Dynamics of electromagnetic fields in nonlinear Klein-Gordon systems

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Localized electromagnetic fields in $(1+1)$ dimensional nonlinear Klein-Gordon systems, i.e., solitons, are treated by transforming Maxwell equations into the equations of motion. It is shown that the force between them is produced by the distortion of space-time coordinates caused by equivalent gravitational fields. The details of local electromagnetic field interactions as extended particles are described for a soliton-antisoliton interaction and a soliton-soliton interaction by the equations of motion. Finally, it is shown that the fields without real charges radiated from a certain system are associated with photons.

The nonlinear Klein-Gordon systems not only play a crucial role for many phenomena appearing in condensed matter physics,[1] but also attract our attention in the area of elementary particles, since the solutions are regarded as extended particles.[2]

In the model field theory on nonlinear Klein-Gordon systems, fields are treated in association with extended particles with distinct masses. So the equations need to be quantized to grasp the real feature as extended particles. However, in this paper, we do not originally treat the systems by regarding them as such extended particles, but rather treat them as localized electromagnetic fields. To do so, we first start with an electromagnetic field equation in a nonlinear medium, and then we derive the fields as extended particles having distributed masses, which are identical to the electromagnetic fields, investigating their dynamic properties.

It was shown by Tateno[3,4] that the exact dynamic behavior of soliton solutions of such nonlinear Klein-Gordon systems can be treated geometrically by a state plane technique using the relation between the field ϕ and its derivative, with respect to either x or t. With the technique, the original equation is transformed into an effective ordinary differential equation. So we can easily investigate the necessary solutions using techniques on ordinary deferential equations.

In the local Riemannian space, the line element ds is represented by

$$ds^2=\sum_{i,j} g_{ij}dx^i dx^j , \tag{1}$$

where g_{ij} is the Riemannian metric and the sums are carried out in such a way that i and j each independently take the values 0–3. For convenience, consider a localized electromagnetic field in a stationary state in a $(1+1)$ dimensional nonlinear Klein-Gordon system. Then, Eq. (1) is expressed by $(x^0,x^1)=(-ut,x)$, where u is the phase velocity (kink velocity) of the field in the stationary state measured by the unit in which the light velocity in vacuum is unity, and $g_{ij}=0$, if $i\neq j$. Since we are treating an electromagnetic field here, it is reasonable to set ds equal to zero, if we observe the field at the inside of the system. Then, we treat the equation as

$$0=-u^2dt^2+dx^2 .$$

However, we can also recognize that the field constructs extended particles if we observe them in the vacuum outside the system, since they move with the velocity u less than unity. Then, the above expression is rewritten as

$$ds^2=-dt^2+dx^2 , \tag{2}$$

where $ds^2=-d\tau^2=-(1-u^2)dt^2$, and τ is the proper time in the moving reference. In this case, the field is Lorentz invariant. Moreover, the existence of ds^2 suggests that the field should have a certain mass.

In the Maxwell equations, we set the field $U(x,y,z,t)=\partial\phi/\partial t$, where $U(x,y,z,t)$ is the potential. Then, $\phi(x,y,z,t)$ corresponds to the magnetic flux, and satisfies

$$\Box\phi+\int_{-\infty}^{t} q(x,y,z,t)dt+\mathrm{div}\mathbf{A}+\frac{\partial^2\phi}{\partial t^2}=0 , \tag{3}$$

where

$$\Box=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial t^2} ,$$

$\mathbf{A}$ is the vector potential, and $q(x,y,z,t)$ is the charge, consisting of the real and polarized charges. Then every quantity in Eq. (3) is normalized so that both the permittivity and the permeability, i.e., the light velocity in the vacuum, is equal to unity in the rationalized meter-kilogram-second-ampere system of units. We postulate the following condition in Eq. (3):

$$\int_{-\infty}^{t} q(x,y,z,t)dt+\mathrm{div}\mathbf{A}+\frac{\partial^2\phi}{\partial t^2}=-F(\phi) , \tag{4}$$

where $F(\phi)$ is a function specializing the nonlinearity of the system. If it is possible for the Lorentz condition to be adopted, $F(\phi)$ is produced by $q(x,y,z,t)$. On the other hand, if $q(x,y,z,t)\equiv 0$, then $F(\phi)$ is produced by the vector potential. Taking Eq. (4) into consideration, Eq. (3) reduces to the nonlinear Klein-Gordon systems as follows:

$$\Box\phi=F(\phi) . \tag{5}$$

For convenience, we limit Eq. (5) to the $(1+1)$ dimensional problems. Then, we treat the following expression:

$$\phi_{xx}-\phi_{tt}=F(\phi) . \tag{6}$$

Equation (6) is also described by an equivalent transmission line.[5,6] Then, $-\phi_X$ and ϕ_t represent the current along the line and the voltage between the line, respectively.

We express ϕ_t and ϕ_x by introducing positive quantities $g(t)$ and $h(x)$ as

$$\phi_t = V(x,t)g(t) , \tag{7}$$

$$\phi_x = -V(x,t)h(x)/u , \tag{8}$$

where $V(x,t)$ is the traveling-wave component of ϕ_t and is defined as

$$V(x,t) = \phi_T = -u\phi_X \tag{9}$$

by introducing nonlinear coordinates $X(x,t)$ and $T(x,t)$.[3,4] From Eq. (9), the phase velocity in this coordinate system is written as

$$\frac{dX}{dT} = -\frac{\phi_T}{\phi_X} = u , \tag{10}$$

which indicates that the field travels with a constant velocity u in the (X,T) coordinate system. Equation (10) also indicates that there is no force acting on ϕ in that coordinate system. On the other hand, in the (x,t) coordinate system, ϕ is expressed by

$$d\phi = \phi_x dx + \phi_t dt . \tag{11}$$

The phase velocity in the (x,t) coordinate system is defined by setting $d\phi = 0$ in Eq. (11) as dx/dt, and it is written by using Eqs. (7) and (8) as

$$\frac{dx}{dt} = -\frac{\phi_t}{\phi_x} = u\frac{g(t)}{h(x)} . \tag{12}$$

From Eq. (12), we obtain the Riemannian metrics as

$$g_{00}(x,t) = -u^2 g^2(t) ,$$
$$g_{11}(x,t) = h^2(x) , \tag{13}$$

and the other components are zero, where $ds^2=0$, since we are observing an electromagnetic field in the system.

Next, regard the field as an extended particle by observing the field in vacuum outside the system. Then, Eq. (10) is transformed into the Minkowsky world. On the other hand, Eq. (12) is transformed into the local Riemannian coordinates. Thus, we obtain the following expressions:

$$ds^2 = -dT^2 + dX^2 , \tag{14}$$

$$ds^2 = -g^2(t)dt^2 + h^2(x)dx^2 , \tag{15}$$

where

$$ds^2 = -(1-u^2)dT^2 = -(1-u^2)g^2(t)dt^2 ,$$

and $g_{00}(x,t)$ in Eq. (13) is altered to be $-g^2(t)$ in Eq. (15), on account of $ds^2 \neq 0$. It is possible to consider that Eq. (14) represents the coordinates for the local inertia system equivalent to Eq. (15), satisfying the demand of the principle of equivalence on the gravitational system by regarding Eq. (15) as an equivalent gravitational system.

Comparing Eqs. (14) and (15), we obtain

$$X(x,t) = \int^x h(x')dx' + X_0(t) ,$$
$$T(x,t) = \int^t g(t')dt' + T_0(x) ,$$

where $X_0(t)$ and $T_0(x)$ are associated with singularities of solutions.[3,4]

Consider an extended particle having a distributed mass $m(x,t)$. In our units, the net energy ε is equal to the net mass m' since the light velocity in vacuum is chosen to be unity. Then $m(x,t)$ satisfies the following equation:

$$\frac{d}{dt}[m(x,t)] = \pm \mathcal{H}(x,t)\frac{dx}{dt} , \tag{16}$$

where $\mathcal{H}(x,t) = (\phi_x)^2$ is the net energy density[6] of the field consisting of the rest energy density and the kinetic energy density, and the plus sign is used for the field traveling in the plus direction, i.e., $dx/dt > 0$, and the minus sign in the minus direction, i.e., $dx/dt < 0$, so that the right-hand side in Eq. (16) is always kept positive. Integrated Eq. (16), ε, i.e., m', is given by

$$\varepsilon = m' = \int_{-\infty}^{+\infty} \mathcal{H}(x,t)dx = m_0/(1-u^2)^{1/2} , \tag{17}$$

where m_0 is the rest mass.

The kinetic energy density of the field, $\mathcal{H}^{(k)}(x,t)$, is divided into two components; the component prescribing the interaction with the other fields, $\mathcal{H}^{(a)}(x,t)$, and the component prescribing the kinetic energy density of the field itself, the $\mathcal{H}^{(e)}(x,t)$, i.e.,

$$\mathcal{H}^{(k)}(x,t) = \mathcal{H}^{(a)}(x,t) + \mathcal{H}^{(e)}(x,t) .$$

The equation on the kinetic energy density is written as

$$\frac{d}{dt}\left[m(x,t)\frac{dx}{dt}\right] = \mathcal{H}^{(k)}(x,t) . \tag{18}$$

From Eqs. (16) and (18), $\mathcal{H}^{(a)}(x,t)$ is written as

$$\mathcal{H}^{(a)}(x,t) = m(x,t)\frac{d^2x}{dt^2} , \tag{19}$$

where

$$m(x,t) = \int_{-\infty}^{x} \mathcal{H}(x',t)dx' ,$$
$$\frac{d^2x}{dt^2} = \frac{u}{h(x)}\frac{dg}{dt} - \frac{u^2g^2(t)}{h^3(x)}\frac{dh}{dx} , \tag{20}$$

and $\mathcal{H}^{(e)}(x,t)$ is written as

$$\mathcal{H}^{(e)}(x,t) = \left[\frac{dx}{dt}\right]^2 \mathcal{H}(x,t) . \tag{21}$$

It is possible for the nonlinear Klein-Gordon systems to have two kinds of solutions referred to as soliton and antisoliton solutions. Under the condition that the soliton asymptotically approaches the stationary solitary-wave solution as $|x|$ and $|t|$ approach infinity, the forms of $h(x)$ and $g(t)$ are clarified.[3,4] The soliton-soliton interaction and the soliton-antisoliton interaction in two local field problems are included in the above condition. Thus, the dynamics of such interactions are described by Eqs. (19) and (20).

We assume that $F(\phi)$ repeats exactly the same shapes having a certain period with increasing ϕ. First, consider the soliton-antisoliton interaction. In Eqs. (7) and (8) we take, for $t<0$, the soliton to be in the region $x<0$ and moving forward ($u>0$), and the antisoliton to be in the region $x>0$ and moving backward ($u<0$), and for $t>0$, we take the soliton to be in the region $x>0$ and moving forward ($u>0$), and the antisoliton to be the region $x<0$ and moving backward ($u<0$), as a result of going through each other. $h(x)$ and $g(t)$ are then given by[3,4]

$$h(x)=\pm\tanh(a_0x/u)\,, \tag{22}$$

$$g(t)=\pm\coth(a_0t)\,, \tag{23}$$

where a_0 is $|\partial V/\partial\phi|$ at

$$|\Xi(x,t)|=|X(x,t)-uT(x,t)|\to\infty\,.$$

The $\pm$ signs in Eq. (22) are applied for $x>0$ and $x<0$, respectively, and the $\pm$ signs in Eq. (23) are applied for $t>0$ and $t<0$, respectively. Moreover, in Eqs. (22) and (23), if u is negative, then it is replaced by $-u$ so that both $h(x)$ and $g(t)$ are always kept positive. On account of this, it then becomes convenient to replace dx/dt in Eq. (12) by $d'x/d't$, where

$$\frac{dx}{dt}=\pm\frac{d'x}{d't}\,,\quad \frac{d^2x}{dt^2}=\pm\frac{d}{dt}\left[\frac{d'x}{d't}\right]. \tag{24}$$

$d'x/d't$ is always positive, and the plus sign is for $u>0$ while the minus sign is for $u<0$.

From Eqs. (20) and (22)–(24), we obtain the following expression:

$$\frac{d}{dt}\left[\frac{d'x}{d't}\right]=\mp ua_0[1/\tanh(a_0x/u)\sinh^2(a_0t)+\cosh(a_0x/u)/\tanh^2(a_0t)\sinh^3(a_0x/u)]\,, \tag{25}$$

where the minus sign is applied for either $x<0$ and $t<0$, or $x>0$ and $t>0$ so that $u>0$, and the plus sign for either $x>0$ and $t<0$, or $x<0$ and $t>0$, so that $u<0$. It is noted that the net sign in Eq. (25) is determined only by the sign of x.

First, consider the situation $t<0$ in Eq. (25). At $t\to-\infty$, as $|x|$ approaches infinity, d^2x/dt^2 approaches zero. Thus, the situation approaches the stationary state. If t is fixed to be a negative value, $d(d'x/c't)/dt$ is positive for either $x<0$ or $x>0$ from Eq. (25). Then, $dx/dt>0$ for $x<0$ and $dx/dt<0$ for $x>0$ from Eqs. (24) and (19). This means that the force acting between the soliton and the antisoliton is attractive. In this situation, $d(d'x/d't)/dt$ is increased with decreasing $|x|$, and approaches infinity as x approaches 0. Thus, the attractive force becomes infinity there. Next, x is fixed, and t is increased from minus infinity. Then, $d(d'x/d't)/dt$ is increased and becomes infinity at $t=-0$. Thus, the attractive force also becomes infinity in that instance. Consider the situation of $t>0$. If t is fixed, $d(d'x/d't)/dt$ is negative for either $x<0$ or $x>0$. From Eqs. (24) and (19), this means that the force between the soliton and the antisoliton is still attractive. As $|x|$ approaches infinity, dx^2/dt^2 approaches zero, i.e., the stationary state.

If we simply set $u=iv$ in Eq. (25), where $i=(-1)^{1/2}$ and v is real, then we can explain the breather oscillation.

In the soliton-soliton interaction, it is realized that for $t<0$, one soliton is in the region $x<0$ and moving forward ($u>0$), and the other soliton is in the region $x>0$ and moving backward ($u<0$), and for $t>0$, one soliton is moving backward ($u<0$) and still in the region $x<0$, while the other is moving forward ($u>0$) and still in the region $x>0$, as a result of bouncing off each other after the collision of their centers at $t=0$.

In this case, tanh in Eq. (22) is replaced by coth, and coth in Eq. (23) by tanh.[3,4] Thus, $d(d'/d't)/dt$ is written as

$$\frac{d}{dt}\left[\frac{d'x}{d't}\right]=\pm ua_0[\tanh(a_0x/u)/\cosh^2(a_0t)+\tanh^2(a_0t)\sinh(a_0x/u)/\cosh^3(a_0x/u)]\,. \tag{26}$$

In Eq. (26), the plus sign is applied for either $x<0$ and $t<0$, or $x>0$ and $t>0$, so that $u>0$, and the minus sign is applied for either $x>0$ and $t<0$, or $x<0$ and $t>0$, so that $u<0$.

As $t\to-\infty$, $d(d'x/d't)/dt$ approaches zero from Eq. (26). In $t<0$, $d(d'x/d't)/dt$ is negative for either $x<0$ or $x>0$. From Eqs. (24) and (19), this means that both solitons are decelerated as they approach each other, producing the repulsive force. At $x=0$, $|d(d'x/d't)/dt|$ again becomes zero. Thus, the repulsive force becomes zero in that instance. When $t>0$, $d'x/d't$ is positive for either $x<0$ or $x>0$. This means that the force between the two solitons is still a repulsive force.

Consider a nonlinear Klein-Gordon system. The net energy of the field $\varepsilon=m'$ is given by Eq. (17). We set $\lambda=(1-u^2)^{1/2}$. It is known from such well-known systems as the ϕ^4 system, the pure sine-Gordon system, double sine-Gordon system, and so on, that λ represents the width of the field.[1] Thus, Eq. (17) is rewritten as $\varepsilon=m_0/\lambda$. If we set $m_0=h'$ and $v=1/\lambda$ for convenience, then

$$\varepsilon=h'v\,, \tag{27}$$

where v corresponds to the number of vibrations of electromagnetic fields in vacuum if we suppose a train of waves with the wavelength λ. The field that has a mass of m_0 has a momentum of $p^{(0)}$. Since we are using the unit where the velocity of light is equal to unity, $p^{(0)}$ should be equal to ε. Thus, we can also set

$$p^{(0)}=h'/\lambda . \tag{28}$$

It is noted that $p^{(0)}$ is different from the momentum deduced from Eq. (18) on the kinetic energy density. It is suspected that h' in Eqs. (27) and (28) corresponds to Plank's constant. Since the values of m_0 are adjusted by changing the functional form of $F(\phi)$, it is possible to find the form of $F(\phi)$ to deduce Plank's constant. Suppose that the fields without real charges are radiated from the system into a vacuum. The fields then have no mass, since they follow exactly the equations of electromagnetic fields in vacuum, where $u=1$.

The properties of the magnetic flux Φ in a long Josephson junction are well known.[5] Φ is specified by the 2π kink solution in the sine-Gordon system, i.e., $F(\phi)=\sin\phi$, not depending upon the magnitude of the energy of the field, and constructing a quantum denoted by $\Phi=h/2e$, where h is Plank's constant and e is the magnitude of the electronic charge. Φ is permitted to take two states, referred to as the fluxon and the antifluxon, respectively, produced by a vortex of currents. This property is associated with the spin in our subject, specified by Eqs. (27) and (28). From the above facts, the electromagnetic fields without real charges radiated from our system are associated with a photon.

In summary, the electromagnetic fields in nonlinear Klein-Gordon systems are expressed by the equations of motion by introducing a distributed mass corresponding to the energy of the electromagnetic field. Then, the force on the field interactions is produced by the distortion of the space-time coordinates according to the principle of equivalence on an equivalent gravitational system. The detailed dynamics in the field interactions are described by the equations of motion for the soliton-antisoliton and the soliton-soliton interactions. Finally, it is shown that the radiation without real charges from the nonlinear Klein-Gordon systems into vacuum is associated with a photon.

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[1]See, for example, *Solitons and Condensed Matter Physics*, edited by A. R. Bishop and T. Schneider, Springer Series in Solid-State Science, Vol. 8 (Springer-Verlag, Berlin, 1978); *Solitons*, edited by R. K. Bullough and P. J. Caudrey, Topics in Current Physics, Vol. 17 (Springer-Verlag, Berlin, 1980).

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