# Orbital magnetoconductance in the variable-range-hopping regime

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The orbital magnetoconductance (MC) in the variable-range-hopping (VRH) regime is evaluated using a model proposed by Nguyen, Spivak, and Shklovskii (NSS), which approximately takes into account the interference among random paths in the hopping process. The results are shown to be valid in more general situations. The MC is obtained using the critical-percolating-resistor method, which is proven to be equivalent to a modified logarithmic averaging. The behavior of the MC is analyzed in detail neglecting backscattering. The small-field MC is quadratic in $H$, is positive deep in the VRH regime, and changes sign when the zero-field conductivity is high enough. Very deep in the VRH regime a quasilinear intermediate-field dependence develops. The calculated MC is always positive for strong fields and is predicted to saturate at sufficiently large fields. This behavior and the relevant magnetic-field scale are in agreement with recent experiments.

## I. INTRODUCTION

The study of the magnetoconductance (MC) in disordered metals in the weak-localization regime has given valuable insights into the interference processes in such systems. Various electronic relaxation times have also been determined using this method. In systems of mesoscopic sizes the sample-specific interference is very important, but is expected to average out in a macroscopic network of such systems. There is no detailed understanding of the magnetotransport in the strongly localized regime. Recent studies[1-3] have focused on the magnetotransport in the mesoscopic range where the finite size of the sample is relevant. It is, however, also of interest to study the MC in a large, macroscopic sample in the regime where thermal hopping dominates the transport. It turns out that the analysis does produce a definite macroscopic effect due to the fluctuations in the bond conductances.

A recent experimental study[4] of transport properties of indium oxide samples in the variable-range-hopping regime (VRH) reveals a positive MC. In the absence of a magnetic field, the conductance of these specimens[4] obeys Mott's VRH law, $\sigma=\sigma_0\exp[-(T_0/T)^{1/(d+1)}]$, in two and three dimensions ($d=2,3$) with $200\le T_0/T\le 1000$. The hopping distance $R_M$ extracted from the data is typically of the order of several $\xi$, where $\xi$ is the localization length. The behavior of the MC at low fields is as follows: after an initial, fast dependence (perhaps $H^4$) at extremely small fields, the MC becomes quadratic in the magnetic field for a rather large range of the latter. Studies of the dependence of the MC on the magnetic-field orientation relative to that of the film strongly indicate that it results from an orbital, rather than a spin effect. The change in the conductance due to the magnetic field is characterized by the flux $\Phi_M\equiv HR_M^{3/2}\xi^{1/2}$ through an effective area of the order of $R_M^{3/2}\xi^{1/2}$. More recently[5] large, mostly positive, MC was found by Assadullaev and Ciric in amorphous $Si_3N_4$, but with large measurement voltages and by Benzaquen *et al.* for GaAs. The positiveness of the MC and the anisotropy with respect to the magnetic field orientation were also observed in earlier measurements on 2D Si inversion layers.[6]

There have been theoretical approaches to the MC in the VRH regime but they could not fully account for these experimental data. Shklovskii and Efros[7] and Suprapto and Butcher[8] predict a negative MC due to the shrinkage of the wave functions in the presence of a magnetic field. Nguyen, Spivak, and Shklovskii (NSS) (Refs. 9 and 10) are the first to consider in this connection the effect of the interference among the various paths associated with the hopping between two sites at a distance $R_M$ apart and a small energy separation of the order of $k_BT^{3/4}T_0^{1/4}$ in three dimensions. They find that the interference between all possible paths within a cigar-shaped domain[11] of length $R_M$ and width $(R_M\xi)^{1/2}$ might considerably change the hopping probability between two sites. Averaging numerically the logarithm of the conductivity over many random impurity realizations, in the presence of a magnetic field, NSS obtain under certain conditions a positive MC which is *linear* in the field in the whole relevant field range.

A theory for the orbital MC in the VRH regime, based on the NSS model but employing the critical percolation path picture,[7,12–14] was presented in Ref. 15 [Sivan, Entin-Wohlman, and Imry (SEI)]. It yields the sign of the MC and the *quadratic* field dependence in the weak-field range where the field scale was determined by the parameter $\Phi_M/\Phi_0$ ($\Phi_0 = hc/e$ being the quantum-flux unit). Furthermore, that model predicted a saturation of the MC for $\Phi_M/\Phi_0 \gg 1$. However, the situation very deep in the VRH regime ($R_M/\xi \gtrsim 10$) was not considered in detail.

In this paper we present a comprehensive discussion of the orbital MC in the VRH model, and argue that the results hold quite generally. We find the regimes in which the various behaviors found by NSS and SEI hold and discuss the various ways to obtain the macroscopic MC from that of the elementary bonds.

## II. ANALYSIS OF THE MC WITHIN THE PERCOLATION MODEL

The conductivity of a sample in the hopping regime may be analyzed in terms of an equivalent resistor network.[7,12–14] Any two sites between which the electron hops are taken to be connected by a conductance $\sigma_{if}$,

$$\sigma_{if} \approx \sigma_0 |I_{if}|^2 e^{-\beta\varepsilon_{if}} , \quad (1)$$

where $\sigma_0$ is a constant having dimensions of conductance and $\varepsilon_{if} = (|\varepsilon_i| + |\varepsilon_f| + |\varepsilon_i - \varepsilon_f|)/2$, where $\varepsilon_i$ and $\varepsilon_f$ are the initial and final site energies measured from the Fermi energy. In (1), $I_{if}$ is the effective overlap integral between the initial and final sites. $|I_{if}|^2$ depends exponentially on the distance $r_{if}$ between the two sites, $|I_{if}|^2 \propto \exp(-r_{ij}\alpha)$ where $\alpha$ is of the order of the inverse localization length.

Quite generally, $\sigma_{if}$ is a random variable, depending upon the site energies, the distance between them, $r_{if}$, and the strength of the overlap integral. The conductivity of the macroscopic sample is determined by the percolation threshold condition.[7,12] Given the probability distribution $P(\sigma/\sigma_0, r)$ for the dimensionless conductivity $\sigma/\sigma_0$ at a spatial separation $r$, the sample conductivity $\sigma_c$ is given by the requirement that the conductors of $\sigma > \sigma_c$ occupy a certain finite fraction $Z_c$ of the system volume $\Omega$

$$\int_{\sigma_c}^{\infty} d\sigma \int_0^{\infty} d(r^d) r^d P\left[\frac{\sigma}{\sigma_0}, r\right] = Z_c \Omega , \quad (2)$$

where $d$ is the system dimensionality. Thus, knowledge of the distribution function will yield an implicit equation for the "critical" conductivity $\sigma_c$. In the variable-range-hopping regime, the critical conductivity obeys Mott's law, $\sigma_c = \sigma_0 \exp(-T_0/T)^{1/(d+1)}$, with $T_0/T \gtrsim 10^2$, and where $k_B T_0$ is on the order of the average energy spacing between levels localized within a localization volume, $\xi^d$. The critical conductance connects sites whose spatial separation, $R_M$, is typically at least several $\xi$'s (depending upon the temperature and impurity concentration) and whose site energies are of the order of a few $k_B T$, much smaller than $k_B T_0$. The percolation argument thus assumes that the conducting properties of the system are determined by hops of elementary conductance $\sigma_c$ which span a critical network throughout the macroscopic system.

The quantum aspect of the elementary hopping is brought about by the overlap integral $I_{if}$. It results from electron tunneling between the initial and final sites around which the wave functions are localized, with localization length $\xi$. The spatial separation $r_{if}$ between two sites belonging to the critical network is larger than $\xi$. There are, therefore, many different paths connecting the initial and final sites, along which the electron traverses other sites, of site energies in general lying far away from the Fermi level. The effective overlap integral $I_{if}$ thus contains the effects of interference among the various possible paths.

We adopt here the picture of Nguyen, Spivak, and Shklovskii (NSS) (Refs. 9, 10, and 15) according to which the important contribution to $I_{if}$ comes from all oriented paths within a cigar-shaped region of length $r_{if}$ and width $(r_{if}\xi)^{1/2}$. One thus may visualize the electron to perform a random walk of step $\xi$ perpendicular to the hopping distance $r_{if}$. In the specific geometrical model employed by NSS the random walk has an elementary step equal to the microscopic length. The appearance of the $\xi$ in the more general case follows from the discussion of the statistics of the path contributing to $I_{if}$ given by Shklovskii and Spivak.[11] The resulting effective overlap integral is the sum over the contributions from the various paths

$$I_{if} = e^{-\alpha r_{if}/2} \sum_{\gamma} J_{\gamma} . \quad (3)$$

As the number of oriented paths is exponential in the path length, a redefinition of the localization length in the expression for the elementary conductance [Eq. (1)] yields

$$\sigma_{if} = \sigma_0 y_{if}^2 e^{-\alpha r_{if} - \beta\varepsilon_{if}}, \quad y_{if}^2 = \frac{1}{n} \left| \sum_{\gamma} J_{\gamma} \right|^2 , \quad (4)$$

where $n$ is the number of oriented paths and $y_{if}^2$ is a random variable of order unity.

In the presence of a constant magnetic field the individual path overlap integral is multiplied by a phase factor $e^{i\phi_\gamma}$, $\phi_\gamma$ being the phase acquired from the magnetic field along the $\gamma$th path. This is the only modification due to the magnetic field at small enough fields such that the flux through an elementary area $\xi^2$ is much smaller than the quantum-flux unit $\Phi_0$.[16] (Spin effects which change the site energies are ignored in our discussion of the orbital effects.) Equation (3) is thus modified to read

$$I_{if} = e^{-\alpha r_{if}/2} \sum_{\gamma} J_{\gamma} e^{i\phi_\gamma} . \quad (5)$$

The elementary conductance of the system [Eq. (1)] then becomes

$$\sigma_{if} = \sigma_0 y_{if}^2 e^{-\alpha r_{if} - \beta\varepsilon_{if}}, \quad y_{if}^2 = \frac{1}{n} \left| \sum_{\gamma} J_{\gamma} e^{i\phi_\gamma} \right|^2 , \quad (6)$$

where $y_{if}^2$ represents the quantum interference effect upon the conductance. It depends upon the hopping distance $r_{if}$ and, in particular, upon the magnetic field.

The interference factor $y^2$ is a random variable. It is governed by a distribution function which varies under the effect of the magnetic field. Consequently, the percolation condition (2) yields a field-dependent critical conductivity $\sigma_c(H)$ and, in turn, the magnetoconductivity of the sample.

Consider first the probability distribution of $y$ in the absence of the magnetic field. The individual path overlap integrals may be assumed to be real. Then, for mutually independent path contributions, $|y|$ is distributed normally,[17] i.e.,

$$F_0(y^2)dy^2=\sqrt{1/2\pi}e^{-y^2/2}d|y| \ , \tag{7}$$

where for simplicity it was assumed that $\langle J_\gamma^2\rangle=1$. Under the assumptions above, the $y^2$ distribution is independent of the hopping distance and is peaked around $y=0$. Correlations among the paths were recently considered in Ref. 18.

In the presence of the magnetic field, it is convenient to construct the $y^2$ distribution from the probability distributions of $(J')^2=1/n(\sum_\gamma J_\gamma \cos\phi_\gamma)^2$ and $J''^2=(1/n)(\sum_\gamma J_\gamma \sin\phi_\gamma)^2$, where $|J|^2=(J')^2+(J'')^2$. Assuming that for any path with a phase $\phi_\gamma$ there is also a symmetric path with a phase $-\phi_\gamma$ (an assumption which is valid in the NSS model), the central-limit theorem[17] for both $J'$ and $J''$ yields (see the Appendix)

$$P(J',J'')=\left[\frac{1}{a\sqrt{2\pi}}e^{-(J')^2/2a^2}\right]\left[\frac{1}{b\sqrt{2\pi}}e^{-(J'')^2/2b^2}\right] , \tag{8}$$

where

$$a^2=\frac{1}{n}\sum_\gamma \cos^2\phi_\gamma,\quad b^2=\frac{1}{n}\sum_\gamma \sin^2\phi_\gamma \ . \tag{9}$$

Thus $P(J',J'')$ is a product of two normal distributions, for the real ($J'$) and imaginary ($J''$) parts of the field-dependent effective overlap, with standard deviations $a$ and $b$, respectively. The resulting $y^2$ distribution is

$$\begin{aligned}F(y^2)dy^2&=\int dJ'dJ''\delta(y^2-(J')^2-(J'')^2)P(J',J'')dy^2\\&=\frac{1}{2ab}e^{-y^2/4a^2b^2}I_0\left[\frac{y^2(a^2-b^2)}{4a^2b^2}\right]dy^2 \ ,\end{aligned} \tag{10}$$

where $I_0$ is the modified Bessel function of order zero.

Equations (8) and (9) are most easily obtained in the simple case where the $J_\gamma$ are independent random variables having a variance of unity. Assuming a symmetric distribution for $\sin 2\phi_\gamma$ one immediately finds that $J'$ and $J''$ are independent Gaussian random variables distributed according to Eq. (8). This is really the central-limit theorem. A fuller analysis leading to Eqs. (8) and (9) under much more general conditions is presented in the Appendix.

The effect of the field upon the $y^2$ distribution is clearly seen from (8) and (10): As the field tends to zero, $a\to 1$ and $b\to 0$. As a result, the second factor in (8) tends to $\delta(J'')$ and the $y^2$ distribution becomes a normal one, i.e., it is peaked at $y=0$ [Eq. (7)]. However, as the magnetic field is switched on, the weight of the distribution shifts towards finite values of $y$. The distribution is zero at $y=0$, increases linearly with $y$ and is peaked at $y$ of the order of $b$. Thus the effect of the magnetic field is to narrow the distribution of $y^2$ by favoring intermediate values of $y$, of order $b$, instead of those $\lesssim b$, and reducing the probability for larger $y$, as compared to the zero field distribution. This behavior is depicted in Fig. 1.

The magnetic field enters the distribution through the phase factors $a$ and $b$ [Eq. (9)]. In the strong-field limit, such that the magnetic flux through the hopping "area" [of length of the typical hopping distance $R_M$ and width of the order of $(R_M\xi)^{1/2}$] becomes larger than the quantum-flux unit, $a^2$ becomes comparable to $b^2$, $a^2\sim b^2\sim\frac{1}{2}$, and the distribution saturates to the form

$$F(y^2)dy^2\sim e^{-y^2}|y|d|y| \ . \tag{11}$$

with a broad peak located at $y\sim(\frac{1}{2})^{1/2}$, and vanishing linearly at $y=0$. We note that the way the $b\to 0$ limit is achieved is by the shrinking of the range (of order $b$ in $y$) where $F$ vanishes at $y\ll b$ and has a peak for $y\sim b$ (see Fig. 1).

In the presence of a magnetic field, the distribution function of $y^2$ depends upon the hopping distance $r$ through the phase factors $a$ and $b$. This complicates the calculation of $\sigma_c(H)$ by the percolation condition [Eq. (2)],[15] but is not expected to be a major effect since everything is governed by the critical bonds whose length does not vary by orders of magnitude. For the sake of simplicity it will be assumed now that the relevant area for the phase factors is characterized by the typical hopping distance $R_M$. The elementary conductance [Eq. (6)] is then a product of the "quantum" factor $y^2$ and the "classical" exponential factor, $\exp(-\alpha r-\beta\varepsilon)$, each obeying its independent distribution function. Assuming the site energies to be distributed uniformly in a band of typical width

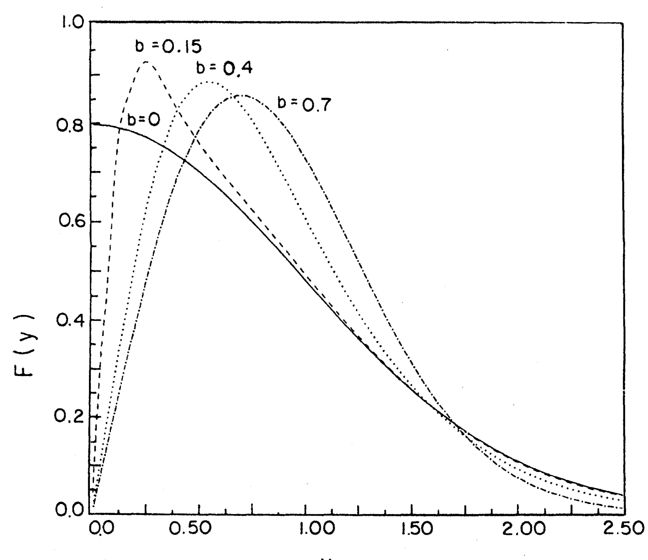


FIG. 1. The distribution of $|y|$ [Eq. (10)] for various values of the magnetic field parameter $b$.

$W$, the distribution function of the classical factor of the conductance is [see Eq. (2)]

$$P(z)dz=\int d(r^d)r^d\int d\varepsilon\, P(\varepsilon)\delta(z-e^{\beta\varepsilon+\alpha r})dz$$
$$=\frac{1}{2\beta^2 W^2}\xi^{2d}\frac{1}{2d(2d+1)}\frac{1}{z}(\ln z)^{2d+1}dz, \quad 1\le z\ . \tag{12}$$

This form for $P(z)$ is valid for $\varepsilon$ smaller than the bandwidth (the relevant range for the variable-range-hopping regime). The upper bound on $z$ is determined in fact by the convolution with the $y^2$ distribution (see below) and hence is of no importance.

With the explicit expressions for the distributions, Eqs. (10) and (12), the percolation condition for $\sigma_c(H)$ reads[15]

$$\int_{\sigma_c(H)/\sigma_0}^{\infty}d(\sigma/\sigma_0)\int_1^{\infty}dz\, P(z)\times\int dy^2F(y^2)\delta(\sigma/\sigma_0-y^2/z)=Z_c\ . \tag{13}$$

The zero-field conductivity, $\sigma_c(0)$, is given by the same expression with $F$ replaced by its field-free counterpart, $F_0$ [Eq. (7)]. The relative magnetoconductivity (MC) is therefore

$$\frac{\sigma_c(H)-\sigma_c(0)}{\sigma_c(0)}\simeq\int_1^{\infty}dz\, P(z)\int_{z\sigma_c(0)/\sigma_0}^{\infty}dy^2[F(y^2)-F_0(y^2)]\left[\int_1^{\infty}dz\, P(z)z\frac{\sigma_c(0)}{\sigma_0}F_0(z\sigma_c(0)/\sigma_0)\right]^{-1}\ . \tag{14}$$

Using the explicit expressions for $P(z)$ and $F_0$, one finds that deep in the VRH regime the denominator on the right-hand side of Eq. (14) is of the order of $[\ln\sigma_0/\sigma_c(0)]^{2d+1}$, and in particular is independent of the zero-field conductivity $\sigma_c(0)$. The sign and magnitude of the MC will thus be determined by the difference $F-F_0$ of the $y^2$ distributions, appearing in the numerator on the right-hand side of (14).

Integration by parts and rearrangement lead to the following form for the numerator of the MC:

$$\int_{\sigma_c(0)/\sigma_0}^{\infty}[\ln z\sigma_0/\sigma_c(0)]^{2d+2}[F(z)-F_0(z)]dz\ , \tag{15}$$

which may be evaluated as follows. For very small magnetic fields such as $b^2\sigma_0/\sigma_c(0)\ll 1$ the argument of the Bessel function appearing in $F(z)$ [see Eq. (10)] is very large. One may then use its asymptotic form to obtain

$$\frac{\sigma_c(H)-\sigma_c(0)}{\sigma_c(0)}\sim\frac{b^2}{[\ln\sigma_0/\sigma_c(0)]^{2d+1}}\times\int_{\sigma_c(0)/\sigma_0}^{\infty}dz\, e^{-z/2}(z^{-3/2}-z^{-1/2})\times[\ln z\sigma_0/\sigma_c(0)]^{2d+1}\ ,\quad b^2\ll\sigma_c(0)/\sigma_0\ . \tag{16}$$

Hence at very low fields the MC is quadratic in the magnetic field.[15] For high enough values of $\sigma_c(0)/\sigma_0$ [$\sigma_c(0)/\sigma_0>\sigma_l/\sigma_0\sim 10^{-4}$, see Ref. 15] the low-field MC is negative as a result of the reduction of $F$ below $F_0$ at large values of $z$. At lower values of $\sigma_c(0)/\sigma_0$ the small-field MC is positive,

$$\frac{\sigma_c(H)-\sigma_c(0)}{\sigma_c(0)}\sim b^2[\sigma_0/\sigma_c(0)]^{1/2}/[\ln\sigma_0/\sigma_c(0)]^{2d+1}\ ,\quad b^2\ll\sigma_c(0)/\sigma_0\ll 1\ . \tag{17}$$

We emphasize that the negative MC at $\sigma>\sigma_l$, while being a correct result of the model, is a weaker prediction for real systems. For such large $\sigma$'s, the neglect of winding paths is not clearly justified. Their inclusion should lead for $\sigma\sim\sigma_0$ to the usual weak-localization MC. The magnitude of the relative MC decreases with increasing temperature ($[\sigma_0/\sigma_c(0)]^{1/2}\simeq\exp[(T_0/T)^{1/(d+1)}/2]$).

We now turn the case $\sigma\ll\sigma_l$, not treated in Ref. 15. Here, at intermediate values of the magnetic fields such that $1\gg b^2>\sigma_c(0)/\sigma_0$ the main contribution to the integration in (15) comes from the region $z>b^2$ and is positive, dominating the negative contribution from the region $\sigma_c(0)/\sigma_0\le z\le b^2$. Using again the asymptotic form for the Bessel function one finds for $d=2$ with $A\equiv\ln b^2\sigma_0/\sigma_c(0)$ (a related expression applies at $d=3$):

$$\frac{\sigma_c(h)-\sigma_c(0)}{\sigma_c(0)}\sim b[A^5+(2)5A^4+(2^2)(5)4A^3+(2^3)5(4)3A^2+(2^4)5!A+(2^5)5!]/[\ln\sigma_0/\sigma_c(0)]^{2d+1}\ ,\quad \sigma_c(0)/\sigma_0<b^2\ll 1\ . \tag{18}$$

In this regime the MC is quasilinear in the field with important logarithmic corrections. These cause the MC to deviate from linearity (Fig. 2) and to appear quadratic over much broader ranges of $b$. For $b^2$ sufficiently larger than $\sigma_c(0)/\sigma_0$ the MC has an approximately linear behavior (Fig. 2). The temperature dependence of the MC arises from the effective hopping area which determines the phase factor $b$. This area is proportional to $R_M^{3/2}$, i.e., to $(T_0/T)^{3/2(d+1)}$, again yielding a decrease of the MC with temperature. Finally, when the magnetic field is strong enough such that the flux through the hopping domain is larger than the quantum-flux unit (i.e., $b^2\gtrsim\frac{1}{2}$, the MC saturates[15] to a relative value of order unity, independent of the magnetic field. The behavior of the MC in the various regimes as computed from Eq. (14) is shown in Fig. 2. One notes the quadratic-type behavior

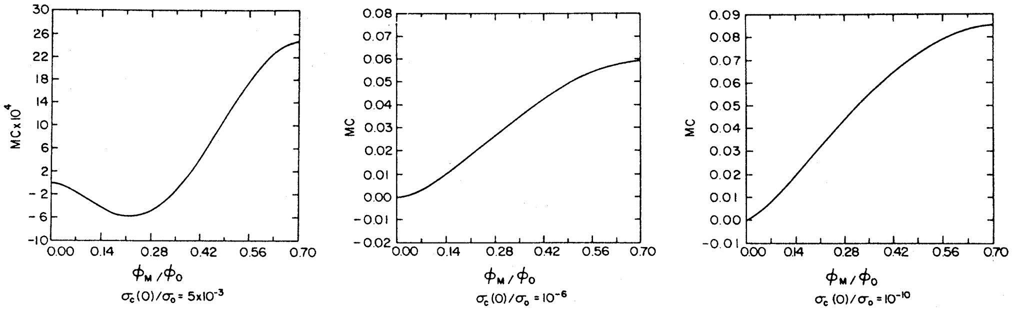


FIG. 2. The behavior of the relative MC as a function of the magnetic field (parametrized by $\Phi_M/\Phi_0$) in the various regimes [from Eq. (14)] for three values of $\sigma_c(0)/\sigma_0$.

for moderate values of $\sigma_c(0)/\sigma_0$ which crosses over to quasilinearity as $\sigma_c(0)/\sigma_0$ decreases.

## III. EQUIVALENCE OF RESTRICTED LOGARITHMIC AVERAGING WITH THE PERCOLATION ANALYSIS FOR THE MC

In this section we consider the generic problem of a broad distribution of the conductance, $e^{-x}$, on top of which the individual bond values of $x$ are randomly changed each by a small $\Delta x$. This is evidently a simplified picture; in the VRH model the bonds have varying lengths and energies. $\Delta x$ can be induced, for example, by a small magnetic field. We denote the distribution of $x$ by $P_0(x)$ and the distribution of $\Delta x$ by $P_\Delta(x,\Delta x)$ (clearly, moments such as $\langle \Delta x \rangle$ depend on $x$). The distribution of $u \equiv x+\Delta x$, is given by

$$P(u)=\int dx \int d\Delta x \, P_0(x) P_\Delta(x,\Delta x)\delta(u-x-\Delta x) \, . \tag{19}$$

The critical value of $u$ is given by the percolation condition

$$\int_{-\infty}^{u_c} P(u)du = p_c \, . \tag{20}$$

We substitute (19) into (20) and perform the integration over $u$

$$p_c=\int dx \int d\Delta x \, P_0(x)P_\Delta(x,\Delta x)\Theta(u_c-x-\Delta x) \, . \tag{21}$$

Expanding the $\Theta$ function (this is justified for $P_\Delta$ which is smooth on the scale of $\Delta x$) and writing $u_c=u_c^0+\Delta u_c$, where $u_c^0$ is the critical value before $\Delta x$ was introduced, we obtain after a further expansion in $\Delta u_c$:

$$\Delta u_c = \int d(\Delta x) \int dx \, P_0(x)\delta(u_c^0-x) \times \Delta x P_\Delta(x,\Delta x) \Big/ \int dx \, P_0(x)\delta(x-u_c^0) \equiv \langle \Delta x \rangle_{x=u_c^0} \tag{22}$$

i.e., the change $\Delta u$ is given by the average of $\Delta x$ over the critical bonds before the change. In terms of the bond conductances $e^{-x}$ this is a (restricted) logarithmic averaging *over the critical network* (we emphasize that this is *not* averaging over the whole network which may give undue weight to, e.g., very small conductances). This demonstration is straightforwardly generalizable to the case where $P(x)$ is governed by a joint distribution of several variables (such as $r$, $\varepsilon$, $y$ in our VRH problem). These issues are discussed in Ref. 19.

## IV. CONCLUSIONS

The conductance of each bond in the VRH model depends on a sum over paths and is thus sensitive to interference, and depends on an applied magnetic field. Each bond, similar to a mesoscopic system, has a MC of a random sign. The surprising result is that the resultant low-field orbital MC of the macroscopic system is *not* averaged out but instead shows a well-defined, substantial, value. This follows from both the critical path analysis and from a modified logarithmic averaging[19] on the critical network.

The MC is governed by a flux in a typical area given by $R_M^{3/2}\xi^{1/2}$.[9,11] At (relatively) large fields, such that the above flux is much larger than a flux quantum, $\Phi_0$, the relative MC tends to a positive constant on the order of 0.1 which depends on the zero-field conductivity $\sigma_c$. The extremely low-field MC depends on $\sigma_c$ as well. For $\sigma_c \gtrsim \sigma_l \sim 10^{-4}\sigma_0$, the MC starts negative and crosses over to positive. For $\sigma_c \sim \sigma_l$, it is positive, proportional to $H^2$ for $\Phi_M \ll \Phi_0$ and saturates at $\Phi_M \gg \Phi_0$. For $\sigma_c \ll \sigma_l$, there is an additional crossover of the (positive) MC from $H^2$ to "quasilinear" behavior for $(\Phi_M/\Phi_0)^2 \sim \sigma_c/\sigma_0$. In the latter, there is a wide range where the logarithmic corrections cause the MC to be roughly quadratic with an effective linear behavior at much larger $\Phi_M/\Phi_0$ values. These features are in agreement with experiment.

We have proven that these results are due to a central-limit theorem and should thus be quite general. An interesting, still open question is how they crossover to the usual weak-localization results at $\sigma_c \sim \sigma_0$ when backscattering becomes important. The latter show the usual sensitivity to, e.g., spin-orbit coupling which is typical of the "coherent backscattering" mechanism and is not expected to occur in our model. This crossover may also influence the negative MC found in our model for the largest $\sigma_c$'s. For very strong disorder it is possible that the direct path for hopping will become dominant and the MC will vanish. These questions deserve further study.

*Note added in proof.* The probability distribution $F(y^2)$ [Eq. (10)] was evaluated for a specific, simplified model. Yet, its main features (Fig. 1), namely, the narrowing of the distribution in the presence of magnetic field and the limiting form for strong enough fields, are far more general and reflect the symmetry crossover of the Hamiltonian from the orthogonal to the unitary ensemble. It is well known from the theory of conductance fluctuations (closely related to random matrix theory) that the fluctuations in the wave-function amplitude at a given point are smaller for the unitary ensemble compared with the orthogonal one ($y^2$ is proportional to the wave-function amplitude squared). Strong spin-orbit coupling should reduce the symmetry to the symplectic one and hence narrow the distribution even further. An application of strong enough magnetic field should then result in broadening of $F(y^2)$ and probably negative magnetoconductance.

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## APPENDIX: DERIVATION OF THE OVERLAP PROBABILITY DISTRIBUTION

Here we outline the derivation of the probability distribution $P(J',J'')$ [Eq. (8)]. Given the probability density $P(J_1,J_2,\ldots,J_n)dJ_1\cdots dJ_n$ that the $i$th path contributes $J_i$ to the effective overlap ($i=1,\ldots,n$), the explicit form of $P(J',J'')$ is

$$P(J',J'')=\int dJ_1\cdots dJ_n P(J_1,\ldots,J_n)\times\delta\left[J'-\frac{1}{\sqrt{n}}\sum_\gamma J_\gamma\cos\phi_\gamma\right]\times\delta\left[J''-\frac{1}{\sqrt{n}}\sum_\gamma J_\gamma\sin\phi_\gamma\right]. \tag{A1}$$

Expressing the $\delta$ functions in the form of Fourier integrals, Eq. (A1) takes the form

$$P(J',J'')=\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}dx_1dx_2e^{ix_1J'+ix_2J''}\times\prod_\gamma\int dJ_1\cdots dJ_nP(J_1,\ldots,J_n)\exp(-ix_1J_\gamma\cos\phi_\gamma/\sqrt{n}-ix_2J_\gamma\sin\phi_\gamma/\sqrt{n}). \tag{A2}$$

One then expands the exponentials appearing in the second factor of (A2). Assuming a symmetric distribution of the $J_i$'s and that each path characterized by $\phi_\gamma$ has its counterpart with $-\phi_\gamma$, the second factor in (A2) yields

$$\prod_\gamma\left[1-\frac{1}{2}\frac{\langle J_\gamma^2\rangle}{n}(x_1^2\cos^2\phi_\gamma+x_2^2\sin^2\phi_\gamma)+\cdots\right]. \tag{A3}$$

The higher-order terms include factors of order $n^{-2}$, $n^{-3}$, etc. For large[17] $n$, Eq. (A3) becomes $\exp[-\frac{1}{2}(x_1^2a^2+x_2^2b^2)]$, to leading order, where $a^2=\sum_\gamma\langle J_\gamma^2\rangle\cos^2\phi_\gamma/n$, $b^2=1-a^2$. For simplicity, and without substantial loss of generality, we choose $\langle J_\gamma^2\rangle=1$. Inserting this form into (A2), and performing the $x_1$ and $x_2$ integrations, we obtain Eq. (8).

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