Partition function and the density of states for an electron in the plane subjected to a random potential and a magnetic field

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The object of the study is the random Landau model, that is, the Schrödinger operator in two dimensions with a perpendicular constant magnetic field and a random potential. With the help of upper and lower bounds the averaged partition function, either unrestricted or restricted to the Hilbert space of the nth Landau level, is discussed. Based on the lower bounds, approximations to the corresponding densities of states are proposed. For Gaussian random potentials the leading low-energy behavior of the unrestricted density of states is derived.

INTRODUCTION

As discussed in recent publications,[1,2] there are shortcomings in some previously proposed approximations to the density of states of an electron in two dimensions subjected to a perpendicular constant magnetic field and a random potential. These shortcomings are the lack of uniqueness or positivity of the approximate density of states,[3–6] or the nonexistence of the corresponding partition function,[7] even for non-δ-correlated random potentials. These problems are avoided in the self-consistent Born-approximation (SCBA),[8] however, it is difficult to get rid of its unphysical sharp cutoffs at the band edges[9] in the case of δ-correlated random potentials. Therefore, after years of research on this topic it is still interesting to propose a new approximation to the density of states which does not share these shortcomings. This is the main purpose of the present paper.

Our approach to the density of states is based on upper and lower bounds to the averaged partition function especially adapted to nonvanishing magnetic field. This may be viewed as a first attempt in the direction of works done for vanishing magnetic field; see, for example, Refs. 10–13. Because of its simple form, the proposed approximation can, in principle, be calculated for any kind of homogeneous random potential. However, in this paper we will mainly concentrate on the most often discussed case of Gaussian random potentials.

The resulting approximations to the density of states and to its restriction to the Hilbert space of the nth Landau level are compared to exact results obtained in some limiting cases of random potentials or energy range.[14–20]

To the best of our knowledge, two of these results are presented in this paper for the first time. In the case of Gaussian random potentials, these are the following.

(i) Asymptotic expansions of the partition function and the density of states restricted to the nth Landau level up to the order of the squared inverse correlation length of a Gaussian covariance function.

(ii) The leading behavior of the unrestricted partition function and density of states in the low-temperature and the low-energy limits, respectively.

Finally, we give a quantitative argument as to why, for the partition function, a possible effect of Landau-level mixing can be neglected for sufficiently high fields. On the other hand, this argument raises doubts regarding the physical relevance of the density of states restricted to the lowest Landau level in the case of δ-correlated random potentials.

I. THE RANDOM LANDAU MODEL

Choosing an asymmetric gauge, in the Schrödinger representation the random Landau model is characterized by the Hamiltonian

$$H := H_0 + V\ ,\qquad H_0 := \frac{1}{2}\left[\frac{\partial}{i\partial x_1}\right]^2 + \frac{1}{2}\left[\frac{\partial}{i\partial x_2} + \omega x_1\right]^2\ , \tag{1.1}$$

for one (spinless) electron in the infinite $x := (x_1, x_2)$ plane subjected to a perpendicular constant magnetic field of strength $\omega > 0$ and a random potential $V(x)$. We fix units such that Planck's constant $\hbar$, Boltzmann's constant k_B, the elementary charge e, and the (effective) electron mass m equal 1.

We write the spectral resolution of the unperturbed Hamiltonian H_0 as

$$H_0 = \omega \sum_{n=0}^{\infty} (n + \tfrac{1}{2}) E_n\ , \tag{1.2}$$

where E_n is the projection operator on the Hilbert subspace of the nth Landau level $(n+\frac{1}{2})\omega$. One has

$$E_n^\dagger E_{n'} = \delta_{n,n'} E_n, \quad \sum_{n=0}^{\infty} E_n = 1\ . \tag{1.3}$$

In the position representation E_n is given by

$$\langle x|E_n|y\rangle = \frac{\omega}{2\pi}\exp[-\tfrac{1}{2}i\omega(x_2-y_2)(x_1+y_1)] \times \exp[-\tfrac{1}{4}\omega(x-y)^2] L_n(\tfrac{1}{2}\omega(x-y)^2)\ . \tag{1.4}$$

Here,

$$L_n(\kappa) := \frac{1}{n!} e^{\kappa} \frac{d^n}{d\kappa^n}(e^{-\kappa}\kappa^n) \qquad (1.5)$$

denotes the Laguerre polynomial of nth order.[21,22]

In the following we are only concerned with random potentials which are homogeneous and have, without loss of generality, the property

$$\overline{V(x)} = 0 \ . \qquad (1.6)$$

The overbar denotes the average with respect to the probability distribution of the potential. For further information about the characterization of random potentials see, for example, Ref. 19.

The problem in view is to estimate the averaged partition function (per area), $\langle x|\overline{e^{-\beta H}}|x\rangle$, and the averaged partition function restricted to the Hilbert space of the nth Landau level, $\langle x|\overline{E_n e^{-\beta E_n H E_n} E_n}|x\rangle$, by upper and lower bounds. Subsequently, approximations to the corresponding densities of states (per area), $\langle x|\overline{\delta(\varepsilon - H)}|x\rangle$ and $\langle x|\overline{E_n \delta(\varepsilon - E_n H E_n) E_n}|x\rangle$, are introduced. The approximations are related to the lower bounds by inverse Laplace transformation. Of course, here $\varepsilon \in \mathbb{R}$ is an energy and $\beta \in \mathbb{R}_+$ can be interpreted as an inverse temperature.

II. BASIC INEQUALITIES FOR THE RESTRICTED PARTITION FUNCTION

To begin with, we state the following chain of inequalities,

$$\begin{aligned} \frac{\omega}{2\pi} &\le \frac{\omega}{2\pi} \overline{\exp\left[-\frac{2\pi\beta}{\omega}\langle x|E_n V E_n|x\rangle\right]} \\ &\le \langle x|\overline{E_n e^{-\beta E_n V E_n} E_n}|x\rangle \\ &\le \frac{\omega}{2\pi} \overline{e^{-\beta V(0)}} , \end{aligned} \qquad (2.1)$$

valid for any nonzero magnetic field. We think that these inequalities for the first time allow a controlled approximation to the restricted partition function. They are essential for the following discussion.

Before giving a proof, we mention as an immediate consequence of the second inequality in (2.1) that

$$Z_n(\beta) := \frac{\omega}{2\pi} e^{-(n+1/2)\beta\omega} \overline{\exp\left[-\frac{2\pi\beta}{\omega}\langle x|E_n V E_n|x\rangle\right]} \qquad (2.2)$$

is a lower bound to the restricted partition function:

$$Z_n(\beta) \le \langle x|\overline{E_n e^{-\beta E_n H E_n} E_n}|x\rangle \ . \qquad (2.3)$$

Because of the translational invariance of the random potential, all terms in (2.1)–(2.3) are independent of the site x.

The first inequality in (2.1) is due to the property (1.6) and to Jensen's inequality[23,24] applied to the average over the random potential.

For the second and the third inequalities we need the following noncommutative version[24] of Jensen's inequality due to Peierls, Bogoliubov, and Berezin. It may be written as

$$\mathrm{Tr}(Ee^{-\beta EAE}E) \le \mathrm{Tr}(Ee^{-\beta A}E), \quad A = A^\dagger, \quad E = E^\dagger E \qquad (2.4)$$

and is valid for any self-adjoint operator A and any projection operator E.

The second inequality in (2.1) follows from an application of (2.4) to

$$A = E_n V E_n, \quad E = \frac{2\pi}{\omega} E_n|x\rangle\langle x|E_n \ . \qquad (2.5)$$

For the third inequality we first introduce a nonrandom confining potential $V_c := \Omega^2 x^2/2$ and apply (2.4) to

$$A = V + V_c, \quad E = E_n \ . \qquad (2.6)$$

Finally, after averaging we take the limit $\Omega \downarrow 0$.

III. APPROXIMATION FOR THE RESTRICTED DENSITY OF STATES AND COMPARISON TO EXACT RESULTS

The lower bound $Z_n(\beta)$ to the restricted partition function can uniquely be represented as the two-sided Laplace transform

$$Z_n(\beta) = \int d\varepsilon\, \rho_n(\varepsilon) e^{-\beta\varepsilon} \qquad (3.1)$$

of the function

$$\rho_n(\varepsilon) := (\omega/2\pi)\overline{\delta(\varepsilon - (2\pi/\omega)\langle x|E_n H E_n|x\rangle)} \ . \qquad (3.2)$$

The restricted partition function and the restricted density of states are connected in the same way. Thus, in view of inequality (2.3), the approximation

$$\langle x|\overline{E_n \delta(\varepsilon - E_n H E_n) E_n}|x\rangle \approx \rho_n(\varepsilon) \qquad (3.3)$$

suggests itself. This approximation can also be applied if the restricted partition function does not exist.

Obviously, ρ_n is correctly normalized and nonnegative; that is,

$$\int d\varepsilon\, \rho_n(\varepsilon) = \frac{\omega}{2\pi}, \quad \rho_n(\varepsilon) \ge 0 \ . \qquad (3.4)$$

Moreover, it satisfies the consistency requirements demanded in Ref. 1 and can easily be computed for each choice of the random potential. In conclusion, (3.2) is a simple and tractable approximation to the density of states restricted to the nth Landau level for arbitrary n.

The remainder of this section deals with the comparison of approximation (3.3) to exact results.

(i) We mention the trivial case of a constantly correlated random potential characterized by

$$\overline{\exp\left[-i\int d^2y\, V(y)J(y)\right]} = \int dv\, w(v) \exp\left[-iv\int d^2y\, J(y)\right] . \qquad (3.5)$$

Here, $w(v)$ is a probability density on the real line which can be interpreted as the single-site density $\overline{\delta(v-V(x))}$ of the random potential. For these potentials the correlations of their fluctuations at various sites do not depend on these sites at all. In this case, approximation (3.2) equals the exact result[19]

$$\langle x|\overline{E_n\delta(\varepsilon-E_nHE_n)E_n}|x\rangle=\rho_n(\varepsilon)=\frac{\omega}{2\pi}w(\varepsilon-(n+\tfrac{1}{2})\omega)\ . \tag{3.6}$$

Therefore, within the approximation (3.3) one might hope to obtain reasonable results for correlation lengths which are large compared to the magnetic length $1/\sqrt{\omega}$.

(ii) Next, we consider δ-correlated random (or white-noise) potentials defined by

$$\overline{\exp\left[-i\int d^2y\, V(y)J(y)\right]}=\exp\left[\int d^2y\, g(J(y))\right]\ , \tag{3.7}$$

where $e^{Ag(a)}$ has to be the Fourier transform of a probability distribution on the real line for all $A>0$.[25] δ-correlated random potentials model physical situations in which all correlation lengths are small on the magnetic length scale $1/\sqrt{\omega}$.

For the Cauchy-Lorentz white-noise potential, that is, $g(a)=-v|a|<0$, the approximation $\rho_n(\varepsilon)$ coincides with the exact result[17,19]

$$\langle x|\overline{E_n\delta(\varepsilon-E_nHE_n)E_n}|x\rangle=\rho_n(\varepsilon)=\frac{\omega}{2\pi}\frac{v/\pi}{v^2+[\varepsilon-(n+\frac{1}{2})\omega]^2}\ . \tag{3.8}$$

Wegner[14] has obtained the density of states restricted to the lowest Landau level (see also Refs. 15 and 17),

$$\langle x|\overline{E_0\delta(\varepsilon-E_0HE_0)E_0}|x\rangle=\frac{\sqrt{2\omega}}{\pi^2 v}\frac{\exp\left[\frac{2\pi}{v^2\omega}\left[\varepsilon-\frac{\omega}{2}\right]^2\right]}{1+\frac{8}{v^2\omega}\left[\int_0^{\varepsilon-\omega/2}d\mu\exp\left[\frac{2\pi}{v^2\omega}\mu^2\right]\right]^2} \tag{3.9}$$

for the Gaussian white-noise potential, that is, $g(a)=-(v^2/2)a^2$. In contrast, in this case the approximation (3.2) gives a simple Gaussian

$$\rho_0(\varepsilon)=\frac{\sqrt{2\omega}}{2\pi v}\exp\left[-\frac{2\pi}{v^2\omega}\left[\varepsilon-\frac{\omega}{2}\right]^2\right]\ . \tag{3.10}$$

While $\rho_0(\omega/2)$ exceeds the exact value at the band center by the factor $\pi/2$, the leading asymptotic behavior for $|\varepsilon|\to\infty$,

$$\lim_{|\varepsilon|\to\infty}\left[\frac{1}{(\varepsilon-\omega/2)^2}\ln\langle x|\overline{E_0\delta(\varepsilon-E_0HE_0)E_0}|x\rangle\right]=-\frac{2\pi}{v^2\omega}\ , \tag{3.11}$$

is reproduced by the approximation. For comparison we have plotted the exact result, Ando's SCBA,[8,9]

$$\langle x|\overline{E_0\delta(\varepsilon-E_0HE_0)E_0}|x\rangle\approx\frac{\sqrt{2\omega}}{4\pi^{3/2}v}\left[4-\frac{2\pi}{v^2\omega}\left[\varepsilon-\frac{\omega}{2}\right]^2\right]^{1/2}\times\Theta\left[4-\frac{2\pi}{v^2\omega}\left[\varepsilon-\frac{\omega}{2}\right]^2\right]\ , \tag{3.12}$$

and our $\rho_0(\varepsilon)$ in Fig. 1. Here, Θ denotes the unit-step function.

IV. APPLICATION FOR GAUSSIAN RANDOM POTENTIALS

In the following we calculate the lower bound $Z_n(\beta)$ and the approximation $\rho_n(\varepsilon)$ for Gaussian random potentials which are homogeneous and also isotropic and have zero mean. Hence the characteristic functional of the random potential is given by

$$\overline{\exp\left[-i\int d^2y\, V(y)J(y)\right]}=\exp\left[-\tfrac{1}{2}\int d^2x\int d^2y\, J(x)C(|x-y|)J(y)\right]\ , \tag{4.1}$$

where the covariance

$$C(|x-y|)=\overline{V(x)V(y)} \tag{4.2}$$

has the Fourier representation

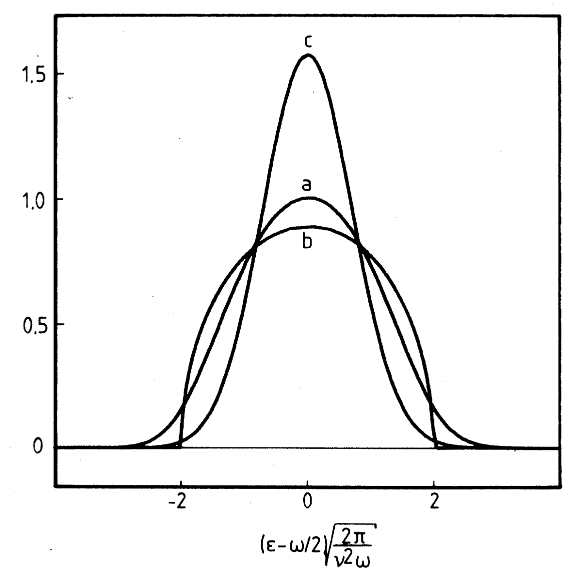

FIG. 1. Plot of (a) the density of states restricted to the lowest Landau level, given in (3.9), of (b) the SCB approximation, given in (3.12), and of (c) our approximation ρ_0, given in (3.10), as a function of the energy ε for the Gaussian white-noise potential. The ordinate is given in units of $\sqrt{2\omega}/\pi^2 v$, the value of the density of states at the band center.

$$C(|x|)=\int \frac{d^2k}{(2\pi)^2}\tilde{C}(|k|)e^{ikx} \tag{4.3}$$

with a non-negative function $\tilde{C}$: $\mathbb{R}_+ \to \mathbb{R}_+$.

In this case the lower bound (2.2) takes the simple form

$$Z_n(\beta)=\frac{\omega}{2\pi}e^{-(n+1/2)\beta\omega}e^{\beta^2\gamma_n^2/2} , \tag{4.4}$$

and therefore one obtains a Gaussian-shaped approximation to the restricted density of states,

$$\rho_n(\varepsilon)=\frac{\omega}{2\pi}\frac{1}{(2\pi\gamma_n^2)^{1/2}}e^{-[\varepsilon-(n+1/2)\omega]^2/2\gamma_n^2} . \tag{4.5}$$

Here the widths γ_n are defined by

$$\gamma_n^2 := \left[\frac{2\pi}{\omega}\right]^2 \int d^2x \int d^2y \,|\langle 0|E_n|x\rangle|^2 \times C(|x-y|)\,|\langle y|E_n|0\rangle|^2 , \tag{4.6}$$

which can be simplified to

$$\gamma_n^2=\frac{\omega}{2\pi}\int_0^\infty d\kappa\, \tilde{C}(\sqrt{2\omega\kappa})e^{-2\kappa}[L_n(\kappa)]^4 \tag{4.7}$$

using the identity (7.377 in Ref. 22)

$$\frac{\omega}{2\pi}|\langle x|E_n|0\rangle|^2=\int \frac{d^2k}{(2\pi)^2}|\langle k/\omega|E_n|0\rangle|^2 e^{ikx} \tag{4.8}$$

for the squared kernel (1.4) of the projection operator E_n.

For Gaussian random potentials the band tails of the density of states restricted to the lowest Landau level have been calculated by Apel,[18] extending a work of Affleck.[16] In particular, the leading asymptotic behavior has been found to be

$$\lim_{|\varepsilon|\to\infty}\left[\frac{1}{(\varepsilon-\omega/2)^2}\ln\langle x|\overline{E_0\delta(\varepsilon-E_0HE_0)E_0}|x\rangle\right] = -\frac{1}{2\gamma_0^2} , \tag{4.9}$$

which is reproduced by the approximation $\rho_0(\varepsilon)$.

Introducing the abbreviation

$$\sigma_n^2 := \frac{\omega}{2\pi}\int_0^\infty d\kappa\, \tilde{C}(\sqrt{2\omega\kappa})e^{-\kappa}[L_n(\kappa)]^2 , \tag{4.10}$$

the widths γ_n can be estimated to

$$\frac{\sigma_n^4}{C(0)} \le \gamma_n^2 \le \sigma_n^2 \le \min\left\{\frac{\omega}{2\pi}\sup_{\kappa\ge 0}[\tilde{C}(\sqrt{2\omega\kappa})],\ C(0)\right\} . \tag{4.11}$$

The first inequality is due to the positive variance of $\exp[-\kappa][L_n(\kappa)]^2$ with respect to the probability density $\omega\tilde{C}(\sqrt{2\omega\kappa})/2\pi C(0)$ on $\mathbb{R}_+$. The other inequalities follow from (22.14.12 in Ref. 21)

$$e^{-\kappa}[L_n(\kappa)]^2 \le 1 \quad \text{for} \quad \kappa \ge 0 , \tag{4.12}$$

and (7.414.3 in Ref. 22)

$$\int_0^\infty d\kappa\, e^{-\kappa}[L_n(\kappa)]^2 = 1 . \tag{4.13}$$

As a consequence of (4.11), the widths γ_n for a Gaussian random potential with continuous covariance $C(|x|)$ take their maximum value $\sqrt{C(0)}$ in the constantly correlated case, that is, $C(|x|)=C(0)$.

V. APPLICATION FOR A GAUSSIAN RANDOM POTENTIAL WITH A GAUSSIAN COVARIANCE

We now focus the discussion on a Gaussian random potential with the Gaussian covariance function

$$C(|x|)=\sigma^2 e^{-x^2/2\lambda^2}, \quad \tilde{C}(|k|)=2\pi\sigma^2\lambda^2 e^{-k^2\lambda^2/2} . \tag{5.1}$$

Here, σ^2 is the single-site variance and λ is the correlation length of the fluctuations of the random potential. Inserting (5.1) into expression (4.7), the widths γ_n can be written as

$$\gamma_0^2=\sigma^2\frac{\lambda^2\omega}{\lambda^2\omega+2} , \qquad \gamma_n^2/\gamma_0^2=\int_0^\infty d\kappa\, e^{-\kappa}[L_n(\kappa/(\lambda^2\omega+2))]^4 . \tag{5.2}$$

In Fig. 2 we give a plot of the ratio γ_n^2/γ_0^2 as a function of $(\lambda^2\omega)^{-1}$ for different values of n.

For $\lambda^2\omega \gg 1$ the widths γ_n are given by (7.414.2 in Ref. 22)

$$\gamma_n^2=\sigma^2\left[1-2\frac{2n+1}{\lambda^2\omega}+O\left[\frac{1}{\lambda^4\omega^2}\right]\right] . \tag{5.3}$$

Upon insertion into (4.4) this leads to an expansion of $Z_n(\beta)$, in which the term of order $1/\lambda^2\omega$ exceeds the corresponding one in the expansion

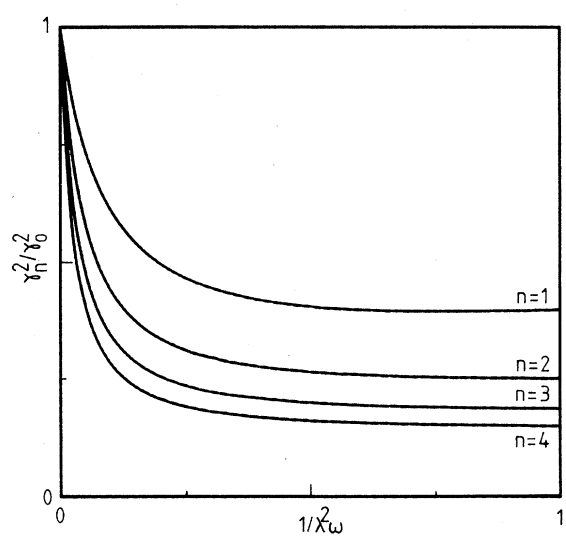

FIG. 2. Plot of the ratio γ_n^2/γ_0^2 of the level widths for the Gaussian covariance function (5.1) as a function of the squared inverse correlation length $1/\lambda^2$.

$$\langle x|\overline{E_n e^{-\beta E_n H E_n} E_n}|x\rangle = \frac{\omega}{2\pi} e^{-(n+1/2)\beta\omega} e^{\beta^2\sigma^2/2} \times \left[1-\left[\frac{\beta^2\sigma^2}{2}\right]\frac{2n+1}{\lambda^2\omega}+O\left[\frac{1}{\lambda^4\omega^2}\right]\right] \quad (5.4)$$

by the factor 2.

To derive the relation (5.4) we employ the Lie-Trotter-Chernoff decomposition[26,27]

$$E_n e^{-\beta E_n V E_n} E_n = \lim_{N\to\infty} (E_n e^{-\beta V/N} E_n)^N , \quad (5.5)$$

and expand the resulting expression for the restricted partition function up to the order $1/\lambda^2\omega$ after averaging. Equation (5.4) extends the result (4.12) in Ref. 28 to nonzero Landau-level indices. Expression (5.4) takes negative values for large β, if corrections of order $1/\lambda^4\omega^2$ are neglected. Therefore the truncated expansion yields negative values of the corresponding restricted density of states for large energies $|\varepsilon|$.

In the limit of Gaussian white noise ($\lambda^2\omega \to 0$, $\sigma/\omega \to \infty$, $v^2 := 2\pi\sigma^2\lambda^2 = \text{const}$), the covariance (5.1) reduces to

$$C(|x|)=v^2\delta(x) \quad (5.6)$$

and the widths γ_n are simplified to

$$\gamma_0^2=\frac{v^2\omega}{4\pi} , \quad \gamma_n^2/\gamma_0^2=\int_0^\infty d\kappa\, e^{-\kappa}[L_n(\kappa/2)]^4 = \frac{1}{\pi}\int_0^\pi d\theta [P_n(\cos\theta)]^2 \le 1 , \quad (5.7)$$

where P_n denotes the nth Legendre polynomial.[21,22] The last equation is due to the convolution theorem for the Laplace transformation and 7.141.2 in Ref. 22. Of course, one recovers (3.10) when γ_0^2 given in (5.7) is inserted into (4.5), and analogously (4.9) reduces to (3.11).

In the Gaussian white-noise limit the asymptotic behavior of the width γ_n for large Landau-level index n can be written as

$$\gamma_n^2 \sim b v^2 \omega \frac{\ln n}{n} \quad \text{as } n\to\infty \quad \text{for } C(|x|)=v^2\delta(x) , \quad (5.8)$$

with some numerical constant $b>0$. This can be deduced from a work by Salomon.[29] Comparing this to a result of Benedict[20] for the Gaussian white-noise potential,

$$\lim_{|\varepsilon|\to\infty}\left[\frac{1}{[\varepsilon-(n+\frac{1}{2})\omega]^2}\ln\langle x|\overline{E_n\delta(\varepsilon-E_nHE_n)E_n}|x\rangle\right] \sim -\frac{2\pi\sqrt{n\pi}}{v^2\omega} \quad \text{as } n\to\infty , \quad (5.9)$$

one recognizes that the approximation (4.5) agrees with the Gaussian decay, but the width is decreasing faster with increasing n.

Finally, we remark that for all $\lambda^2\omega<\infty$ the width γ_n vanishes, too, when the Landau-level index n tends to infinity; that is,

$$\lim_{n\to\infty}\gamma_n^2=0 \quad \text{for } \lambda^2\omega<\infty . \quad (5.10)$$

This can be seen by

$$\gamma_n^2/\gamma_0^2 \le \frac{\lambda^2\omega+2}{2}\int_0^\infty d\kappa\, e^{-\kappa}[L_n(\kappa/2)]^4 , \quad (5.11)$$

Eq. (5.7), and the asymptotic formula (5.8). In general, we have no analytical proof that $\gamma_n^2 \le \gamma_0^2$, although there is numerical evidence that even $\gamma_{n+1}^2 \le \gamma_n^2$.

VI. IMPLICATIONS FOR THE UNRESTRICTED PARTITION FUNCTION AND DENSITY OF STATES

In the preceding sections, quantities restricted to the Hilbert space of the nth Landau level have been discussed. Next we are concerned with the unrestricted quantities $\langle x|\overline{e^{-\beta H}}|x\rangle$ and $\langle x|\overline{\delta(\varepsilon-H)}|x\rangle$, which determine the static properties of the unrestricted model (1.1).

Again we start with a chain of inequalities,

$$\frac{\omega}{4\pi\sinh(\beta\omega/2)} \le \sum_{n=0}^{\infty} Z_n(\beta) \le \langle x|\overline{e^{-\beta H}}|x\rangle \le \frac{\omega}{4\pi\sinh(\beta\omega/2)}\overline{e^{-\beta V(0)}} , \quad (6.1)$$

valid for homogeneous and isotropic random potentials with zero mean and intended to control the unrestricted partition function.

The first inequality follows from the corresponding one in the chain (2.1) by multiplying with $\exp[-(n+\frac{1}{2})\beta\omega]$ and summing over n.

The second inequality is a consequence of (2.3), of the inequality

$$\langle x|\overline{E_n e^{-\beta E_n H E_n} E_n}|x\rangle \le \langle x|E_n\overline{e^{-\beta H}}|x\rangle , \quad (6.2)$$

and the fact (1.3) that the projectors $\{E_n\}$ are resolving the identity operator.

The inequality (6.2), in its turn, follows from an application of (2.4) to

$$A=H+V_c, \quad E=E_n , \quad (6.3)$$

by removing the confining potential V_c after averaging, and by noting that the commutativity

$$\overline{e^{-\beta H}}E_n = E_n\overline{e^{-\beta H}} \quad (6.4)$$

is valid for homogeneous and isotropic random potentials.[30,7,2]

The third inequality in (6.1) is proven with the help of the Golden-Thompson inequality,[31,24]

$$\text{Tr}(e^{-\beta(H+V_c)}) \le \text{Tr}(e^{-\beta H_0}e^{-\beta V}e^{-\beta V_c}) , \quad (6.5)$$

and the subsequent removal of the confining potential after averaging.

As the approximation (3.3) has been encouraged by the inequality (2.3), the second inequality in (6.1) supports the approximation

$$\langle x|\overline{\delta(\varepsilon-H)}|x\rangle \approx \rho(\varepsilon) \qquad (6.6)$$

to the unrestricted density of states, where we have defined

$$\rho(\varepsilon):=\sum_{n=0}^{\infty}\rho_n(\varepsilon)\ . \qquad (6.7)$$

By construction, effects of level mixing are neglected within this approximation. For all constantly correlated random potentials and the Cauchy-Lorentz white-noise potential, level mixing does not occur,[17,19] because in these cases $E_n\overline{\delta(\varepsilon-H)}=E_n\overline{\delta(\varepsilon-E_nHE_n)}E_n$. In view of Eqs. (3.6) and (3.8) the approximation (6.6) is therefore exact in these cases.

For Gaussian random potentials one explicitly gets

$$\rho(\varepsilon)=\frac{\omega}{2\pi}\sum_{n=0}^{\infty}\frac{1}{(2\pi\gamma_n^2)^{1/2}}e^{-[\varepsilon-(n+1/2)\omega]^2/2\gamma_n^2}\ , \qquad (6.8)$$

with the widths γ_n defined in (4.6). This approximation to the density of states is plotted in Fig. 3 for the Gaussian covariance (5.1). The plot should be compared with a corresponding one in Ref. 2 coming from a first-order cumulant approximation proposed in Ref. 6.

A long time ago Gerhardts[7] also proposed a series of Gaussians to approximate the unrestricted density of states for Gaussian random potentials:

$$\langle x|\overline{\delta(\varepsilon-H)}|x\rangle \approx\frac{\omega}{2\pi}\sum_{n=0}^{\infty}\frac{1}{(2\pi\sigma_n^2)^{1/2}}e^{-[\varepsilon-(n+1/2)\omega]^2/2\sigma_n^2}\ . \qquad (6.9)$$

Here each width σ_n, given in (4.10), is larger than the corresponding γ_n [see (4.11)]. Gerhardts has obtained (6.9) within his first-order cumulant approximation by an additional long-time approximation. For a critical discussion of the derivation and the interpretation of (6.9), see Ref. 2.

Eventually we state that for Gaussian random potentials the low-energy behavior of ρ is governed by

$$\lim_{\varepsilon\to-\infty}\left[\frac{1}{\varepsilon^2}\ln\rho(\varepsilon)\right]=-\frac{1}{2\gamma_0^2} \qquad (6.10)$$

under the assumption $\gamma_n^2\le\gamma_0^2$; refer to the end of Sec. V. This will be compared to the asymptotics of $\langle x|\overline{\delta(\varepsilon-H)}|x\rangle$ derived in the next section.

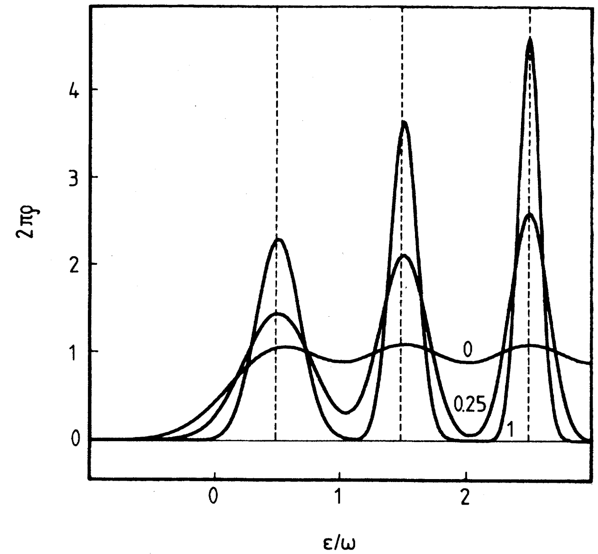

FIG. 3. Plot of the approximate density of states ρ given in (6.8) for the Gaussian covariance function (5.1) as a function of the energy ε for three different values of the correlation length λ and a fixed variance $\sigma^2=0.15\omega^2$. Numbers attached to the curves are values of $1/\lambda^2\omega$.

VII. LOW-ENERGY BEHAVIOR OF THE UNRESTRICTED DENSITY OF STATES FOR GAUSSIAN RANDOM POTENTIALS

In this section we are concerned with the exact asymptotic behavior of the unrestricted density of states for low energies. At this we will only consider Gaussian random potentials as characterized in (4.1).

In contrast to the result (4.9) for the density of states restricted to the lowest Landau level, we assert that the leading low-energy behavior of the unrestricted one is given by

$$\lim_{\varepsilon\to-\infty}\left[\frac{1}{\varepsilon^2}\ln\langle x|\overline{\delta(\varepsilon-H)}|x\rangle\right]=-\frac{1}{2C(0)} \qquad (7.1)$$

independent of the field strength ω. For $\omega=0$ the result (7.1) is well known.[11,32,13]

Due to a saddle-point argument an equivalent statement to (7.1) is

$$\lim_{\beta\to\infty}\left[\frac{1}{\beta^2}\ln\frac{\langle x|\overline{e^{-\beta H}}|x\rangle}{\langle x|e^{-\beta H_0}|x\rangle}\right]=\frac{C(0)}{2} \qquad (7.2)$$

for the large-β limit of the averaged partition function.

Since the upper bound in (6.1) has the same limit, there remains only the construction of a suitable lower bound. To this end we employ the bound

$$\mathrm{Tr}(e^{-\beta(H+V_c)})\ge\frac{e^{-\beta\langle\psi|H_0|\psi\rangle}}{2\pi\beta}\int d^2x\,\exp\left[-\beta\int d^2y\,|\langle\psi|y\rangle|^2[V(x+y)+V_c(x+y)]\right]\ , \qquad (7.3)$$

which is true for any $|\psi\rangle$ with

$$\langle x|\psi\rangle=\langle -x|\psi\rangle\in\mathbb{R},\quad \langle\psi|\psi\rangle=1\ . \qquad (7.4)$$

The inequality (7.3) follows from an application[26,33,10] of (2.4) to

$$A=H+V_c,\quad E=\int d^2x \int d^2y\, e^{ip(x-y)}|x\rangle\langle x-q|\psi\rangle\langle\psi|y-q\rangle\langle y| \tag{7.5}$$

and integrating over p and q.

After averaging and removal of the confining potential V_c, the inequality (7.3) transforms into

$$\langle x|\overline{e^{-\beta H}}|x\rangle \geq \frac{e^{-\beta\langle\psi|H_0|\psi\rangle}}{2\pi\beta}\exp\left[\frac{\beta^2}{2}\int d^2y \int d^2z\, |\langle\psi|y\rangle|^2 C(|y-z|)\,|\langle z|\psi\rangle|^2\right], \tag{7.6}$$

and therefore one has

$$\liminf_{\beta\to\infty}\left[\frac{1}{\beta^2}\ln\frac{\langle x|\overline{e^{-\beta H}}|x\rangle}{\langle x|e^{-\beta H_0}|x\rangle}\right] \geq \tfrac{1}{2}\int d^2y \int d^2z\, |\langle\psi|y\rangle|^2 C(|y-z|)\,|\langle z|\psi\rangle|^2 . \tag{7.7}$$

Choosing $\langle x|\psi\rangle=(2\pi a^2)^{-1/4}\exp(-x^2/4a^2)$, the right-hand side of (7.7) tends to $C(0)/2$ as $a\downarrow 0$. This proves the assertion.

In addition, the limits (7.1) and (7.2) are true for general homogeneous Gaussian random potentials, since the assumption of isotropy has not been used in the above derivation. As an aside we remark that for the proof of Eq. (7.2) the usual large-deviation methods for Wiener path integrals[34] do not apply in the presence of a magnetic field.

The low-energy limit (6.10) of the approximate density of states does not coincide with the result (7.1) except for the constantly correlated case $C(|x|)=C(0)$. This fact is mainly due to the neglect of level mixing.

More seriously, in the case of the Gaussian white-noise potential, the decay of the density of states for low energies must be weaker than that of its Gaussian-shaped approximation (6.8). In fact, for vanishing magnetic field it has been calculated[35] that

$$\lim_{\varepsilon\to-\infty}\left[\frac{1}{|\varepsilon|}\ln\langle x|\overline{\delta(\varepsilon-H)}|x\rangle\right]=-\frac{c}{v^2},\quad \omega=0\ , \tag{7.8}$$

with a numerical constant $c>0$. In analogy to (7.1), this is presumably true for $\omega\neq 0$, too.

VIII. HIGH-FIELD BEHAVIOR OF THE PARTITION FUNCTION

In this section we are concerned with the exact asymptotic behavior of the partition function for high magnetic fields. Again we only will consider homogeneous Gaussian random potentials.

The inequalities (2.1) and (6.1) combined with

$$\lim_{\omega\to\infty}\gamma_n^2=C(0) \tag{8.1}$$

imply

$$\lim_{\omega\to\infty}\frac{\langle x|\overline{E_n e^{-\beta E_n H E_n}E_n}|x\rangle}{\langle x|E_n e^{-\beta H_0}E_n|x\rangle}=e^{\beta^2 C(0)/2} \tag{8.2}$$

and

$$\lim_{\omega\to\infty}\frac{\langle x|\overline{e^{-\beta H}}|x\rangle}{\langle x|e^{-\beta H_0}|x\rangle}=e^{\beta^2 C(0)/2}\ . \tag{8.3}$$

Of course, Eqs. (8.2) and (8.3) reflect the fact that in the high-field limit all correlation lengths of the random potential are infinite compared to the magnetic length $1/\sqrt{\omega}$.

An immediate consequence of Eqs. (8.2) and (8.3) is

$$\lim_{\omega\to\infty}\frac{\langle x|\overline{e^{-\beta H}}|x\rangle}{\sum_{n=0}^{\infty}\langle x|\overline{E_n e^{-\beta E_n H E_n}E_n}|x\rangle}=1\ , \tag{8.4}$$

which means that, at least for the partition function, level mixing can be neglected for sufficiently high fields. For high fields and random potentials with short correlation lengths it is inconsistent to restrict the model (1.1) to the Hilbert space of the lowest Landau level as follows from Eqs. (8.2) and (8.4).

CONCLUDING REMARKS

In the present work we have introduced a new and simple approximation to the averaged partition function and density of states for an electron in the plane subjected to a perpendicular magnetic field and a random potential.

There are two drawbacks of the approximation. First, by the application of the Jensen-Peierls inequality, the quantum fluctuations, and therefore probably the broadenings of the Landau levels, are, in general, underestimated. Second, for the unrestricted quantities, a possible influence of level mixing is neglected from the outset. For example, this is crucial for the Gaussian white-noise potential.

On the other hand, among the benefits of the present approximation we list the following. First, it contains nonperturbative corrections to the solvable case of constantly correlated random potentials. These corrections are physically relevant for very high fields and do not lead to the shortcomings of other approaches discussed in Refs. 1 and 2. Second, for general Gaussian random potentials the approximation reproduces the leading behavior in the tail region of the density of states restricted to the lowest Landau level.

In summary, the present approximation yields a rough overall picture of the averaged density of states and provides a controlled estimate for the averaged partition function.

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