated the effective Kapitza resistance R_k for a ^{4}He II-solid interface near T_λ. The result is given by Eqs. (31) and (32). If we compare them with the measured divergence of R_k (Ref. 7) we find an upper bound for the constant B, Eq. (30), which depends on the surface Onsager coefficients a, b, c, and the value of $\rho^s(0)/\rho^s(\infty)$. If the heat current f is not small enough, critical velocity effects become more and more probable near the λ point.

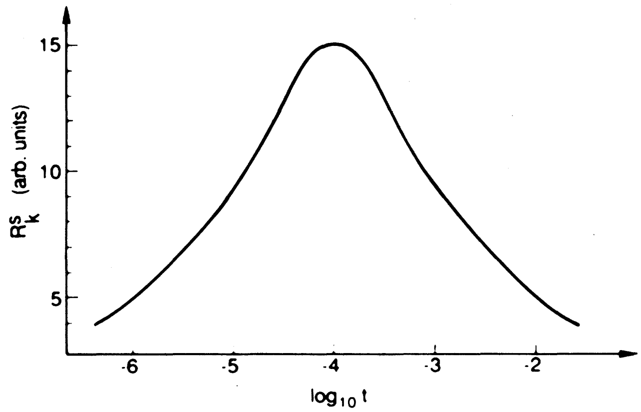

FIG. 1. R_k^s is plotted vs the logarithm of the reduced temperature $t=(T_\lambda - T)/T_\lambda$. The plates distance $d=10^{-4}$ cm and $b=c=0$ is assumed. If the Kapitza resistance of a helium sandwiched between two plates shows a maximum as a function of t, one may confidently conclude that the singular part of the Kapitza resistance indeed comes from temperature gradients within the helium liquid.

If this is the case, the estimate of R_k^s will be given by Eq. (35), which is also consistent with the measured data. The condition that the critical velocity has been reached leads to an upper bound for $\rho^s(0)/\rho^s(\infty) < 10^{-5} j^s(0)/j^s(\infty)$.

The experiment of Ref. 7 gives no clear-cut decision which boundary condition is realized. But it remains a powerful tool to probe both. Therefore it is very desirable to enlarge the temperature range and to lower the heat current in further experiments. In addition, we suggest a modified experimental setup: The Kapitza resistance can also be measured between two parallel plates, where the distance d between them is such that it varies between $d \gg \lambda_0(t)$ and $d \ll \lambda_0(t)$ for the experimental range of t. As long as $d \gg \lambda_0$ holds, R_k^s will diverge as described in the preceding sections. If $d \ll \lambda_0$ the temperature variation lacks space to develop and R_k^s vanishes. To calculate the explicit expression for R_k^s one has to use our boundary conditions at both plates. To simplify the calculation, we assume that ρ^s does not vary in space in equilibrium (as has been done in Sec. III). This should lead to quite reasonable results as long as $d \gg \xi$ holds. A calculation anologous to that of Sec. III gives

$$R_k^s = 2\frac{\lambda}{\kappa} \frac{(1+\sigma c/a)\sinh[d/(2\lambda)]}{\cosh[d/(2\lambda)] + (b-c^2/a)\sigma T_0(\lambda/\kappa)\sinh[d/(2\lambda)]}$$

For $d \gg \lambda$ it becomes just twice the result of Eq. (26), as it should, while it vanishes for $d \ll \lambda$ as

$$R_k^s = d/\kappa(1+\sigma c/a)[1-O(d/\kappa)] \ .$$

Because $(b-c^2/a)\sigma T_0 \lambda/\kappa \ll 1$ (for the experimental temperature range) is one possibility of the experiment,[7] we have plotted R_k^s over t for $b=c=0$ (Fig. 1). The pronounced maximum that results should be precisely determinable in the experiment and gives a precise measure of λ.

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[1] P. L. Kapitza, J. Phys. (Moscow) **4**, 181 (1941).
[2] J. Weber, W. Sandmann, W. Dietsche, and H. Kinder, Phys. Rev. Lett. **40**, 1469 (1978).
[3] W. M. Saslow, Phys. Lett. **35A**, 241 (1971); S. J. Putterman, *Superfluid Hydrodynamics*, Low Temperature Physics Series (North-Holland, Amsterdam, 1974), Vol. 3.
[4] M. Grabinski and M. Liu, Phys. Rev. Lett. **58**, 800 (1987); J. Low Temp. Phys. **73**, 79 (1988).
[5] V. L. Ginzburg and A. A. Sobyanin, Usp. Fiz. Nauk **120**, 153 (1976) [Sov. Phys. Usp. **19**, 773 (1976)]; J. Low Temp. Phys. **49**, 507 (1982); Jpn. J. Appl. Phys. **26**, Suppl. 26-3, 1785 (1987); *Superconductivity, Superdiamagnetism, Superfluids*, Physics Series (MIR, Moscow, 1987), Chap. 6; A. Onuki, Prog. Theor. Phys. **79**, 191 (1984).
[6] D. Frank and V. Dohm, Phys. Rev. Lett. **62**, 1864 (1989).
[7] R. V. Duncan, G. Ahlers, and V. Steinberg, Phys. Rev. Lett. **58**, 377 (1987); R. Duncan and G. Ahlers, Jpn. J. Appl. Phys. **26**, Suppl. 26-3, 363 (1987); and (unpublished).
[8] D. Frank, M. Grabinski, V. Dohm, and M. Liu, Phys. Rev. Lett. **60**, 2336 (1988).
[9] W. Huhn and V. Dohm, Phys. Rev. Lett. **61**, 1368 (1988).
[10] I. M. Khalatnikov, *Theory of Superfluidity* (Benjamin, New York, 1965).
[11] L. P. Pitaevskii, Zh. Eksp. Teor. Fiz. **35**, 408 (1958) [Sov. Phys.—JETP **35**, 282 (1959)].
[12] V. Dohm, J. Low Temp. Phys. **69**, 51 (1987), and references therein.
[13] M. J. Lea, D. S. Spencer, and P. Fozooni, Jpn. J. Appl. Phys. **26**, Suppl. 26-3, 53 (1987).
[14] R. A. Ferrell, N. Menyhard, H. Schmidt, F. Schwable, and P. Szépfalusy, Ann. Phys. (N.Y.) **47**, 565 (1968).
[15] W. Y. Tam and G. G. Ahlers, Phys. Rev. B **32**, 5932 (1985).
[16] W. Y. Tam and G. G. Ahlers, Phys. Rev. B **33**, 183 (1986); **37**, 7898 (1988).
[17] J. Clow and J. D. Reppy, Phys. Rev. Lett. **19**, 291 (1967); J. S. Langer and M. E. Fisher, *ibid.* **19**, 560 (1967).